

TIME–HARMONIC BEHAVIOUR OF CRACKED PIEZOELECTRIC SOLID BY BOUNDARY INTEGRAL EQUATION METHOD*

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ABSTRACT. Anti–plane cracked functionally graded finite piezoelectric solid under time–harmonic electro–mechanical load is studied by a non–hypersingular traction boundary integral equation method (BIEM). Exponentially varying material properties are considered. Numerical solutions are obtained by using Mathematica. The dependance of the intensity factors (IF) – mechanical stress intensity factor (SIF) and electrical field intensity factor (FIF) on the inhomogeneous material parameters, on the type and frequency of the dynamic load and on the crack position are analyzed by numerical illustrative examples.

KEY WORDS: piezoelectric finite solid, exponential variability of material properties, anti–plane cracks, dynamic load, boundary integral equation method, mechanical stress intensity factor, electrical field intensity factor.

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1. Introduction

The idea for development of a new class of active composite piezoelectric materials named functionally graded piezoelectric materials (FGPM) appears in recent years. FGPM are with continuously inhomogeneous electro-mechanical material properties and with a predetermined composition profile due to a special processing technological technique where continuously change of the contents of the constituents in space is realized. These materials are investigated intensively in our days due to their applications in smart intelligent systems. The FGPM are brittle, so subjected to dynamic electro-mechanical load in manufacturing and service conditions can cause cracks and other type of defects, which lead to damage.

Mathematical modelling of finite cracked solids of FGPM leads to boundary value problem (BVP) for coupled electro-mechanical system of partial differential equations with non-constant coefficients.

The solution of problems for inhomogeneous piezoelectric solids requires advanced numerical methods because of the high mathematical complexity. The dual integral equations approach used in the most of the papers is restricted to the problems with a simple geometry and boundary conditions. This method is developed in CHAN Y. S. et al. [1], CHEN, J. et al. [2], LI C. and WENG G. [3], MA L. et al. [4], SINGH B. M. et al. [5] for investigating anti-plane cracks in a plane or in a strip with exponentially varying properties in a parallel or in a perpendicular direction to the crack line. SIF is evaluated solving a suitable Fredholm integral equation transforming the BVP to dual integral equations along the crack. Modern computational techniques like finite element method (FEM) and BIEM are applied for more general fracture analysis of piezoelectric solids. So far, most commercial software is available mainly for the static piezoelectric analysis and mostly is based on FEM. FEM results for a solution of fracture problems of piezoelectric solids are shown in KUNA M. [6], ABENDROTH M. et al. [7], GRUEBNER O. et al. [8], WANG C. Y. et al. [9]. Application of FEM for solution of dynamic fracture problems of PEM is discussed in ENDERLEIN M. et al. [10], ENDERLEIN M. [11], ENDERLEIN M. et al. [12]. A survey of the FEM results for cracks in piezoelectric structures is presented in KUNA M. [13]. Most of the results concern cracks in infinite homogeneous piezoelectric solids and only few papers consider finite bounded solids. The numerical example for a finite rectangular in-plane cracked plate under transient loading conditions is solved by FEM in ENDERLEIN M. et al. [12]. However, numerical techniques for 2D and 3D cracked finite piezoelectric solids are still under development, as it is concluded by KUNA M. [13]. By

our opinion, it is even more true for inhomogeneous bounded cracked solids subjected to dynamic electro-mechanical loading. The BIEM is an efficient alternative method. The main difficulty in application of the BIEM to this type of materials is the derivation of the fundamental solution of the governing equation describing wave propagation in anisotropic piezoelectric inhomogeneous media. The difficulty stems from the combination of the material anisotropy, the coupled character of the field variables, the quasi-static approximation of the electric field and the dependence of the material properties on the space variables. Additionally, when cracks are considered the conventional displacement BIEM degenerates. The alternative techniques proposed for the elastic case as sub-region technique, established in BLANDFORD G. E. et al. [14], dual BIEM (SOLLERO P. and ALIABADI M. H. [15]), hypersingular (ZHANG C. et al. [16]) and non-hypersingular traction BIEM (ZHANG C. and GROSS D. [17]) should be applied. The elastodynamic fundamental solution for the general inhomogeneous transversely-isotropic piezoelectric solid is still not available. In RANGELOV T. et al. [18] were derived by the Radon transform elastodynamic fundamental solutions for certain classes of FGPM, of quadratic, exponential and sinusoidal type, correspondingly. The results are mainly for cracks in infinite inhomogeneous solids. The fracture dynamic behaviour of the cracked finite solid for quadratic type of material gradient is studied in DINEVA P. et al. [19]. When the BIEM formulation is applied, the BVP is transformed to an equivalent integro-differential equation along the crack line and along the external boundary of the considered solid. After obtaining numerical solution at every point of the domain, it is evaluated SIF – the leading coefficient in the asymptotic of the generalized displacement and stress solutions near the crack tips.

The present paper concerns the field of computational dynamic fracture mechanics. It is a continuation of the authors' previous studies (RANGELOV T. et al. [18], DINEVA P. et al. [19], MARINOV M. and RANGELOV T. [20], MARINOV M. and RANGELOV T. [21]) as well as a continuation of the efforts of another authors such as SAEZ A. et al. [22], ZHANG C. et al. [23, 24, 16]. These works are on the cracks in elastic and piezoelectric solids using BIEM, well-knowing the advantages of this numerical tool for a solution of problems in the dynamic fracture mechanics. To the authors' knowledge, there are no results for a finite cracked transversely isotropic piezoelectric exponentially inhomogeneous solid subjected to anti-plane mechanical and/or in-plane electrical dynamic time-harmonic load. This is the motivation of the authors to focus mainly on the aim to propose an efficient non-hypersingular traction BIEM for the solution of anti-plane dynamic fracture problems concerning piezoelectric

transversely isotropic exponentially inhomogeneous finite solid.

The solution of the posed problem is based on the following authors' results discussed in their previous papers: a) the fundamental solution for an exponentially inhomogeneous piezoelectric material under dynamic anti-plane mechanical and in-plane electrical load was derived in a closed form in RANGELOV T. et al. [18]; b) the dynamic problem for anti-plane cracks in an infinite exponentially inhomogeneous plane was solved in MARINOV M. and RANGELOV T. [20]; c) the time-harmonic behaviour of a homogeneous finite anti-plane cracked piezoelectric plate was studied in MARINOV M. and RANGELOV T. [21]. The new points in the current work are: (i) an efficient numerical BIEM scheme is proposed for solving an anti-plane dynamic problem of a finite cracked exponentially inhomogeneous transversely isotropic piezoelectric solid; (ii) the material gradient has an arbitrary direction with respect to the crack line, while in the most published cases the material properties vary along or perpendicular to the crack line; (iii) the dynamic behaviour of the cracked solid depends on the frequency of the applied load, on the reference material properties and on the magnitude of the material gradient. Two types of solutions are considered for: frequencies higher than the defined in the paper critical frequency, where the dynamic behaviour is described by wave propagation process, and frequencies lower than the defined critical frequency, where only simple vibration occurs.

The paper is organized as follows: The formulations of the BVP for a finite cracked piezoelectric transversely isotropic solid is presented in Section 2. A fundamental solution of the governing equation for exponential type of inhomogeneity together with the non-hypersingular traction based BIEM formulations and the corresponding solutions is described in Section 3. Series of numerical results for different examples are shown in Section 4, followed by a discussion and some conclusions in Section 5.

2. Statement of the problem

In a Cartesian coordinate system $Ox_1x_2x_3$ in R^3 consider a finite transversely isotropic piezoelectric solid $G \in R^2$, with boundary S and poled in Ox_3 direction. Let $\Gamma = \Gamma^+ \cup \Gamma^-$, $\Gamma \subset G$ is an internal linear crack – an open segment, see Fig. 1. Assume that G is subjected to anti-plane mechanical and in-plane electrical time-harmonic load. The only non-vanishing displacements are the anti-plane mechanical displacement $u_3(x, t)$ and the in-plane electrical displacement $D_i(x, t)$, $i = 1, 2$, $x = (x_1, x_2)$. Since all fields are time-harmonic with frequency ω , the common multiplier $e^{i\omega t}$ is suppressed here and in the following. Assuming quasi-static approximation of piezoelectricity, the field

equation in absence of a body force and an electric charge is presented by the balance equations:

$$(1) \quad \sigma_{i3,i} + \rho\omega^2 u_3 = 0, \quad D_{i,i} = 0,$$

where the summation convention over repeated indices is applied. The strain-displacement and electric field-potential relations are:

$$(2) \quad s_{i3} = u_{3,i}, \quad E_i = -\Phi_{,i},$$

and the constitutive relations, see LANDAU D. L. and LIFSHITZ E. M. [25] are:

$$(3) \quad \begin{aligned} \sigma_{i3} &= c_{44}s_{i3} - e_{15}E_i, \\ D_i &= e_{15}s_{i3} + \varepsilon_{11}E_i, \end{aligned}$$

where subscript $i = 1, 2$ and comma denotes partial differentiation. Here σ_{i3} , s_{i3} , E_i , Φ are the stress tensor, strain tensor, electric field vector and electric potential, respectively. Furthermore, $\rho(x) > 0$, $c_{44}(x) > 0$, $e_{15}(x)$, $\varepsilon_{11}(x) > 0$ are the inhomogeneous mass density, the shear stiffness, piezoelectric and dielectric permittivity characteristics. We assume that the mass density and material parameters vary in the same manner with x , through function $h(x) = e^{2\langle a, x \rangle}$, where $\langle \cdot, \cdot \rangle$ means the scalar product in R^2 , and $a = (a_1, a_2)$, such that

$$(4) \quad c_{44}(x) = c_{44}^0 h(x), \quad e_{15}(x) = e_{15}^0 h(x), \quad \varepsilon_{11}(x) = \varepsilon_{11}^0 h(x), \quad \rho(x) = \rho^0 h(x).$$

Introducing Eqs (3) and (2) into Eq. (1) leads to the coupled system:

$$(5) \quad \begin{aligned} (c_{44}u_{3,i})_{,i} + (e_{15}\Phi_{,i})_{,i} + \rho\omega^2 u_3 &= 0, \\ (e_{15}u_{3,i})_{,i} - (\varepsilon_{11}\Phi_{,i})_{,i} &= 0. \end{aligned}$$

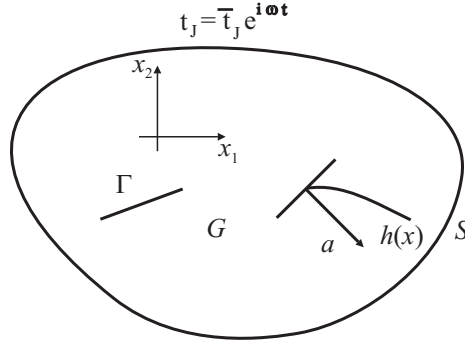


Fig. 1. Cracked inhomogeneous finite solid G with boundary S , subjected to the load t_J . The inhomogeneity direction is a , $h(x)$ is the exponential inhomogeneity function and Γ is an internal linear crack

The basic equations can be written in a more compact form if the notation $u_J = (u_3, \Phi)$, $J = 3, 4$ is introduced. Then, the constitutive equations (3) take the form:

$$(6) \quad \sigma_{iJ} = C_{iJKl} u_{K,l}, \quad i, l = 1, 2,$$

where $C_{iJKl} = C_{iJKl}^0 h(x)$ and $C_{i33l}^0 = \begin{cases} c_{44}^0, & i = l \\ 0, & i \neq l \end{cases}$, $C_{i34l}^0 = C_{i43l}^0 \begin{cases} e_{15}^0, & i = l \\ 0, & i \neq l \end{cases}$,
 $C_{i44l}^0 = \begin{cases} -\varepsilon_{11}^0, & i = l \\ 0, & i \neq l \end{cases}$ and Eq. (5) is reduced to:

$$(7) \quad L(u) \equiv \sigma_{iJ,i} + \rho_{JK} \omega^2 u_K = 0, \quad J, K = 3, 4,$$

where $\rho_{JK} = \begin{cases} \rho, & J = K = 3 \\ 0, & J = 4 \text{ or } K = 4 \end{cases}$.

The boundary conditions on the outer boundary S are given as a prescribed traction $\bar{t}_J$:

$$(8) \quad t_J = \bar{t}_J \quad \text{on } S,$$

where $t_J = \sigma_{iJ} n_i$ and $n = (n_1, n_2)$ is the outer normal vector. The boundary condition along the crack is:

$$(9) \quad t_J = 0 \quad \text{on } \Gamma^+.$$

It means that the crack is free of mechanical traction as well as of surface charge, i.e. the crack is electrically impermeable.

Following AKAMATSU M. AND NAKAMURA G. [26] can be proved that the BVP, Eqs (7) - (9) admits continuously differentiable solution if the usual smoothness and compatibility requirements for the boundary data are satisfied. Consider the following BVPs:

$$(10) \quad \left| \begin{array}{l} L(u^1) = 0 \quad \text{in } G, \\ t_J^1 = \bar{t}_J \quad \text{on } S, \end{array} \right.$$

$$(11) \quad \left| \begin{array}{l} L(u^2) = 0 \quad \text{in } G \setminus \Gamma, \\ t_J^2 = -t_J^1 \quad \text{on } \Gamma^+, \\ t_J^2 = 0 \quad \text{on } S. \end{array} \right.$$

Since the BVP, Eqs (7)–(9) is linear its solution is a superposition of BVPs, Eqs (10) and (11), so $u_J = u_J^1 + u_J^2$ and $t_J = t_J^1 + t_J^2$. The fields u_J^1, t_J^1 are

obtained by the dynamic load on S in the crack free domain G , while u_J^2, t_J^2 are produced by the load $t^2 = -t_J^1$ on Γ^+ and zero boundary conditions on S .

3. Non-hypersingular BIEM

The system of BVPs presented by Eqs (10) and (11) is transformed into an equivalent system of integro-differential equations on $S \cup \Gamma$ following after WANG C. Y. and ZHANG C. [27], RANGELOV T. et al. [18].

$$(12) \quad \begin{aligned} \frac{1}{2}t_J^1(x) &= C_{iJKl}n_i(x) \int_S [(\sigma_{\eta PK}^*(x, y)u_{P,\eta}^1(y) \\ &\quad - \rho_{QP}\omega^2 u_{QK}^*(x, y)u_P^1(y))\delta_{\lambda l} - \sigma_{\lambda PK}^*(x, y)u_{P,l}^1(y)]n_\lambda(y)dS \\ &\quad - C_{iJKl}n_i(x) \int_S u_{PK,l}^*(x, y)t_P^1(y)dS, \quad x \in S, \end{aligned}$$

$$(13) \quad \begin{aligned} t_J^c(x) &= C_{iJKl}n_i(x) \int_{\Gamma^+} [(\sigma_{\eta PK}^*(x, y)\Delta u_{P,\eta}^2(y) \\ &\quad - \rho_{QP}\omega^2 u_{QK}^*(x, y)\Delta u_P^2(y))\delta_{\lambda l} - \sigma_{\lambda PK}^*(x, y)\Delta u_{P,l}^2(y)]n_\lambda(y)d\Gamma \\ &\quad + C_{iJKl}n_i(x) \int_S [(\sigma_{\eta PK}^*(x, y)u_{P,\eta}^2(y) - \rho_{QP}\omega^2 u_{QK}^*(x, y)u_P^2(y))\delta_{\lambda l} \\ &\quad - \sigma_{\lambda PK}^*(x, y)u_{P,l}^2(y)]n_\lambda(y)dS, \quad x \in S \cup \Gamma. \end{aligned}$$

Here, $t_J^c(x) = \begin{cases} -t_J^1(x), & x \in \Gamma^+ \\ 0, & x \in S \end{cases}$, u_{JK}^* is the fundamental solution of Eq.

(7), $\sigma_{iJQ}^* = C_{iJKl}u_{KQ,l}^*$ is the corresponding stress, and $\Delta u_J^2 = u_J^2|_{\Gamma^+} - u_J^2|_{\Gamma^-}$ is the generalized COD on the crack Γ , $x = (x_1, x_2)$ and $y = (y_1, y_2)$ denote the position vector of the observation and source point, respectively. The functions $u_J, t_J, u_{JK}^*, \sigma_{iJQ}^*$ additionally depend on the frequency ω , which is omitted in the list of arguments for simplicity. Equations (12) and (13) constitute a system of integro-differential equations for the unknown Δu_J^2 on the line Γ , t_J^1 on Γ^+ and u_J^1, u_J^2 on the external boundary S of the piezoelectric solid G . From its solution the generalized displacement u_J at every internal point of G can be determined by using the corresponding representation formulae, see WANG C. Y. and ZHANG C. [27], GROSS D. et al. [28].

It is necessary to know the fundamental solution u_{JK}^* and the corresponding stress σ_{iJQ}^* in a closed form in order to solve the system of Eqs (12) and (13). The fundamental solution of Eq. (7) is defined as a solution of the equation:

$$(14) \quad \sigma_{iJM,i}^* + \rho_{JK}\omega^2 u_{KM}^* = -\delta_{JM}\delta(x, \xi),$$

where δ is the Dirac distribution, x, ξ are the source and the field points, respectively and δ_{JM} is the Kronecker symbol. The fundamental solution for the piezoelectric inhomogeneous solids under anti-plane mechanical and in-plane electrical loading using the Radon transform is derived in RANGELOV T. et al. [18], see also MARINOV M. and RANGELOV T. [20]. Here, we shortly present fundamental solutions underlying their different kinds that depend on some critical value of the frequency ω .

Firstly, the equation (14) by a suitable change of functions is transformed to an equation with constant coefficients. Then, we apply Radon transform which allows the construction of a set of fundamental solutions depending on the roots of the characteristic equation of the obtained system of ordinary differential equations. Finally, using both the inverse Radon transform and the inverse change of functions, the fundamental solutions of Eq. (14) is obtained in a closed form. In the first step the smooth transformation:

$$(15) \quad u_{KM}^* = h^{-1/2} U_{KM}^*,$$

applied to Eq. (14) gives:

$$(16) \quad C_{iJKi}^0 U_{KM,ii}^* + [\rho_{JK}^0 \omega^2 - C_{iJKi}^0 a_i^2] U_{KM}^* = -h^{-1/2} (\xi) \delta_{JM} \delta(x, \xi).$$

The equation (16) is solved by the usage of the Radon transform, see ZAYED A. [29]. In R^2 it is defined for function $f \in \mathfrak{F}$ – the set of rapidly decreasing C^∞ functions as:

$$\widehat{f}(s, m) = R[f(x)] = \int_{\langle m, x \rangle = s} f(x) dx = \int f(x) \delta(s - \langle m, x \rangle) dx,$$

with the inverse transform:

$$f(x) = \frac{1}{4\pi^2} \int_{|m|=1} K(\widehat{f}(s, m)|_{s=\langle m, x \rangle}) dm, \quad K(\widehat{f}) = \int_{-\infty}^{\infty} \frac{\partial_\sigma \widehat{f}(\sigma, m)}{s - \sigma} d\sigma.$$

Applying the Radon transform to both sides of Eq. (16) we get with $p_{JK} = C_{iJKi}^0 a_i^2$ the following equation:

$$(17) \quad [C_{iJKi}^0 m_i^2 \partial_s^2 + (\rho_{JK}^0 \omega^2 - p_{JK})] \widehat{U}_{KM}^* = -h^{-1/2} (\xi) \delta_{JM} \delta(s - \langle \xi, m \rangle).$$

These two systems of two linear second order ordinary differential equations are solved following VLADIMIROV V. [30]. Let us denote:

$$(18) \quad \gamma = (\rho^0 \omega^2 - a_0 p) a_0^{-1}, \quad a_0 = c_{44}^0 + \frac{e_{15}^{02}}{\varepsilon_{11}^0}, \quad \omega_0 = \sqrt{\frac{a_0}{\rho^0}} |a|.$$

We obtain the solutions of Eq. (17) with respect to the frequency ω of the applied load, as follows:

(i) for $\gamma > 0$, i.e. $\omega > \omega_0$, $k = \sqrt{\gamma}$,

$$(19) \quad \begin{aligned} \hat{U}_{33}^* &= h^{-1/2}(\xi) \frac{i}{2ka_0} e^{ik|s-\tau|}; \quad \hat{U}_{34}^* = h^{-1/2}(\xi) \frac{i}{2ka_0} \frac{e_{15}^0}{\varepsilon_{11}^0} e^{ik|s-\tau|}; \\ \hat{U}_{43}^* &= \hat{U}_{34}^*, \quad \hat{U}_{44}^* = h^{-1/2}(\xi) \left[\frac{i}{2ka_0} \left(\frac{e_{15}^0}{\varepsilon_{11}^0} \right)^2 e^{ik|s-\tau|} + \frac{1}{\varepsilon_{11}^0} \frac{1}{2|a|} e^{|a||s-\tau|} \right]; \end{aligned}$$

(ii) for $\gamma = 0$, i.e. $\omega = \omega_0$,

$$(20) \quad \begin{aligned} \hat{U}_{33}^* &= -h^{-1/2}(\xi) \frac{1}{2a_0} |s-\tau|, \quad \hat{U}_{34}^* = -h^{-1/2}(\xi) \frac{1}{2a_0} \frac{e_{15}^0}{\varepsilon_{11}^0} |s-\tau|, \quad \hat{U}_{43}^* = \hat{U}_{34}^*, \\ \hat{U}_{44}^* &= \frac{h^{-1/2}(\xi)}{2|a|} \left(\frac{1}{\varepsilon_{11}^0} - \frac{e_{15}^0}{a_0 \varepsilon_{11}^0} + \frac{e_{15}^{02}}{a_0 \varepsilon_{11}^0} \right) e^{|a||s-\tau|} - \frac{h^{-1/2}(\xi)}{2a_0} \left(\frac{e_{15}^0}{\varepsilon_{11}^0} \right)^2 |s-\tau|. \end{aligned}$$

(iii) for $\gamma < 0$, i.e. $\omega < \omega_0$, $k = \sqrt{|\gamma|}$,

$$(21) \quad \begin{aligned} \hat{U}_{33}^* &= -h^{-1/2}(\xi) \frac{1}{2ka_0} e^{k|s-\tau|}, \quad \hat{U}_{34}^* = -h^{-1/2}(\xi) \frac{1}{2ka_0} \frac{e_{15}^0}{\varepsilon_{11}^0} e^{k|s-\tau|}, \\ \hat{U}_{43}^* &= \hat{U}_{34}^*, \quad \hat{U}_{44}^* = -h^{-1/2}(\xi) \left[\frac{1}{2ka_0} \left(\frac{e_{15}^0}{\varepsilon_{11}^0} \right)^2 e^{k|s-\tau|} - \frac{1}{\varepsilon_{11}^0} \frac{1}{2|a|} e^{|a||s-\tau|} \right]. \end{aligned}$$

The fundamental solution is finally derived applying the inverse Radon transform to $\hat{U}_{KJ}^*$. Since the functions $\hat{U}_{KJ}^*$ are linear combinations of $e^{iq|s-\tau|}$, $e^{q|s-\tau|}$ and $|s-\tau|$, for the first part of the inverse Radon transform the following formulas:

$$(22) \quad \begin{aligned} K(e^{iq|s-\tau|}) &= -iq \{ i\pi e^{iq\beta} - 2[\text{ci}(q\beta) \cos(q\beta) + \text{si}(q\beta) \sin(q\beta)] \}_{|\beta|=|s-\tau|}, \\ K(e^{q|s-\tau|}) &= q \{ 2[\text{chi}(q\beta) \cosh(q\beta) - \text{shi}(q\beta) \sinh(q\beta)] \}_{|\beta|=|s-\tau|}, \\ K(|s-\tau|) &= 2 \ln |\beta|_{|\beta|=|s-\tau|}, \end{aligned}$$

are used, where $\text{ci}(\eta) = -\int_{\eta}^{\infty} \frac{\cos t}{t} dt$, $\text{si}(\eta) = -\int_{\eta}^{\infty} \frac{\sin t}{t} dt$ are the cosine integral and sine integral functions and $\text{chi}(\eta) = -\int_0^{\eta} \frac{\cosh t - 1}{t} dt + \ln \eta + C$, $\text{shi}(\eta) = -\int_0^{\eta} \frac{\sinh t}{t} dt$ are the hyperbolic cosine and sine integral functions with Euler's constant C , see BATEMAN H. and A. [31].

So, now the final form of the fundamental solution is obtained from the smooth transformation presented by Eq. (15).

The derived fundamental solutions show clearly that the dynamic behaviour of an exponentially inhomogeneous piezoelectric material can be governed by the frequency of the dynamic load in the following way: (a) at frequency $\omega > \omega_0$, the fundamental solution is expressed by oscillating functions presenting wave propagation process; (b) at frequency $\omega < \omega_0$, the fundamental solution loses its wave nature and shows simple vibration with decreasing amplitudes; (c) at frequency $\omega = \omega_0$, the fundamental solution corresponds to this in the static case.

4. Numerical solution and results

4.1. Numerical solution

The numerical procedure for the solution of the defined BVP follows the numerical algorithm developed and validated in RANGELOV T. et al. [18], DINEVA P. et al. [19]. The outer boundary S and the crack Γ are discretized by quadratic boundary elements (BE). Special crack-tip quarter-point BE is used in order to model the correct asymptotic behaviour of the displacement (like $\sqrt{r}$) and the traction (like $1/\sqrt{r}$) near the crack tips. Applying the shifted point scheme, the singular integrals converge in the sense of Cauchy principal value (CPV), since the smoothness requirements of the approximation $\Delta u_J \in C^{1+\alpha}(\Gamma)$ are fulfilled. All integrals in Eqs (12) and (13) are two dimensional due to the form of the fundamental solution as an integral over the unit circle. In general, there appear two types of integrals – regular integrals and singular integrals, the latter including a weak " $\ln r$ " type of singularity and also a strong " $\frac{1}{r}$ " type of singularity. The regular integrals are solved using Quasi Monte Carlo method, while the singular integrals are solved by a combined method – partially numerically and partially analytically as CPV integrals. After the discretization procedure, an algebraic linear complex system of equations is obtained. It is solved with respect to the generalized displacement along the crack and along the external boundary.

The program code based on Mathematica 8, see [32] has been created following the above outlined procedure. The mechanical dynamic SIF K_{III} , the electrical displacement intensity factor K_D and the electric intensity factor K_E are obtained directly from the traction nodal values ahead of the crack-tip, see SUO Z. et al. [33]. In a local polar coordinate system (r, φ) with the origin

the crack edge the formulae are correspondingly:

$$(23) \quad \begin{aligned} K_{III} &= \lim_{r \rightarrow \pm 0} t_3 \sqrt{2\pi r}, \quad K_D = \lim_{r \rightarrow \pm 0} t_4 \sqrt{2\pi r}, \\ K_E &= \lim_{r \rightarrow \pm 0} E_3 \sqrt{2\pi r}, \quad E_3 = \frac{1}{e_{15}^2 + c_{44}\varepsilon_{11}} (-e_{15}t_3 + c_{44}t_4), \end{aligned}$$

where t_J is the generalized traction at the point (r, φ) close to the crack-tip. Formulae (23) is based on the known fact, see LI C. and WENG G. [3], that stresses and electric displacements at the crack-tip in graded materials still possess the inverse square root singularity in terms of a local coordinate at the crack-tip. The angular distribution functions are the same as in the homogeneous material. The IFs depend on the material gradient through the solution of the BVP. Although the structure of the asymptotic crack-tip fields is not influenced by the material gradient.

Note that the integrals in Eqs (12) and (13) over the BE are two-dimensional with regular and singular kernels. In the Mathematic's code the regular integrals are solved using Adaptive Monte Carlo Method with 300 points. The singular integrals are solved analytically with respect to r and numerically with respect to φ (in the intrinsic coordinates in the domain $(z, \varphi) \in [-1, 1] \times [0, 2\pi]$), see DINEVA P. et al. [19], MARINOV M. and RANGELOV T. [21].

4.2. Numerical results

The material used in the numerical examples is PZT-4, whose data are $c_{44}^0 = 2.56 \times 10^{10} \text{ N/m}^2$, $e_{15}^0 = 12.7 \text{ C/m}^2$, $\varepsilon_{11}^0 = 64.6 \times 10^{-10} \text{ C/Vm}$ and $\rho^0 = 7.5 \times 10^3 \text{ kg/m}^3$. The crack Γ is a segment with a length $2c = 5 \text{ mm}$ and its position is determined by the center point (x_1, x_2) and inclination angle ψ with respect to Ox_1 axis. The rectangular domain G is with dimension $20 \text{ mm} \times 40 \text{ mm}$. The crack is discretized by 7 BE with lengths l_j : $l_1 = l_7 = 0.375 \text{ mm}$, $l_2 = l_6 = 0.5 \text{ mm}$, $l_3 = l_5 = 1.0 \text{ mm}$, $l_4 = 1.25 \text{ mm}$. The boundary S is discretized by 20 BE. Time-harmonic load is uniform uniaxial electro-mechanical tension in Ox_2 direction with amplitudes σ_0 in N/m^2 and D_0 in C/m^2 , see Fig. 2.

The proposed method has no computational limitations concerning the frequency interval used for simulations. However, the size of the discretization mesh should satisfy the well known accuracy condition $\lambda > 10l$ where λ is the wave length and l is the maximal size of the boundary elements. Our aim in simulations is to consider scattering and diffraction processes in finite exponentially inhomogeneous piezoelectric solids with internal cracks around critical frequency ω_0 , that depends on the variable material properties.

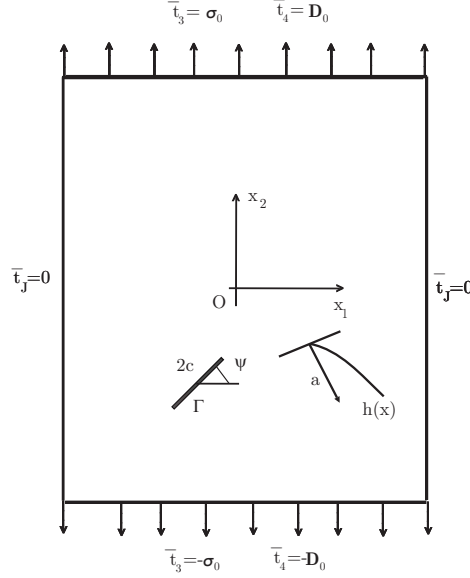


Fig. 2. Rectangular inhomogeneous finite solid with a crack Γ of length $2c$ and angle ψ with respect to Ox_1 axis. The inhomogeneity direction is a , $h(x)$ is the exponential inhomogeneity function and the load on the boundary is $\bar{t}_J$

Denote $a_1 = |a| \cos \alpha$, $a_2 = |a| \sin \alpha$, where α is the direction and $|a| = \sqrt{a_1^2 + a_2^2}$ is the magnitude of the material inhomogeneity. The behaviour of SIF for exponential inhomogeneity depends strongly on the critical frequency ω_0 , defined in (18) as is mentioned in DAROS C. H. [34], MARINOV M. and RANGELOV T. [20] for the whole FGPM plane. The dynamic behaviour for $\omega < \omega_0$ is simple vibration; for $\omega = \omega_0$ it is static and for $\omega > \omega_0$ it is wave propagation. These effects are shown in the numerical examples below in Figs 4–8.

The normalized frequency is $\Omega = c\sqrt{\rho^0/a_0}\omega$ in the presented examples. Denote the amplitude of the mechanical load by $\sigma = 400 \times 10^6 \text{ N/m}^2$. We consider the following types of loads: mechanical with $\sigma_0 = \sigma$, $D_0 = 10^{-5} \text{ C/m}^2$; electro-mechanical with $\sigma_0 = \sigma$, $D_0 = 0.1 \frac{\varepsilon_{11}^0}{e_{15}^0} \sigma$; electrical with $\sigma_0 = 10^{-3} \text{ N/m}^2$,

$$D_0 = 0.1 \frac{\varepsilon_{11}^0}{e_{15}^0} \sigma.$$

It is plotted in the figures the absolute value of the normalized SIF

$K_{III}^* = \frac{K_{III}}{\sigma\sqrt{\pi c}}$ and normalized electrical FIF $K_E^* = \frac{K_E}{\sigma\sqrt{\pi c}}$ versus nondimensional frequency Ω for different values of the normalized inhomogeneity amplitude $\beta = 2|a|c$, the direction of the material inhomogeneity α and the crack location expressed by the crack center (x_1, x_2) and the crack's inclined angle ψ .

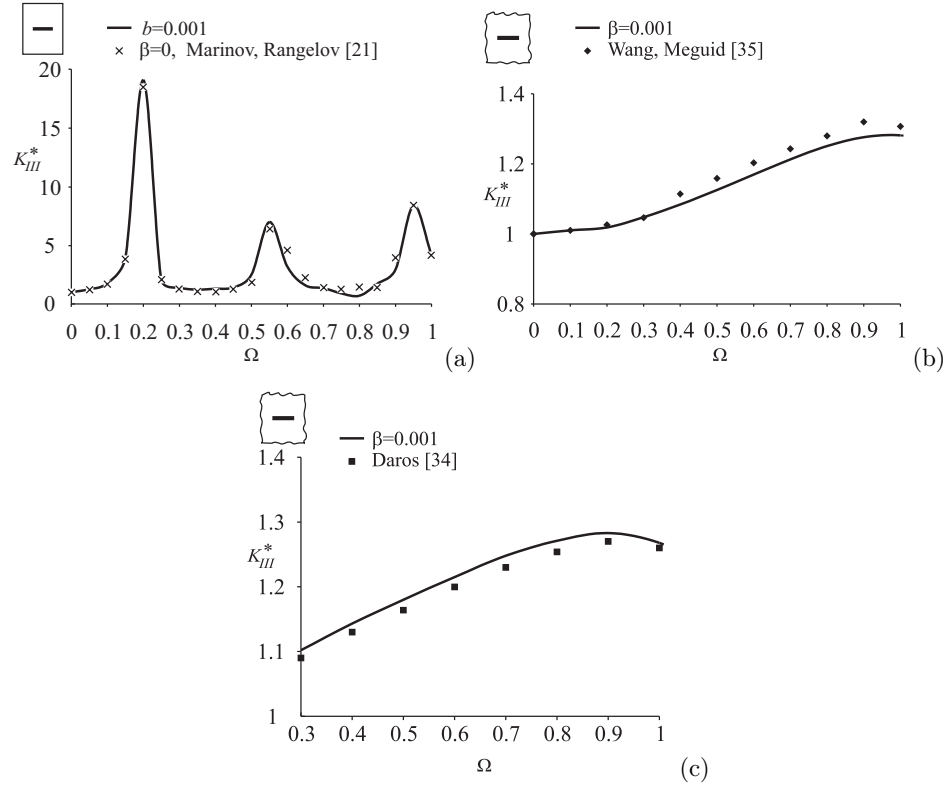


Fig. 3 Comparison of SIF K_{III}^* versus frequency Ω between: (a) the authors' BIEM results for $\beta = 0.001$ and MARINOV M. and RANGELOV T. [21] result for $\beta = 0$; (b) the authors' BIEM results based on truncation approach for $\beta = 0.001$ and WANG X. D. and MEGUID S. A. [35] for $\beta = 0$; (c) the authors' BIEM results based on truncation approach, using $27BE$ and $47BE$ for $\beta = 0.4$ and with DAROS C. H. [34] for $\beta = 0.4$

To the authors' best knowledge, there are still no available results for mechanical SIF and electrical FIF of finite anti-plane cracked piezoelectric solids with exponentially varying properties in both frequency intervals before

and after the critical frequency ω_0 , where the cracked solid changes its dynamic behaviour. Due to that, the validation is based on the comparison of the authors' results with the available ones in the literature for the homogeneous case. It is used the developed software for the inhomogeneous case and the inhomogeneity amplitude is replaced with $\beta = 0.001$. In this homogeneous case, the compare is with the obtained results in DINEVA P. et al. [19], MARINOV M. and RANGELOV T. [21] for a finite rectangular anti-plane cracked solid under uniform uniaxial time-harmonic traction and the maximal percentage difference is smaller than 7%, see Fig. 3(a). Additionally, in order to test the proposed here new BIEM results for a finite piezoelectric solid, they are also compared with the results of WANG X. D. and MEGUID S. A. [35] for an anti-plane crack in a homogeneous plane using the truncation approach where the size of the square plate is 10 times greater than the half-length of the crack, see Fig. 3(b). The same truncation technique is used by the authors in order to compare the authors' results with the results obtained by DAROS C. H. [34], see Fig. 3(c) where anti-plane crack in elastic anisotropic exponentially inhomogeneous plane under time-harmonic load is considered. In both cases the percentage difference between the authors' results and the results obtained in WANG X. D. and MEGUID S. A. [35], DAROS C. H. [34] is not more than 7%. Note here that in WANG X. D. and MEGUID S. A. [35] it is used the dual integral equation method, while in DAROS C. H. [34] it is used the non-hypersingular traction BIEM based on the frequency dependent fundamental solution, obtained by Fourier transform.

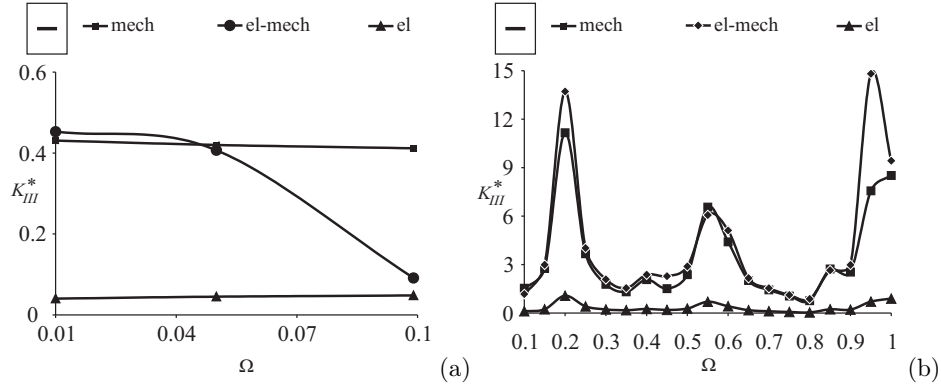


Fig. 4. SIF K_{III}^* versus frequency Ω for different types of applied load:
(a) $\Omega \in (0, 0.1)$; (b) $\Omega \in (0.1, 1.0)$

In the first group of figures (Figs 4 and 5), mechanical SIF K_{III}^* and

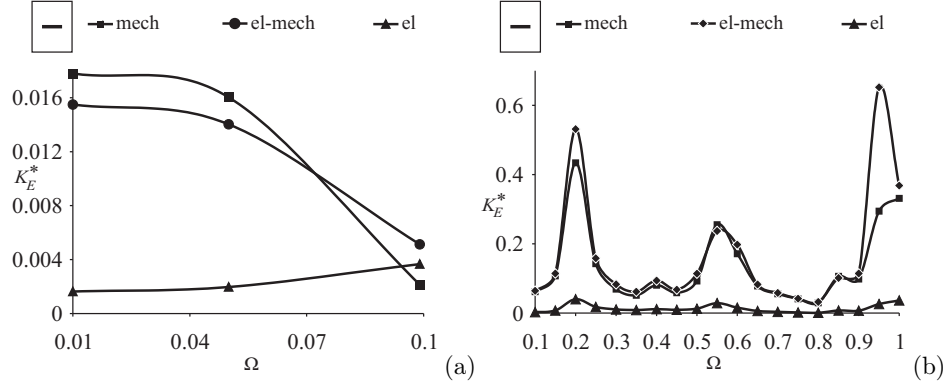


Fig. 5. Electrical FIF K_E^* versus frequency Ω for different types of load:
(a) $\Omega \in (0, 0.1)$; (b) $\Omega \in (0.1, 1.0)$

electrical FIF K_E^* versus frequency Ω for three types of loads: mechanical, electro-mechanical and electrical are plotted. The inhomogeneity is with magnitude $\beta = 0.2$, so the critical frequency is $\Omega = 0.1$. The direction of material inhomogeneity is $\alpha = \pi/2$ and the crack is a segment $(-c, c)$ on Ox_1 axis, i.e. $(x_1, x_2) = (0, 0)$ and $\psi = 0$.

Figures 4(a), (b) and 5(a), (b) demonstrate clearly the sensitivity of the stress and the electric field concentrations to the type of the applied load and to the coupled character of the electro-mechanical continuum. As could be expected, the behaviour of both SIFs strongly differs in two considered frequency intervals before and after the critical frequency ω_0 . The oscillating character and the appearance of the resonance frequencies when $\omega > \omega_0$ is visible when Figs 4(a) and 5(a) with Figs 4(b) and 5(b) are compared. Normalized mechanical SIF K_{III}^* in the frequency interval $\omega < \omega_0$ has different behaviour with respect to the frequency when electro-mechanical load is applied. At very low frequencies, this behaviour approaches the corresponding one in the case of pure mechanical load, while for frequencies near the critical one the values of K_{III}^* are close to those obtained in the case of pure electrical load, see Fig. 4(a). Figure 4(b) reveals that the frequency dependent curve of K_{III}^* almost reserves the places of the resonance frequencies, but the maximal values are obtained at applying electro-mechanical loads. Figs 4 and 5 confirm that the coupled character of the exponentially inhomogeneous piezoelectric material leads to a quite different dynamic stress and electric field concentrations with respect to the prescribed frequency, i.e. there exists different near-field behaviour if the frequency is lower or higher than the critical one.

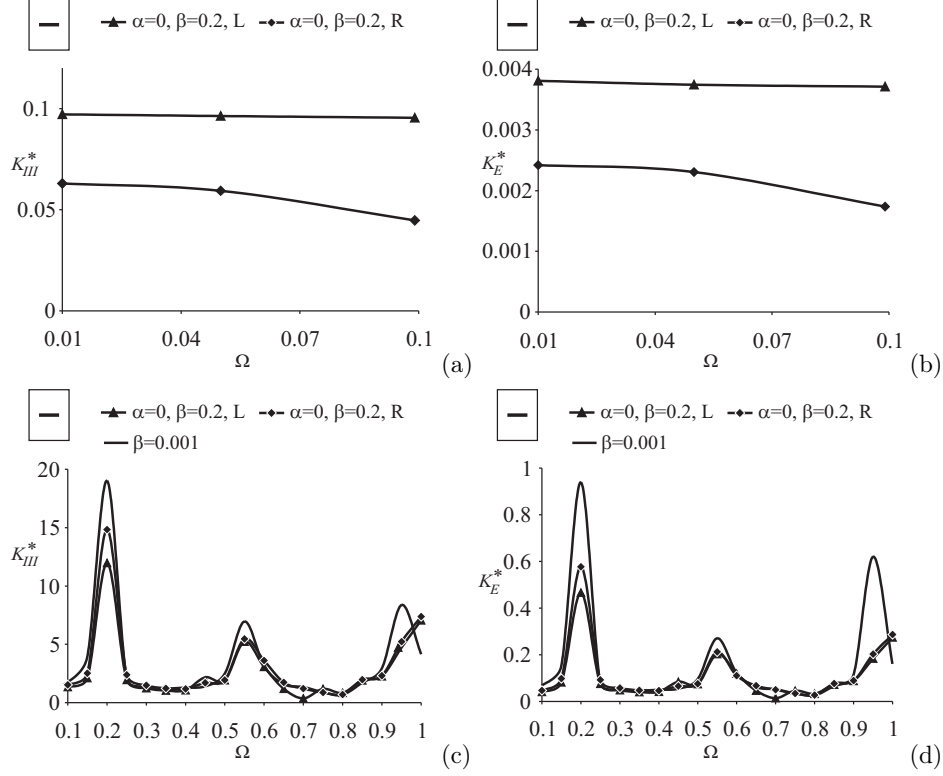


Fig. 6. SIF K_{III}^* and FIF K_E^* versus Ω for $\beta = 0.2$, $\alpha = 0$: (a) K_{III}^* , $\Omega \in (0.0, 0.1)$; (b) K_E^* , $\Omega \in (0.0, 0.1)$; (c) K_{III}^* , $\Omega \in (0.1, 1.0)$; (d) K_E^* , $\Omega \in (0.1, 1.0)$

The second group of figures (Figs 6–8) aims to show the dependence of the mechanical SIF and the electrical FIF on the magnitude of the inhomogeneity for a fixed inhomogeneity direction $\alpha = 0$. Three cases for β are considered: $\beta = 0.2$ in Fig. 6; $\beta = 0.4$ in Fig. 7 and $\beta = 0.8$ in Fig. 8. The crack is situated as in the first group of examples and both IFs at the left and at the right crack-tip are shown in the figures.

Figures 6–8 visualize the following effects: (a) near the critical frequency ω_0 where the character of the dynamic behaviour is changed the value of the mechanical SIF and electrical FIF make a jump; (b) there is a different stress and electric field concentration behaviour at a fixed magnitude at the left and at the right crack tips; (c) with increasing the inhomogeneity magnitude the difference in the near stress fields at left and right crack-tips increases and while

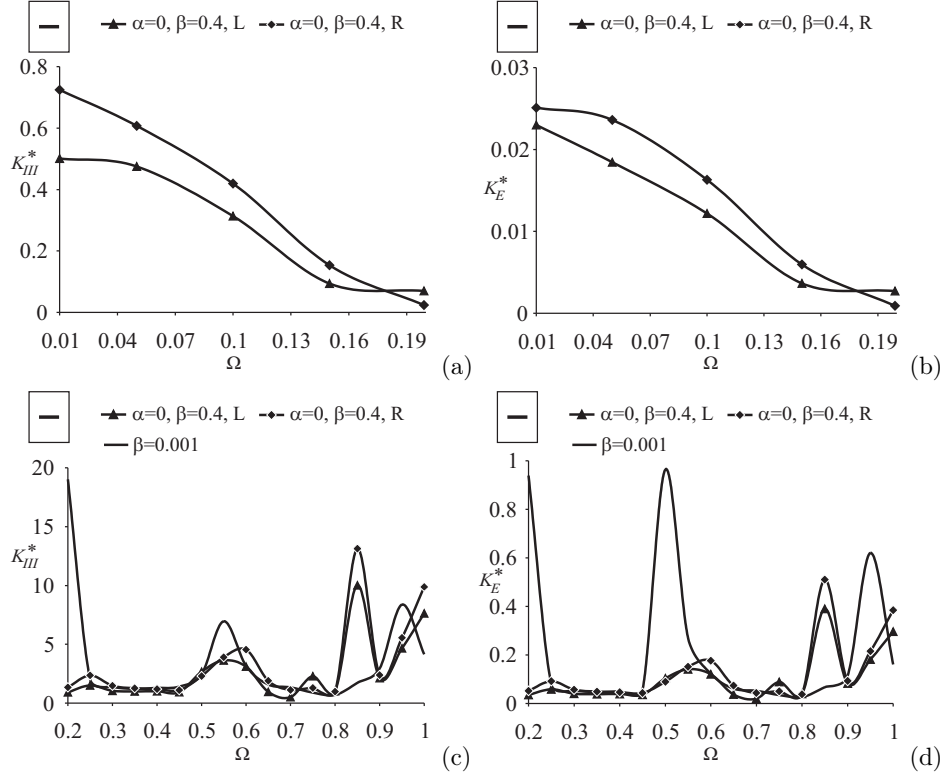


Fig. 7. SIF K_{III}^* and FIF K_E^* versus Ω for $\beta = 0.4$, $\alpha = 0$: (a) K_{III}^* , $\Omega \in (0.0, 0.1)$; (b) K_E^* , $\Omega \in (0.0, 0.1)$; (c) K_{III}^* , $\Omega \in (0.1, 1.0)$; (d) K_E^* , $\Omega \in (0.1, 1.0)$

in Fig. 6(a) the higher values are for the left crack-tip, in Figs 7(a) and 8(a) the right crack-tip values are higher; (d) with increasing of the inhomogeneity magnitude, K_{III}^* and K_E^* decrease, see and compare Fig. 6(c, d) and Fig. 8(c, d). This is only true for the frequencies $\omega > \omega_0$, while the opposite behaviour is observed for the frequencies $\omega < \omega_0$; (e) with changing the magnitude of the material gradient the places of the resonance phenomena are also changed and these changes are different for both crack-tips.

The effect of the crack location presented by the inclination angle ψ and the coordinates of the crack center on the SIF K_{III}^* is presented in Fig. 9 for a fixed inhomogeneity magnitude $\beta = 0.2$ and inhomogeneity direction $\alpha = 0$. Three positions of the crack are considered: a) $\psi_1 = \pi/3$, $(x_1^1, x_2^1) = (0, 0)$; b) $\psi_2 = \pi/4$, $(x_1^2, x_2^2) = (5.17 \cdot 10^{-4}, -3.97 \cdot 10^{-4})$; c) $\psi_3 = \pi/6$, $(x_1^3, x_2^3) =$

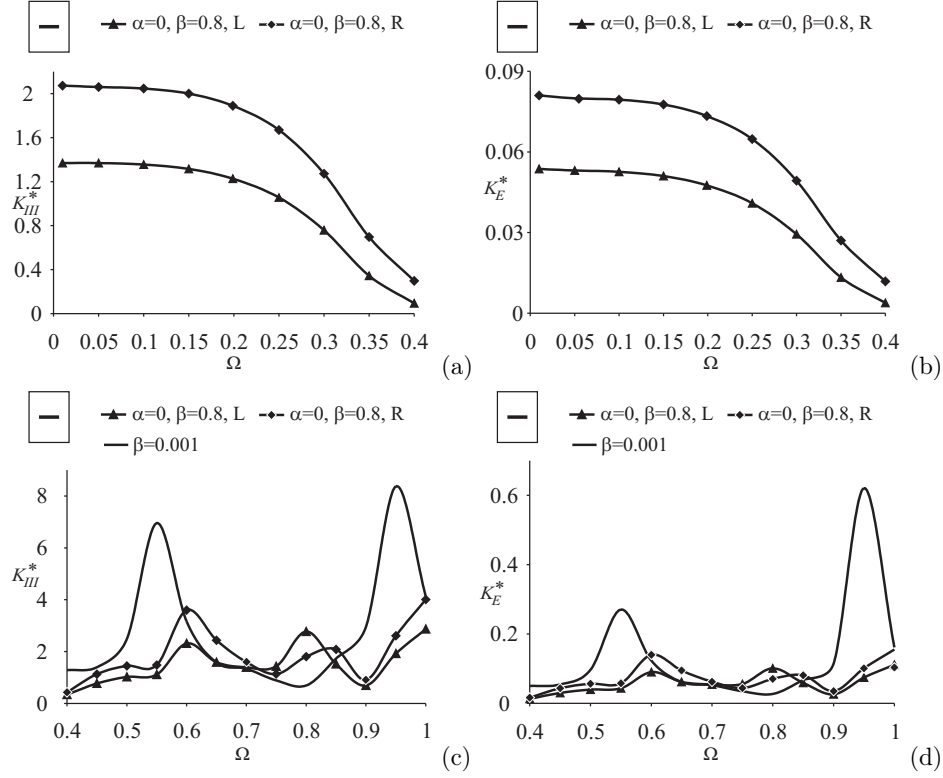


Fig. 8. SIF K_{III}^* and FIF K_E^* versus Ω for $\beta = 0.8$, $\alpha = 0$: (a) K_{III}^* , $\Omega \in (0.0, 0.1)$; (b) K_E^* , $\Omega \in (0.0, 0.1)$; (c) K_{III}^* , $\Omega \in (0.1, 1.0)$; (d) K_E^* , $\Omega \in (0.1, 1.0)$

$(9.15 \cdot 10^{-4}, -9.15 \cdot 10^{-4})$.

Figure 9 shows that with the decreasing of the angle of the crack inclination the values of the SIF increase. There is also not big difference between the behaviour at both crack-tips excluding the resonance values at higher frequencies.

The last group of figures (Figs 10 and 11) presents the combined effect of both load type: mechanical (Fig. 10) or electrical (Fig. 11) and the crack position on the non-uniform stress and the electric field distribution in a finite solid. The magnitude of the material inhomogeneity is $\beta = 0.4$, the fixed frequency $\Omega = 0.6$ and dynamic stress concentration filed for different directions of the material gradient $\alpha = m \frac{\pi}{10}, m = 0, \dots, 9$ is shown. The crack is with a fixed angle $\psi = \pi/3$ with respect to Ox_1 axis and two cases for the

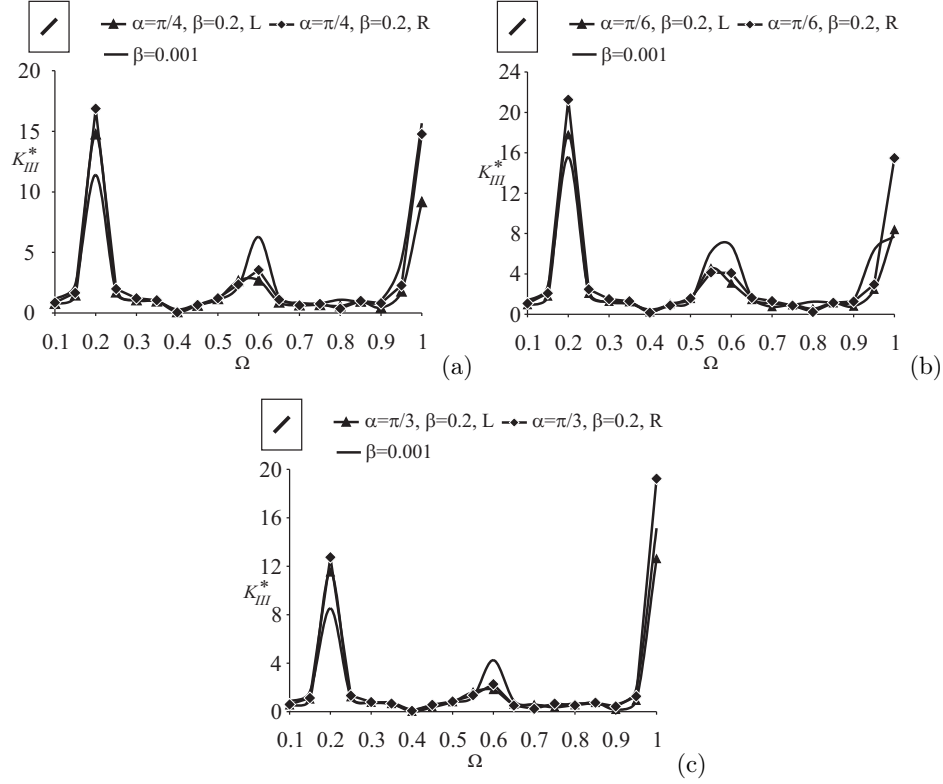


Fig. 9. SIF K_{III}^* versus frequency $\Omega \in (0.1, 1.0)$ for $\beta = 0.2$, $\alpha = 0$, and slanted crack: (a) $\psi_1, (x_1^1, x_2^1)$; (b) $\psi_2, (x_1^2, x_2^2)$; (c) $\psi_3, (x_1^3, x_2^3)$

crack's center: $(x_1^1, x_2^1) = (0, 0)$ and $(x_1^4, x_2^4) = (-7.75 \cdot 10^{-3}, -1.583 \cdot 10^{-2})$ are considered.

The following effects are visible comparing Figs 10 and 11: (a) the coupled material properties provoke the coupled reaction in the obtained stress concentration fields, when mechanical or electrical load are applied. These fields are different at both crack-tips due to the material inhomogeneity; (b) the crack location is an important key factor and it can change the amplitude of the SIF as can be seen in Figs 10(b) and 11(b), where higher values are obtained due to the interaction between the crack and the external boundary of the finite solid; (c) the direction of the material gradient is also key factor and its influence is different for the stress fields at the left and the right crack tip.

The following conclusions can be derived: (i) In most obtained results

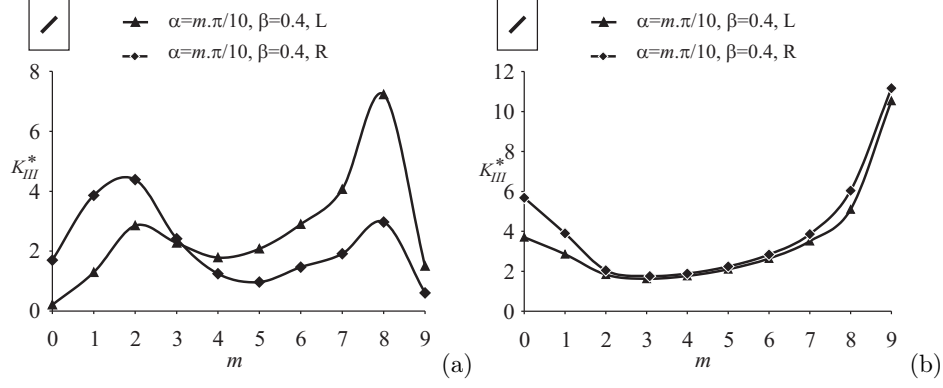


Fig. 10. SIF K_{III}^* for fixed frequency $\Omega = 0.6$, $\beta = 0.4$ versus $\alpha = m\frac{\pi}{10}$, $m = 0, \dots, 9$ under mechanical load for crack angel $\psi = \pi/3$ and crack center: (a) (x_1^1, x_2^1) ; (b) (x_1^4, x_2^4)

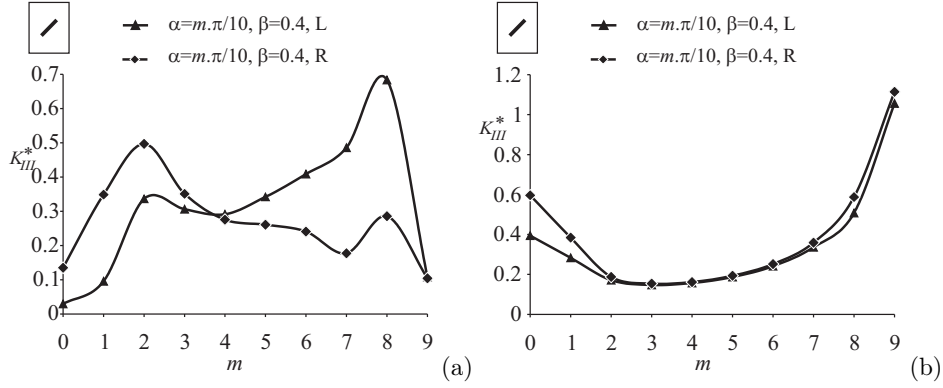


Fig. 11. SIF K_{III}^* for fixed frequency $\Omega = 0.6$, $\beta = 0.4$ versus $\alpha = m\frac{\pi}{10}$, $m = 0, \dots, 9$ under electrical load for crack angel $\psi = \pi/3$ and crack center: (a) (x_1^1, x_2^1) ; (b) (x_1^4, x_2^4)

the IFs for the inhomogeneity case is lower than for the homogeneous one but this effect is frequency dependent; (ii) For the inhomogeneity direction $\alpha = \pi/2$, SIF is equal at both crack-tips; (iii) For the inhomogeneity direction $\alpha = 0$, at the left crack-tip the magnitude of SIF is greater than the magnitude of the SIF at the right crack-tip for frequencies below the critical frequency and vice-versa for frequencies above the critical one. Note, that in the case of the

infinite exponentially inhomogeneous piezoelectric plane, see MARINOV M. and RANGELOV T. [20] the order of the magnitudes of the SIF at the left and at the right crack-tips with increasing of the inhomogeneity magnitude is opposite to those obtained here for the case of finite solids. The cause of changing the order in the case of finite domain can be explained by the influence of the external boundary of the considered finite solid, where additional scattering and reflections of the propagating wave occur and where the scattered waves from both the external solid boundary and from the existing crack interact. The numerical results show that the influence of the external boundary strongly decreases at the following relation $d > 10c$ between the smaller side d of the rectangular plate and the half-length of the crack c .

5. Conclusion

Time-harmonic fracture problem for an anti-plane cracked finite solid of FGPM is solved numerically by means of non-hypersingular traction BIEM and developed and validated by Mathematica code. The BIEM computational tool is based on the derived in a closed form fundamental solution obtained by Radon transform. Presented and discussed are illustrative examples for mechanical SIF and electrical FIF computation. The simulations reveal that the dynamic non-uniform stress and the electric field distribution in a finite inhomogeneous piezoelectric cracked solid is a complex result of mutual play of different key factors as: the type and the characteristics of the applied load; the type of material gradient and its magnitude and direction; the crack location and its interaction with the external boundary of the finite solid. The proposed methodology can be applied further for the solution of dynamic fracture problems in finite FGPM solids with multiple cracks. The proposed approach holds the potential to be used as a base for formulation and solution of inverse problems for identification of cracks shape, sizes and locations as well as for the evaluation of the fracture state of different structural elements with a different type of defects as cracks, holes and inclusions.

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