

Study of Vibration of Viscoelastic Timoshenko Beam on Viscoelastic Foundation Subjected to Axial Force

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1. Introduction*

This paper is devoted to analysing the dynamic response of a simply supported viscoelastic Timoshenko beam on viscoelastic foundation subjected to a given (compressive) axial force under given initial conditions and specified time-dependent boundary conditions. The reaction of the foundation both in horizontal and vertical direction is taken care of by adopting a generalized Winklerian model for the foundation. The viscoelasticity of the beam and foundation is represented by a standard linear solid type model so as to encompass a wide variety of soil conditions encountered in practice. The free, steady state and forced vibrations of the system are studied for three cases of practical importance, namely, steady-state frequency response, free vibrations under velocity impact and forced vibrations of the beam under the action of a moving force. Laplace and Finite Fourier transform are used in solving the problem. The present paper generalizes the earlier studies by Nowacki [5] on vibrations of Euler beam on Winklerian foundation, Leissa, Hwee [4] on the effect of viscous damping on the response of a Timoshenko beam, Veletsos [8] on the response of bridges (idealised as a discrete system) subjected to a moving load, Djodjo [2] on vibrations of elastic Timoshenko beam on elastic foundation subjected to axial force and Pan [6] on the vibrations of simply supported viscoelastic Timoshenko beams.

2. The mathematical model

The viscoelasticity of the beam is represented by a standard linear solid in which the material constants E and G (which are functions of time-derivative operator $\partial_t = \partial/\partial t$) are given by Nowacki [5]:

$$(1) \quad E = E(\partial_t) = E_0(1 + t_2\partial_t)/(1 + t_3\partial_t),$$

* Partial derivatives with respect to time are denoted by ∂_t . Partial derivatives with respect to x are denoted by $w_{,x}$ or $\partial_x w$. SSEB stands for a simply supported Euler — Bernoulli beam subjected to a uniform y distributed load. ' $\leftarrow$ ' denotes the replacement symbol. A δ before a symbol denotes the virtual displacement operator.

$$(2) \quad G = G(\partial_t) = G_0(1 + t_2 \partial_t)/(1 + t_1 \partial_t),$$

in which E_0 and G_0 are elastic constants and t_1 and t_2 are relaxation and retardation time constants of the material of the beam and t_3 is given by

$$(3) \quad t_3 = 2(1 + \nu)t_1/3 + (1 - 2\nu)t_2/3,$$

with ν denoting the Poisson's ratio which is assumed to be constant. If the material of the beam is almost incompressible, then $\nu \approx \frac{1}{2}$ so that $t_3 \approx t_1$ and the elastic and shear moduli operators E and G given above almost coincide. The kinetic energy T and strain energy U for the Timoshenko beam are then given by

$$(4) \quad T = \frac{1}{2} \int_0^L (\rho A \dot{w}^2 + \delta I \dot{\psi}^2) dx,$$

$$(5) \quad U = \frac{1}{2} \int_0^L [EI \psi_{,x}^2 + GA(w_{,x} - \psi)^2] dx,$$

in which

L — length of the beam;

ρ — mass density of the material of the beam;

I — area moment of inertia of the cross-section of the beam about neutral axis;

A — area of cross-section of the beam;

w — total deflection of the beam;

ψ — slope due to bending alone of the beam.

The beam is subjected to lateral loading $q(x, t)$ on the top surface, compressive axial force S and shearing force Q^* and bending moments M^* on the edge faces (Fig. 1). The work done by these external forces is

$$(6) \quad W_1 = \int_0^L \left(qw + \frac{1}{2} S w_{,x}^2 \right) dx + [wQ^*]_0^L - [\psi M^*]_0^L,$$

in which the term with S denotes the work done by the compressive axial force S acting on the beam and is obtained by linearisation and assuming that the beam is freely supported.

Besides the above forces, the beam is subjected to horizontal and vertical reaction loadings F_x and F_z (Fig. 1) on the bottom surface due to the

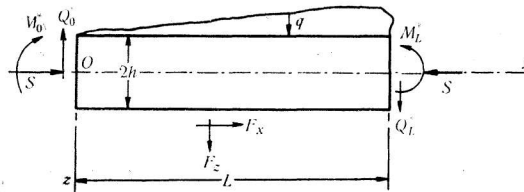


Fig. 1. Conventions of external loading acting on the beam

viscoelastic foundation. Following the Winkler's hypothesis, it will be assumed that these reactions are proportional to local displacements and act in a direction opposite to them, so that

$$(7,8) \quad F_x = k_x h \psi, \quad \text{and} \quad F_z = -k_z w,$$

in which $2h$ denotes the thickness of the beam and $-h\psi$ and w denote respectively the horizontal and vertical components of displacement of particles on the bottom surface of the beam. k_x and k_z are moduli of foundation along horizontal and vertical directions. Such a foundation which exerts both horizontal and vertical reactions is called a 'generalized foundation' by Kerr [3] and a similar foundation model was assumed by Djodjo [2] recently. To account for a wide variety of soil conditions, a standard linear solid model is assumed for vertical component k_z and a Kelvin model for the horizontal component k_x so that

$$(9) \quad k_z = k_z(\partial_t) = k_{z0}(1 + t_4 \partial_t)/(1 + t_5 \partial_t),$$

$$(10) \quad k_x = k_x(\partial_t) = k_{x0}(1 + t_6 \partial_t),$$

where k_{z0} , k_{x0} are the elastic components of the foundation modulus and t_4 , t_5 , t_6 are time constants of the foundation material to be determined from experiments. The work done by these reactive loadings F_x , F_z given by eq. (7) and (8) is

$$(11) \quad W_2 = -\frac{1}{2} \int_0^L (k_z w^2 + k_x h^2 \psi^2) dx.$$

The Hamilton's principle applied to equations (4), (5), (6) and (11) leads to the following equations of motion:

$$(12) \quad \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix} \begin{bmatrix} \ddot{w} \\ \ddot{\psi} \end{bmatrix} + \begin{bmatrix} -\partial_x \cdot (GA - S) \partial_x + k_z & \partial_x \cdot GA \\ -GA \partial_x & (-\partial_x \cdot EI \partial_x + GA + k_x h^2) \end{bmatrix} \begin{bmatrix} w \\ \psi \end{bmatrix} = \begin{bmatrix} q \\ 0 \end{bmatrix}.$$

The corresponding boundary conditions are

$$(13) \quad [\delta w(Q - Q^*)]_0^L = 0 \quad \text{and} \quad [\delta \psi(M - M^*)]_0^L = 0,$$

in which the shearing force Q and bending moment M developed in the beam are defined by

$$(14, 15) \quad Q = (GA - S)w_x - GA\psi, \quad \text{and} \quad M = -EI\psi_x.$$

3. Nondimensionalisation

Taking a Simply Supported Euler — Bernoulli Beam subjected to a uniformly distributed load of intensity q_0 as the basic beam (denoted as SSEB), equation (12) is nondimensionalised by introducing the notations

$$(16) \quad \begin{aligned} x &\leftarrow x/x_0, & t &\leftarrow t/t_0, & w &\leftarrow w/w_0, & S &\leftarrow S/S_0, & G &\leftarrow G/G_0, \\ E &\leftarrow E/E_0, & k_x &\leftarrow k_x/k_{x0}, & \psi &\leftarrow \psi/\psi_0, & k_z &\leftarrow k_z/k_{z0}, & q &\leftarrow q/q_0, \\ M &\leftarrow M/M_0, & Q &\leftarrow Q/Q_0, & t_1 &\leftarrow t_1/t_0, & t_2 &\leftarrow t_2/t_0 \quad \text{etc.} \end{aligned}$$

so that the current symbols $x, t, \dots$ stand for nondimensional variables $x/x_0, t/t_0, \dots$, where $x_0, t_0, \dots$ are given by

$$(17) \quad \begin{aligned} x_0 &= L, \quad t_0 = 2(\rho AL^4/E_0 I)^{\frac{1}{2}}/\pi, \\ w_0 &= 5q_0 x_0^4/(384 E_0 I), \quad \psi_0 = w_0/x_0, \\ S_0 &= \pi^2 E_0 I/L^2, \quad M_0 = q_0 L^2/8, \\ Q_0 &= q_0 L/2. \end{aligned}$$

In the above, the length coordinate is measured in terms of length of the SSEB, time is measured in terms of fundamental time period t_0 of oscillation of SSEB, axial load is measured in terms of critical load S_0 of SSEB and bending moment and shearing forces are measured in terms of maximum bending moment M_0 and maximum shearing force Q_0 developed in SSEB.

From now onwards, the physical variables $x, t, \dots$ will be written with units, e. g., x cm, t s, $\dots$ in order to distinguish them from nondimensional variables $x, t, \dots$.

Introducing the system parameters of axial force S , stiffness of foundation K_x, K_z and size factor s defined by

$$(18) \quad \begin{aligned} S &\leftarrow S/S_0 = \text{axial force/critical load on SSEB}, \\ K_x &= k_{x0} h^2/S_0, \\ K_z &= (k_{z0} L^4/4EI)^{\frac{1}{4}}, \\ s &= I/AL^2, \end{aligned}$$

and using equations (16) and (17), differential equation (12) becomes equation (19) with the notation

$$(19) \quad \begin{aligned} a &= G_0 A/S_0 = G_0/(\pi^2 E_0 s) \\ \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} \ddot{w} \\ \ddot{\psi} \end{bmatrix} + \begin{bmatrix} -4(aG-S)\partial_x^2 + (16K_z^4/\pi^2)k_z & 4aG\partial_x \\ -4aG\partial_x & -(4/\pi^2)E\partial_x^2 + 4aG + 4K_x k_x \end{bmatrix} \begin{bmatrix} w \\ \psi \end{bmatrix} &= \begin{bmatrix} \frac{384 \cdot 4}{5\pi^2} q \\ 0 \end{bmatrix}. \end{aligned}$$

The corresponding expressions for nondimensional bending moment and shearing force are

$$(20) \quad Q = (10\pi^2/384)[(aG-S)w_x - aG\psi],$$

$$(21) \quad M = -(5/48)E\psi_x.$$

4. Solution of Fourier and Laplace transforms

Multiplying both sides of first and second equations in (19) by $\sin n\pi x$ and $\cos n\pi x$ respectively and integrating between limits of $x=(0,1)$ leads to the following Fourier transform of equation (19):

$$(22) \quad \begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} \ddot{w}_n \\ \ddot{\psi}_n \end{bmatrix} + \begin{bmatrix} 4(aG-S)(n\pi)^2 + (16K_z^4/\pi^2)k_z & -4aGn\pi \\ -4aGn\pi & (4/\pi^2)E \cdot (n\pi)^2 + 4aG + 4K_x k_x \end{bmatrix} \begin{bmatrix} w_n \\ \psi_n \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \frac{384}{5} \cdot \frac{4}{\pi^2} & q_n \\ 0 & \end{bmatrix} + \begin{bmatrix} -4(aG-S)(n\pi) & 0 \\ 4aG & -(48 \times 4/5\pi^2) \end{bmatrix} \begin{bmatrix} (\cos n\pi \cdot w_1^* - w_0^*) \\ (\cos n\pi \cdot M_1^* - M_0^*) \end{bmatrix},$$

where w_n , q_n and ψ_n , $n=1, 2, \dots$ denote the sine transforms of w and q , and cosine transform of ψ respectively, and are defined by

$$(23) \quad w_n(t) = \int_0^1 w(x, t) \sin n\pi x \, dx,$$

$$(24) \quad q_n(t) = \int_0^1 q(x, t) \sin n\pi x \, dx,$$

$$(25) \quad \psi_n(t) = \int_0^1 \psi(x, t) \cos n\pi x \, dx$$

and w_0^* , M_0^* , w_1^* , M_1^* , which may be functions of time, denote respectively the prescribed nondimensional displacements and bending moments at the faces $x=0$ and $x=1$ of the beam. Substituting the nondimensional material constants from equations (1), (2), (9), (10) and (16) into equation (22) and simplifying, they can be written in the matrix form as

$$(26) \quad Ax_n = f,$$

where A is a (2×2) characteristic matrix and $x_n = \{w_n, \psi_n\}^T$ is a (2×1) column vector of response and $f = \{f_1, f_2\}^T$ is a (2×1) column vector of excitation. The elements a_{11} , a_{12} , $\dots$, f_2 are given by

$$\begin{aligned} a_{11} &= d_t^2(1+t_1d_t)(1+t_3d_t) + 4(n\pi)^2[a(1+t_2d_t) - S(1+t_1d_t)] \\ &\quad \cdot (1+t_5d_t) + (16K_z^4/\pi^2)(1+t_4d_t)(1+t_1d_t), \\ a_{12} &= -4an\pi(1+t_2d_t)(1+t_5d_t), \\ a_{21} &= -4an\pi(1+t_2d_t)(1+t_3d_t), \\ a_{22} &= sd_t^2(1+t_1d_t)(1+t_3d_t) + 4(n\pi)^2(1+t_1d_t)(1+t_3d_t)/\pi^2 \\ &\quad + 4a(1+t_2d_t)(1+t_3d_t) + 4K_x(1+t_1d_t)(1+t_3d_t)(1+t_6d_t), \\ f_1 &= (384 \times 4/5\pi^2)(1+t_1d_t)(1+t_5d_t)q_n - 4n\pi[a(1+t_2d_t) \\ &\quad - S(1+t_1d_t)] \cdot (1+t_5d_t) \cdot (\cos n\pi \cdot w_1^* - w_0^*), \\ f_2 &= 4a(1+t_2d_t)(1+t_3d_t)(\cos n\pi \cdot w_1^* - w_0^*) - (48 \times 4/5\pi^2) \\ &\quad \cdot (1+t_1d_t)(1+t_3d_t)(\cos n\pi \cdot M_1^* - M_0^*). \end{aligned}$$

Applying the Laplace transform defined by

$$(28) \quad x_n = \int_0^\infty x_n(t) \exp(-pt) \, dt,$$

where p is the Laplace transform parameter, to equation (26) and (27) yields the following transformed equation:

$$(29) \quad \bar{A} \bar{x}_n = \bar{f} + \bar{B} \bar{x}_n(0) + \bar{C} \ddot{\bar{x}}_n(0) + \bar{D} \ddot{\bar{x}}_n(0) + \bar{E} \ddot{\bar{x}}_n(0),$$

where $\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{E}$ are (2×2) matrices with the elements which are polynomials in p in the form $\sum_{k=1}^N \bar{a}_{ij,k} p^{k-1}$, and are given by

$$(30) \quad \begin{aligned} \bar{a}_8 &= a - S, \\ \bar{t}_8 &= at_2 - St_1, \\ \bar{a}_{11,1} &= 4(n\pi)^2 a_8 + 16K_z^4/\pi^2, \\ \bar{a}_{11,2} &= 4(n\pi)^2 (a_8 t_5 + t_8) + (16K_z^4/\pi^2)(t_1 + t_4), \\ \bar{a}_{11,3} &= 1 + 4(n\pi)^2 t_5 t_8 + (16K_z^4/\pi^2) t_1 t_4, \\ \bar{a}_{11,4} &= t_1 + t_5, \\ \bar{a}_{11,5} &= t_1 t_5, \\ \bar{a}_{12} &= -4an\pi - 4an\pi(t_2 + t_5)p - 4an\pi t_2 t_5 p^2, \\ \bar{a}_{21} &= -4an\pi - 4an\pi(t_2 + t_3)p - 4an\pi t_2 t_3 p^2, \\ \bar{a}_{22,1} &= 4(n^2 + a + K_x), \\ \bar{a}_{22,2} &= 4[n^2(t_1 + t_2) + a(t_2 + t_3) + K_x(t_1 + t_3 + t_6)], \\ \bar{a}_{22,3} &= 4[n^2 t_1 t_2 + at_2 t_3 + K_x(t_1 t_3 + t_3 t_6 + t_6 t_1)], \\ \bar{a}_{22,4} &= s(t_1 + t_3) + 4K_x t_1 t_3 t_6, \\ \bar{a}_{22,5} &= st_1 t_3, \\ \bar{b}_{11} &= \bar{a}_{11,2} + \bar{a}_{11,3}p + \bar{a}_{11,4}p^2 + \bar{a}_{11,5}p^3, \\ \bar{b}_{12} &= \bar{a}_{12,2} + \bar{a}_{12,3}p, \\ \bar{b}_{21} &= \bar{a}_{21,2} + \bar{a}_{21,3}p, \\ \bar{b}_{22} &= \bar{a}_{22,2} + \bar{a}_{22,3}p + \bar{a}_{22,4}p^2 + \bar{a}_{22,5}p^3, \\ \bar{c}_{11} &= \bar{a}_{11,3} + \bar{a}_{11,4}p + \bar{a}_{11,5}p^2, \\ \bar{c}_{12} &= \bar{a}_{12,3}, \\ \bar{c}_{21} &= \bar{a}_{21,3}, \\ \bar{c}_{22} &= \bar{a}_{22,3} + \bar{a}_{22,4}p + \bar{a}_{22,5}p^2, \\ \bar{d}_{11} &= \bar{a}_{11,4} + \bar{a}_{11,3}p, \\ \bar{d}_{12} &= \bar{d}_{21} = 0, \end{aligned}$$

$$\begin{aligned}\bar{a}_{22} &= \bar{a}_{22,4} + \bar{a}_{22,5} p, \\ \bar{e}_{11} &= \bar{a}_{11,5}, \quad \bar{e}_{12} = \bar{e}_{21} = 0, \quad \bar{e}_{22} = \bar{a}_{22,5}.\end{aligned}$$

The transfer matrix $\bar{T}$ of the system, being the inverse of the characteristic matrix $\bar{A}$ (i. e., $\bar{T} = \bar{A}^{-1}$) is given by

$$(31) \quad \bar{T} = [\bar{t}_{ij}] = \begin{bmatrix} \bar{a}_{22}/\bar{A} & -\bar{a}_{12}/\bar{A} \\ -\bar{a}_{21}/\bar{A} & \bar{a}_{11}/\bar{A} \end{bmatrix},$$

where the characteristic determinant $\bar{A}$ is given by

$$(32) \quad \bar{A} = \bar{a}_{11}\bar{a}_{22} - \bar{a}_{21}\bar{a}_{12}.$$

Using the transfer matrix, the solution in the transformed plane is expressed in the matrix form as

$$(33) \quad \bar{x}_n = \bar{T} \bar{B} x_n(0) + \bar{T} \bar{C} \dot{x}_n(0) + \bar{T} \bar{D} \ddot{x}_n(0) + \bar{T} \bar{E} \ddot{\ddot{x}}_n(0) + \bar{T} \bar{f}$$

in which the first four terms represent the transient response and the last term corresponds to steady state response of the system. The Laplace inverse of $\bar{x}_n$ given by the equation (33) can be expressed in terms of convolution integrals. Inverting the Laplace and Fourier transforms, the solution can then be expressed as

$$(34) \quad w(x, t) = 2 \sum_{1,2,3}^{\infty} w_n(t) \sin n\pi x,$$

$$\psi(x, t) = \psi_0(t) + 2 \sum_{1,2,3}^{\infty} \psi_n(t) \cos n\pi x.$$

The inverse of equation (33) will now be investigated for a number of special cases.

5. Applications

a. Steady state response: Let the beam be subjected to a sinusoidal load $q = \exp(j\Omega t)$ from rest, where Ω denotes the frequency of excitation ($\Omega = 2\pi$ corresponds to resonance in SSEB), so that the excitation components f_1, f_2 are found from equation (27) to be

$$(35) \quad \begin{aligned}f_1 &= (384 \times 4/5\pi^2)[(1 - \cos n\pi)/n\pi](1 + j\Omega t_1)(1 + j\Omega t_5) \cdot \exp(j\Omega t), \\ f_2 &= 0.\end{aligned}$$

The steady state response of the system under the above excitation will be obtained by replacing d_t in equation (26) by $j\Omega$. This leads to the following expression for the Fourier transform w_n of the deflection:

$$(36) \quad \begin{aligned}w_n &= (384 \times 4/5\pi^2)[(1 - \cos n\pi)/n\pi](1 + j\Omega t_1)(1 + j\Omega t_5) \\ &\quad \cdot [\bar{a}_{22}(j\Omega)/\bar{A}(j\Omega)] \cdot \exp(j\Omega t).\end{aligned}$$

The magnitude of the above complex quantity denotes the dynamic magnifi-

cation factor and its modulus represents the phase angle by which the response lags behind the excitation in the n th mode of vibration.

b. Free vibrations under velocity impact: Let the beam be moving with an initial velocity $\dot{w}(x, 0) = v$ cm/s during a drop test. Let the rigid body motion be arrested at time $t = 0$. Using $\dot{w}(x, 0) = v$ cm/s in equations (27), (29), (31) and (33) yields the following expression for w_n governing the ensuing free vibrations:

$$(37) \quad \bar{w}_n = (vt_0/w_0)(1 - \cos n\pi)(a_{22}c_{11} - \bar{a}_{12}\bar{c}_{21})/\bar{A}n\pi$$

so that $\bar{w}_n = 0$ for $n = 2, 4, 6, \dots$ which means that the antisymmetric modes do not contribute to the resulting free vibrations. The first of equation (34) then yields the following expression for dynamic deflection $w(\frac{1}{2}, t)$ at the midpoint $x = \frac{1}{2}$ of the beam:

$$(38) \quad w\left(\frac{1}{2}, t\right) = 2 \sum_{1,3,5}^{\infty} \bar{w}_n(t) \sin(n\pi/2).$$

c. Beam subjected to a travelling force. When a concentrated force P kg starting from $x = 0$ moves with uniform velocity v cm/s over a SSEB, resonance occurs in the beam at the critical velocity $v_0 = 2x_0/t_0$ cm/s, i. e., when the time taken by the load to cross the beam, $t' = L/v_0$ s is half fundamental time period t_0 s of SSEB. The intensity of external load for this problem is

$$(39) \quad q = P(x - vt)H(t' - t) \text{ kg/cm},$$

where δ and H denote respectively the Dirac delta and Heaviside step function and $t' = L/v$ denote the time taken by the moving force to cross the beam. Nondimensionalisation of load with respect to $P = q_0 x_0$ kg and velocity v with respect to critical velocity v_0 cm/s leads to the following nondimensional intensity of external load:

$$(40) \quad q = \delta(x - 2vt)H(t' - t), \quad t' = 1/2 v.$$

Equation (27) then leads to the following transform of components of excitation f_1, f_2 :

$$(41) \quad \begin{aligned} \bar{f}_1 &= (384 \times 4/5\pi^2)[1 + (t_1 + t_5)p + t_1 t_5 p^2] \bar{q}_n - t_1 t_5 2n\pi v, \\ \bar{f}_2 &= 0, \end{aligned}$$

where

$$(42) \quad \bar{q}_n = L \int_0^1 q \sin n\pi x dx = \{2n\pi v/[p^2 + (2n\pi v)^2]\}(1 - \cos n\pi \cdot \exp(-pt')).$$

Assuming that the beam is initially at rest and free, inversion of Laplace and Fourier transforms in the first of equations (29) and using equations (33), (34) and (41) leads to the following solution:

$$(43) \quad w(x, t) = 2 \sum_{1,2,3}^{\infty} \sin n\pi x [-t_1 t_5 F_n + G_n - \cos n\pi \cdot G_n(t-t')H(t-t')],$$

where F_n and G_n are Laplace inverses of

$$(44) \quad \bar{F}_n = (384 \times 4/5\pi^2) 2n\pi v \cdot \bar{a}_{22}/A,$$

$$(45) \quad \bar{G}_n = \bar{F}_n \cdot [1 + (t_1 + t_5)p + t_1 t_5 p^2] / [p^2 + (2n\pi v)^2].$$

The first two terms in the rectangular bracket in equation (43) correspond to motion when the load is on the beam while the third term in it corresponds to motion after the load leaves the beam. The corresponding solutions for a SSEB are as follows:

1. The steady state solution for a SSEB subjected to a nondimensional sinusoidal load $q = \exp(j\Omega t)$ from rest is obtained from the Fourier transform w_n given by Timoshenko [7], p. 350:

$$(46) \quad w_n = (384 \times 8)/(5n\pi^3) \cdot [1/(n^4 4\pi^2 - \Omega^2)] \exp(j\Omega t),$$

so that the dynamic magnification in fundamental mode is

$$(47) \quad 2w_1 = 1/1 - (\Omega/2\pi^2).$$

2. When a SSEB is moving with an initial velocity $\dot{w}(x, 0) = v$ cm/s during a drop test and the rigid body motion is arrested at $t=0$, the ensuing free vibrations are given by

$$(48) \quad w(x, t) = (vt_0/w_0)(2/\pi^2) \sum_{1,3,5}^{\infty} \sin 2\pi n^2 t \cdot \sin n\pi x/n^3.$$

3. When a concentrated force P kg starts moving from $x=0$ with a speed v cm/s over an SSEB, the nondimensional excitation is (see equation (40)) $q = \delta(x - 2vt)H(t' - t)$ and the nondimensional response of the SEEB is

$$(49) \quad w(x, t) = 2 \sum_{1,2,\dots}^{\infty} w_n(t) \sin n\pi x,$$

where $w_n(t)$ is given by Timoshenko [7], p. 352:

$$(50) \quad w_n(t) = \frac{768}{10\pi^4} \left[\frac{\sin 2n\pi vt}{n^2(n^2 - v^2)} - \frac{v}{n} \frac{\sin 2n^2\pi t}{(n^2 - v^2)} \right].$$

Equations (36), (38), (43), (46)–(50) are programmed for a digital computer and the resulting graphs are given in Figs. 2–8. The next section is devoted to discussion of the effects of various system parameters on the free and forced vibration response of the beam using Figs. 2–8.

6. Discussion of results

a. Free vibrations under velocity impact. Fig. 2 shows the free vibrations of a Timoshenko beam with viscous damping under velocity impact, as computed from equation (38). The curve in chain line in this figure is the corresponding response of SSEB as computed from equation (48). Viscous damping in the beam is accounted for by selecting a low value of foundation K_z (viz. $K_z = 2.2 \times 10^{-2}$, with $v = 0.25$ and size factor $s = 0.4 \times$

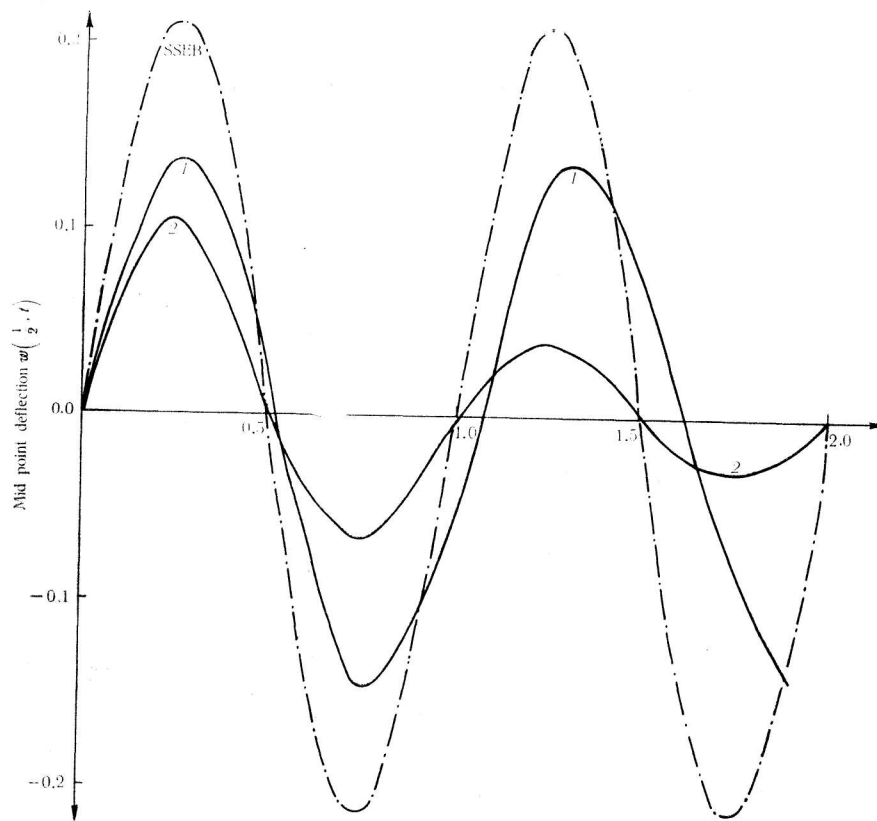


Fig. 2. Effect of viscous damping on the free vibrations of a Timoshenko beam subjected to velocity impact

10^{-4} for curve 1 and 0.04×10^{-4} for curve 2) and a large value of retardation time parameter t_4 ($=4.6 \times 10^4$ for curve 1 and 9.2×10^4 for curve 2) so that the vertical foundation reaction $k_z w = k_{z0}(1 + t_4 \partial_t)/(1 + t_5 \partial_t)w$ effectively reduced to $K_z t_4 \dot{w}$ which characterizes viscous damping in the beam. Comparison of curves in Fig. 3 shows that the amplitude of oscillation dies down quickly with viscous damping.

b. Steady state response. Figs. 3–6 show the effect of various parameters on the frequency response of the Timoshenko beam. Fig. 3 is a plot of the dynamic magnification factor $2|w_1|$ in the first mode of vibration (computed from equation (36)) showing the effect of size factor s and the axial force parameter S on the frequency response of a viscoelastic Timoshenko beam. This graph reveals that as the size factor s increases, the subcritical response increases while the supercritical response decreases, for a given frequency ratio. This implies that the resonant frequency decreases as the size factor s increases. It can also be seen from this graph that for frequencies of excitation approximately twice that of fundamental frequency of SSEB, the effect of size factor on the dynamic magnification factor is negligible. The curves in dotted lines show the effect of axial force. These curves show that an increase in tension increases the resonant frequency (relative to that of SSEB) while an increase in the compressive axial force decreases it. As the axial force approaches the buckling load, $S=1$, the subcritical response tends

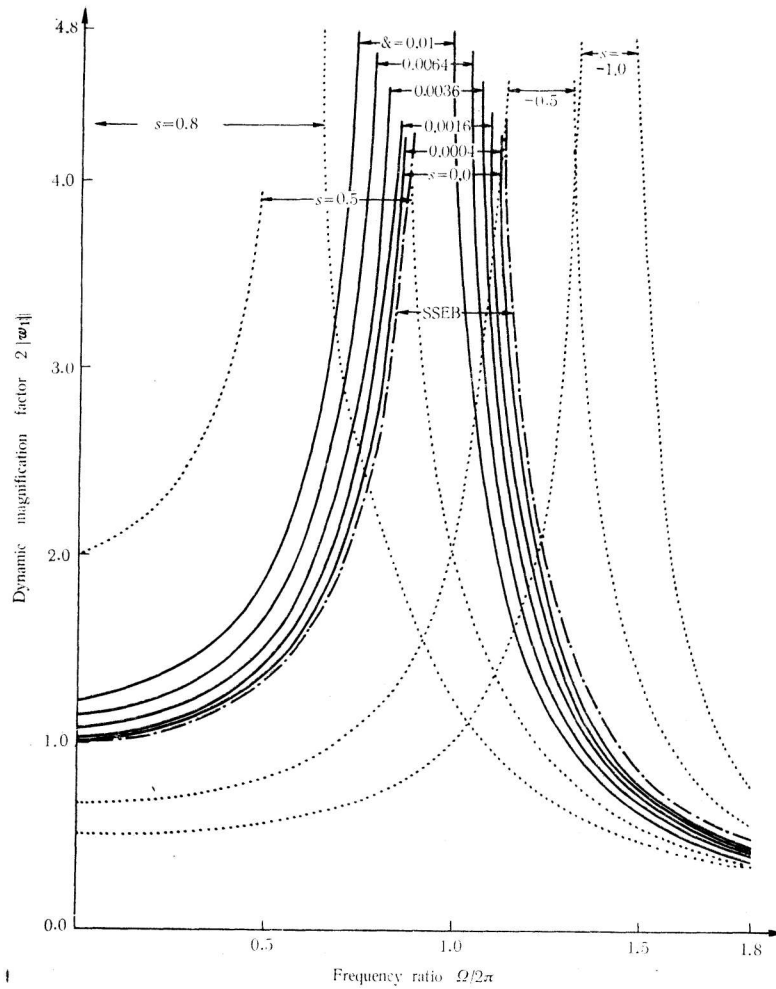


Fig. 3. Frequency response of viscoelastic Timoshenko beam
— effect of size factor. $\&$; $s=0$; $\nu=0.25$; ... effect of axial force factor. s ; $\&=0.0004$, $\nu=0.25$
(rest parameters zero)

to infinity at $\Omega=0$, i. e. the static case. For $S=0.8$, the subcritical response is outside the plotted region in Fig. 3. The curve in chain line shows the corresponding frequency response of SSEB in fundamental mode of vibration, as computed by the formula (47).

Fig. 4 shows the effect of viscoelasticity of Timoshenko beam on its frequency response. Curves in full line correspond to the effect of parameter $\varphi_1 = \pi t_1$ ($\varphi_1=1$ corresponds to critical damping the fundamental mode of vibration of SSEB with viscous damping) while curves in dotted line show the effect of parameter $\varphi_2 = \pi t_2$. This figure shows that the resonant frequency decreases with an increase in φ_1 and φ_2 . The magnification factor at resonant point increases rapidly with an increase in φ_1 while it decreases with a corresponding increase in φ_2 . The figure also shows that the response below resonance is very much sensitive to changes in relaxation time parameter φ_1 while the response beyond resonance is almost unaffected by the changes φ_1 .

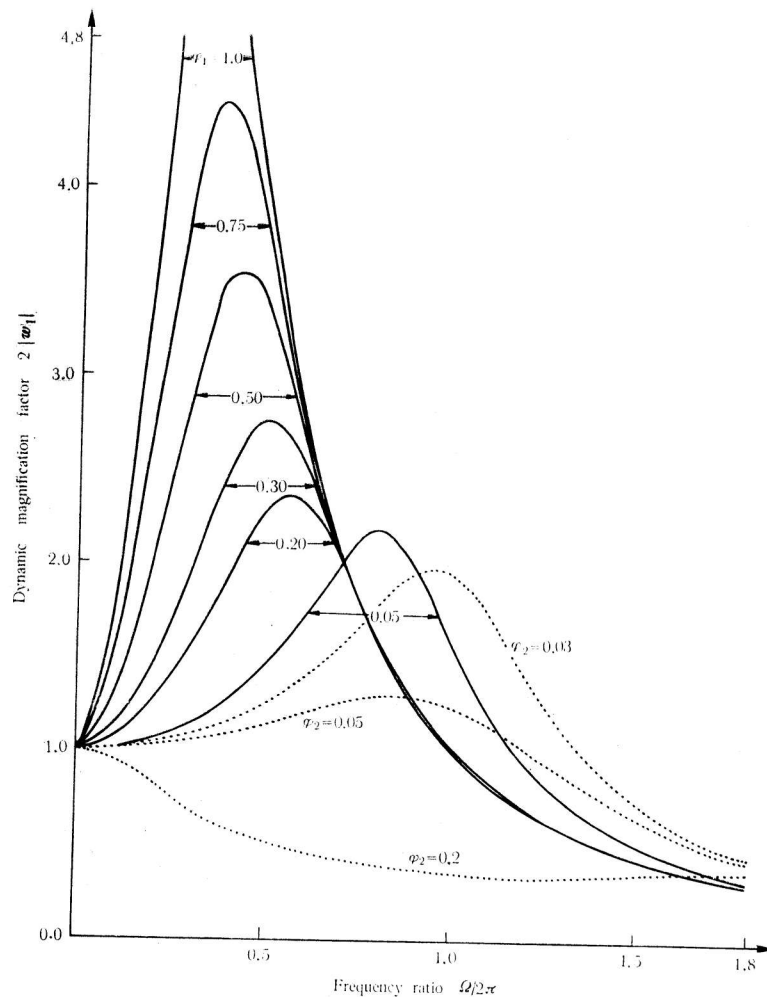


Fig. 4. Effect of viscoelasticity on the frequency response of a Timoshenko beam

$\delta = 0.0004$, rest of the parameters zero; — effect of relaxation time parameter φ_1 ; $\cdots$ effect of retardation time parameter φ_2

in the range (0.2, 1). Comparison of peak values for curves with $\varphi_1 = 0.05$ and $\varphi_2 = 0.05$ show that the resonant magnification factor for $\varphi_2 = 0.05$ is approximately 2/3rd that of $\varphi_1 = 0.05$, i. e., parameter φ_2 has greater influence in damping the response of the beam than the parameter φ_1 .

Fig. 5 shows the effect of foundation parameter K_z on the dynamic magnification factor. In contrast to the effect of the size factor, the effect of an increase in the stiffness K_z of the foundation is to decrease the subcritical response and increase the supercritical response, so that the resonant frequency increases with K_z . For low values of $K_z < 1$, the frequency response is quite insensitive to changes in K_z .

Fig. 6 shows the effect of viscoelasticity of foundation on the response of the system. Curves in full and dotted lines show respectively the effects of foundation viscoelasticity parameters φ_4 and φ_5 . From this figure, it is clear

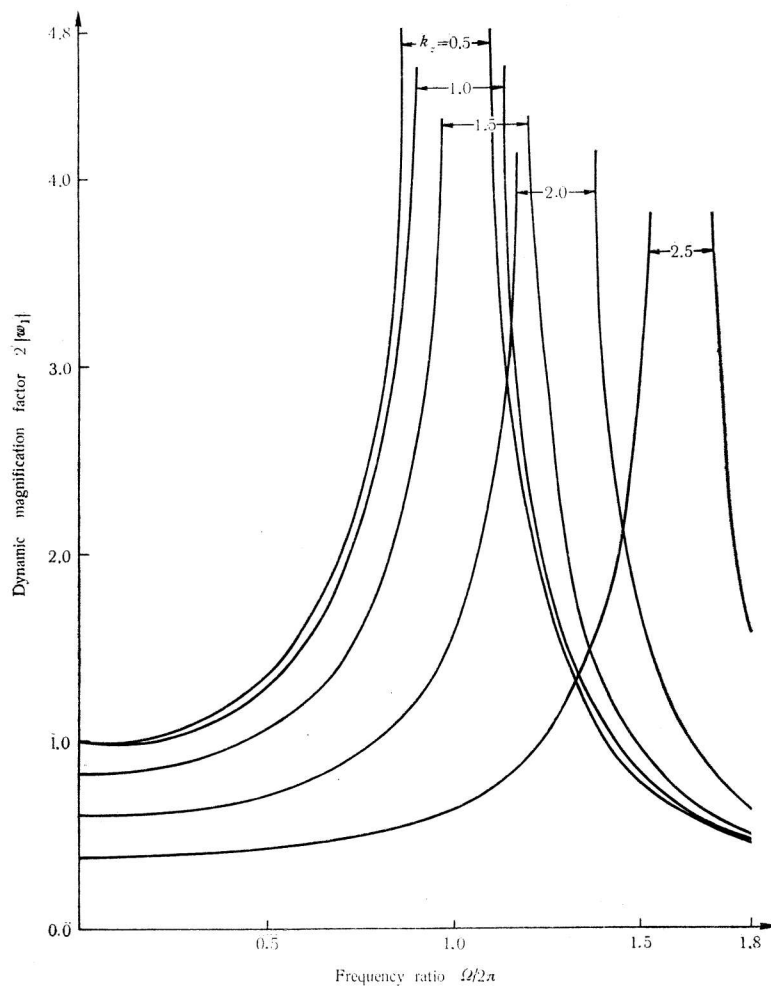


Fig. 5. Effect of foundation modulus on the frequency response of a Timoshenko beam
 $\alpha=0.0004$, rest of the parameters zero

that an increase in retardation time parameter φ_4 decreases the resonant frequency and resonant magnification factor. An increase in relaxation time parameter φ_5 decreases the resonant frequency; however, it can be seen from this figure that φ_5 has insignificant effect on the damping characteristics of the system. This can be explained by the fact that φ_4 and φ_5 occur respectively in numerator and denominator of the foundation viscoelasticity operator. This figure also shows that as φ_5 is increased, response below resonance increases while the response beyond resonance decreases. Further, the nearness of the dotted curves above resonance show that the response above resonance is affected less by φ_5 than the response below resonance. The magnification factor (at $\Omega/2\pi=1$) for $\varphi_5=0.05$ is found from computations (not shown in the figure) to be twice that for $\varphi_4=0.05$, showing that the parameter φ_4 is more sensitive in damping the response of the beam than the parameter φ_5 .

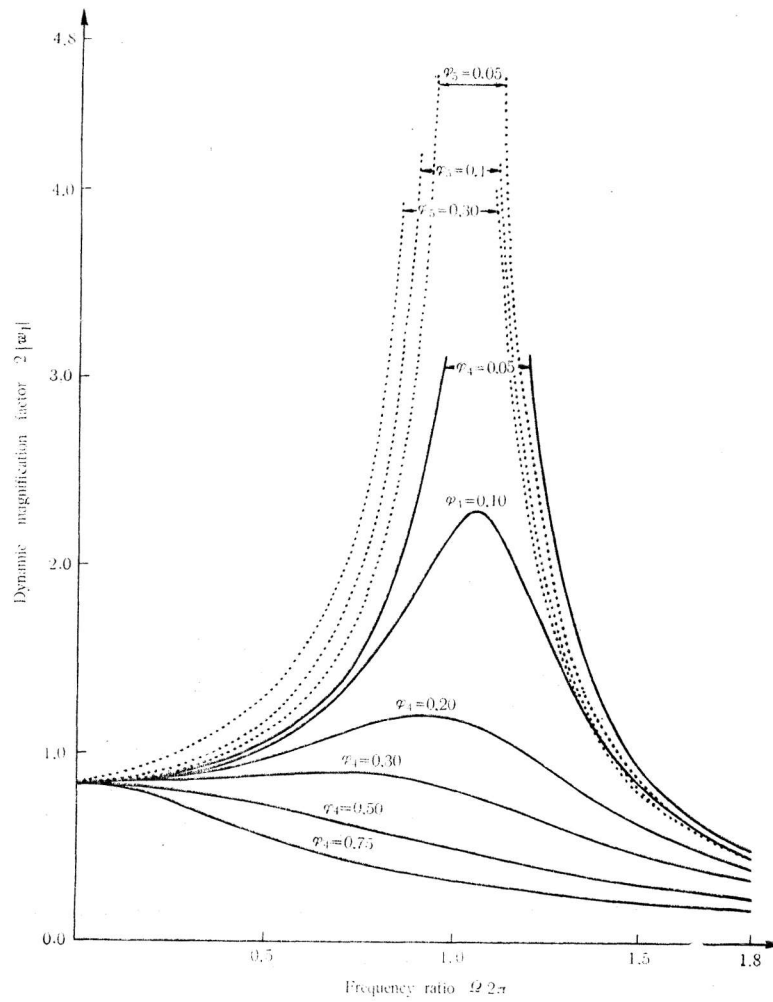


Fig. 6. Effect of foundation viscoelasticity on frequency response of a Timoshenko beam
 $K_2=1.5$, $\nu=0.25$; $\xi=0.0004$, rest parameters zero; ——— effect of retardation time parameter φ_4 ; . . . effect of relaxation time parameter φ_5

c. Response under a moving load. Fig. 7 shows the effect of speed of travelling force on the response of midpoint of a viscoelastic Timoshenko beam, as computed from equation (43). The curve in chain line in this figure is the response of SSEB (plotted by summing the first 20 terms in equation (49)) for the purpose of comparison. It shows that the maximum midpoint deflection of a Timoshenko beam is less than that of SSEB when both are subjected to the same moving load. Comparison of peak-times of different curves in Fig. 7 shows that as the speed of the travelling force increases to critical speed v_0 cm/s, the time at which the maximum midpoint deflection occurs decreases. As the speed of the force increases further, this midpoint time-peak starts increasing again, and the peak value itself decreases.

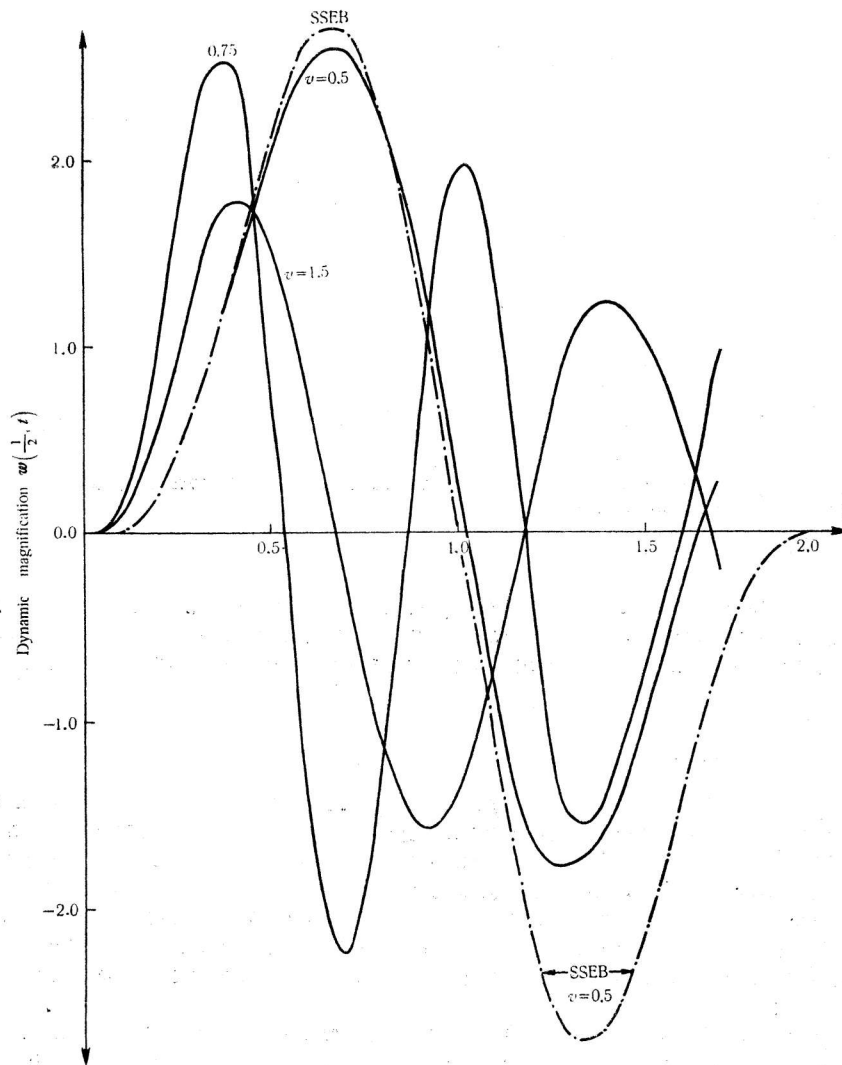


Fig. 7. Effect of speed of travelling force on the response of a viscoelastic Timoshenko beam
 $\xi=0.0004$; $\varphi_2=0.05$

Fig. 8 shows the effect of foundation stiffness on the response of a viscoelastic Timoshenko beam subjected to a travelling force which is $1/4$ th of critical speed of SSEB. Comparison with curve in chain line (corresponding to SSEB with $v=0.25$) shows that the deflection of the Timoshenko beam is less than that of Euler — Bernoulli beam until the response reaches its peak value. For low values of foundation stiffness factor K_z (in the range 0 to 0.5) the response of the beam is almost unaffected by the change in K_z . As K_z increases beyond 0.5, its effect on the response is seen to be quite appreciable. The time at which the peak value occurs is seen to be delayed with increasing

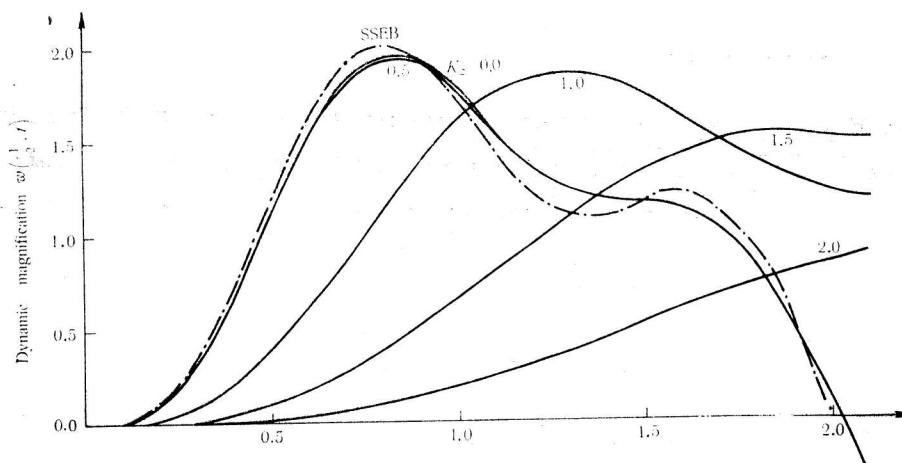


Fig. 8. Effect of foundation stiffness on the response of a Timoshenko beam subjected to a travelling force
 $\nu=0.25$; $\xi=0.0004$; $\varphi_2=0.05$

K_z . For very large $K_z > 2$, it is seen from the figure that the deflection of the beam is negligible even until the load is at midpoint ($t'=2$, $t=1$) and the peak midpoint deflection will occur well after the load has left the beam.

7. Conclusions

In this paper the free and forced vibrations of a viscoelastic Timoshenko beam are investigated when the beam is on a viscoelastic generalized foundation or when it is subjected to an axial force. The equations of motion are derived by using the Hamilton's principle and the symmetric mass and stiffness operators of the problem are displayed by putting the equations of motion in matrix operator form. The solution is studied for steady state response, free vibrations under velocity impact, and vibrations induced under a moving load.

The present study indicates that the subcritical steady state response of a Timoshenko beam increases with size factor while its supercritical response decreases. The resonant frequency is shown to decrease with an increase in viscoelasticity parameters of the beam. The resonant point magnification factor is seen to increase with retardation time parameter φ_1 while it decreases with relaxation time parameter φ_2 . The frequency response is shown to be insensitive for low values of foundation modulus $K_z < 1$. The retardation time parameter φ_4 of the foundation is shown to be more influential in damping response of the beam than the relaxation time parameter φ_5 . The response of a viscoelastic Timoshenko beam subjected to a travelling force is shown to be less than that of SSEB until it reaches its peak value. The stiffness K_z of foundation is shown to be having negligible influence on the response of the beam subjected to moving load until $K_z=0.5$.

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