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The flow field induced by an oscillating fluid drop immersed in another fluid

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1. Introduction

The motion of a liquid drop immersed in another fluid occurs in a wide variety of physico-chemical systems. Interest in this phenomenon which, from a practical point of view, is of considerable importance in various processes such as extraction and atomization has existed for many years. The translation of a liquid drop was first treated independently by Rybczynski [1] and Hadamard [2] under the assumption that the drop remained exactly spherical. Since interfacial tension acting at the junction between the two immiscible fluids tends to maintain a spherical shape against the shearing stresses which are inclined to deform it, the investigations of the shape of a viscous drop settling through a fluid appear to be the most detailed treatment of a free surface. This was attempted by Saito [3], but in view of an error in his work, Taylor and Acrivos [4] re-examined this problem. Some investigators introduce a simplification of the boundary conditions by assuming that one side of the free surface is bounded by a fluid of extremely small viscosity. Various models [5, 6] to describe the rheology of interfaces have been proposed. Many have stressed the importance of the intense local pressures and temperatures. Others [7, 8] have given attention to effects such as shape-mode resonances and their associated violent mechanical deformations at the bubble surface. Nybord [9] has called attention to still another effect which may explain the action of bubbles in certain instances. This is the presence of small-scale acoustic streaming generated by bubbles-scattered sound waves. It has been experimentally established that when sound sources oscillate in a viscous fluid a steady streaming motion will be induced. Such steady motion is called acoustic streaming and it arises from the interaction of viscosity with the nonlinear inertia terms. Elder [10] examined experimentally the steady streaming in the neighbourhood of a small bubble attached to a vibrating piston.

In a recent paper Davidson and Riley [11] have considered the flow induced by an isolated spherical bubble which performs translational harmonic oscillations relative to the liquid in which it is embedded. Their attention was focused upon the microstreaming which is induced.

In the present paper an examination of the flow field induced by a translatory oscillating fluid drop immersed in another fluid is undertaken. Of particular interest is the steady streaming induced both inside and outside the

drop. We assume that the fluids are immiscible and the particle is sufficiently small that the liquid drop maintains a fixed spherical shape. So we neglect the effects such as shape-mode resonances and their associated violent mechanical deformations at the globulan surface. We also do not consider the effects of gravity and transport across the interface due to static diffusion or diffusion associated with the microstreaming.

2. Formulation of the problem

Consider a spherical fluid drop with radius a. Adopt spherical polar coordinates at the center of the globule with the axis oz pointing upstream parallel to the transverse vibration with speed $U_\infty \cos \omega t$ at infinity. Dimensionless variables will be employed throughout the analysis, and physical parameters pertaining to the interior of the drop will be distinguished from the corresponding exterior parameters by a caret. With U_∞ as a typical velocity, ω^{-1} as a typical time, a as a geometrical length and ν and $\hat{\nu}$ as the kinematic viscosities of the internal media and external there are several length scales associated with this problem. From these length scales we can contract the following dimensionless parameters which characterize the motion:

$$R_e = \frac{U_\infty a \varrho}{\mu'}, \quad \hat{R}_e = \frac{U_\infty a \hat{\varrho}}{\hat{\mu}'},$$

$$|M|^2 = \frac{\omega a^2}{\nu}, \quad |\hat{M}|^2 = \frac{\omega a^2}{\hat{\nu}},$$

$$\varepsilon = \frac{U_\infty}{\omega a}.$$

We shall be concerned entirely with the situation in which the amplitude of the oscillation is small compared with a, so that $\varepsilon \ll 1$ and $R_e \ll |M|^2$.

The governing equations of motion in terms of the stream functions χ and $\hat{\chi}$ have the following form:

$$(1) \qquad \frac{1}{|M|^2} D^4 \chi = \varepsilon \left[\frac{1}{r^2} \cdot \frac{\partial(\chi, D^2\chi)}{\partial(r, \mu)} + \frac{2}{r^2} D^2\chi \left(\frac{\mu}{(1-\mu^2)} \cdot \frac{\partial \chi}{\partial r} + \frac{1}{r} \cdot \frac{\partial \chi}{\partial \mu} \right) \right] + \frac{\partial}{\partial \tau} (D^2 \chi)$$

for the exterior region $1 \leq r < \infty$ and

$$(2) \qquad \frac{k}{\gamma |M|^2} D^4 \hat{\chi} = \varepsilon \left[\frac{1}{r^2} \cdot \frac{\partial(\hat{\chi}, D^2\hat{\chi})}{\partial(r, \mu)} + \frac{2}{r^2} D^2\hat{\chi} \left(\frac{\mu}{1-\mu^2} \cdot \frac{\partial \hat{\chi}}{\partial r} + \frac{1}{r} \cdot \frac{\partial \hat{\chi}}{\partial \mu} \right) \right] + \frac{\partial}{\partial \tau} (D^2 \hat{\chi})$$

for the region inside the globule.

Here $\mu = \cos\theta$, $D^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{(1-\mu^2)}{r^2} \cdot \frac{\partial^2}{\partial \mu^2}$, $R_e = \frac{U_\infty a}{\nu}$ is the Reynolds number and M, k and γ are respectively the frequency parameter, the ratio of the viscosity of the interior to the exterior fluid and the ratio of the density of the interior to that of the exterior fluid.

The radial and transverse velocity components u and v are related to χ by

$$(3) \qquad u = \frac{1}{r^2 \sin\theta} \cdot \frac{\partial \chi}{\partial \theta}, \quad v = -\frac{1}{r \sin\theta} \cdot \frac{\partial \chi}{\partial r}.$$

The boundary condition satisfied by χ as $r \to \infty$ is

(4) $$\chi \sim \frac{1}{2} r^2(1-\mu^2)e^{i\tau}.$$

The boundary conditions at the interface are:

1) zero values of the normal velocity both inside and outside of the globule

(5) $$\hat{\chi}=0, \quad \chi=0 \quad \text{at} \quad r=1$$

2) continuity of tangential velocity across the interface, whereupon

(6) $$\frac{\partial \hat{\chi}}{\partial r}=\frac{\partial \chi}{\partial r} \quad \text{at} \quad r=1$$

3) continuity of shear stresses of the two fluids across the interface

(7) $$\mu' \frac{\partial}{\partial r}\left(\frac{1}{r^2}\cdot\frac{\partial \chi}{\partial r}\right)=\hat{\mu}' \frac{\partial}{\partial r}\left(\frac{1}{r^2}\cdot\frac{\partial \hat{\chi}}{\partial r}\right) \quad \text{at} \quad r=1.$$

These conditions hold when fluids are immiscible, the surface tension is constant, the surface viscosity effects are negligible and axial symmetry is postulated. It should be remarked at this point that in [4] there is another condition requiring the balance of normal stresses when the drop changes its form and size. One can use this condition to determine the shape of the fluid drop as a function of Weber number W_e, k, γ and R_e. Uniqueness of the solution for small R_e is ensured by the additional conditions that the velocity be a finite continuous function of position at the center $r=0$:

(8) $$\hat{\chi}=\frac{\partial \hat{\chi}}{\partial r}=0 \quad \text{at} \quad r=0.$$

Furthermore we shall suppose that $|M|\ll 1$. In this diffuse case for the stream function χ one can develop solutions in an "inner" (Stokes) region of scale $O(a)$ and a much larger "outer" (Oseen) region of scale $O(a|M|^{-1})$.

The inner and outer expansions for the region $r\geq 1$ are assumed to be of the form

(9) $$\chi=\chi_1+\varepsilon\chi_2+\cdots$$

(10) $$\psi=\psi_1+\varepsilon\psi_2+\cdots$$

For the region $r\leq 1$ we seek the solution of the form

(11) $$\hat{\chi}=\hat{\chi}_1+\varepsilon\hat{\chi}_2+\cdots$$

3. Construction of the solution

Substituting (9) and (11) in equations (1) and (2) respectively and equating the terms free of ε on both sides we have

(12) $$|M|^2\cdot\frac{\partial}{\partial \tau}(D^2\chi_1)=D^4\chi_1$$

and

(13) $$|M|^2\cdot\frac{\gamma}{k}\cdot\frac{\partial}{\partial \tau}(D^2\hat{\chi}_1)=D^4\hat{\chi}_1.$$

The boundary conditions for χ_1 and $\hat{\chi}_1$ are (4)—(8). We assume that

$$\chi_1=f(r,|M|)(1-\mu^2)e^{i\tau}$$

and

$$\hat{\chi}_1 = \hat{f}(r, |M|)(1-\mu^2)e^{i\tau}.$$

Then from equations (12) and (13) we find

$$f(r, |M|) = a_1 r^2 + a_2 \frac{1}{r} + \frac{b_1}{M^2}\left(\frac{1}{r} - M\right)e^{Mr} + \frac{b_2}{M^2}\left(\frac{1}{r} + M\right)e^{-Mr} \tag{14}$$

$$\hat{f}(r, |M|) = \bar{a}_1 r^2 + \bar{a}_2 \cdot \frac{1}{r} + \frac{\bar{b}_1}{M^2} \cdot \frac{k}{\gamma} \cdot \left(\frac{1}{r} - \sqrt{\frac{\gamma}{k}} M\right)e^{Mr} + \frac{\bar{b}_2}{M^2} \cdot \frac{k}{\gamma}\left(\frac{1}{r} - \sqrt{\frac{\gamma}{k}} M\right)e^{-Mr}, \tag{15}$$

where a_i, b_i, $\bar{a}_i$, $\bar{b}_i$ $(i = 1, 2)$ are constants.

Applying boundary conditions (4)—(8) we obtain

$$\chi_1 = \frac{(1-\mu^2)}{4}\left[2r^2 - \frac{(3k+2)}{(k+1)} \cdot r + \frac{k}{(k+1)} \cdot \frac{1}{r}\right]e^{i\tau} \tag{16}$$

$$+ \frac{|M|}{6}(1-\mu^2) \cdot \frac{(3k+2)}{(k+1)^2}\left[-\frac{(3k+2)}{2} r + \frac{k}{2r} + (k+1) \cdot r^2\right]e^{i(\tau+\pi/4)} + O(|M|^2)$$

$$\hat{\chi}_1 = \frac{(1-\mu^2)(r^2-r^4)}{4(k+1)} e^{i\tau} - \frac{|M|(1-\mu^2)(3k+2)}{12(k+1)^2} \cdot r^2 - r^4 \; e^{i(\tau+\pi/4)} + O(|M|^2). \tag{17}$$

Examining solution (16), we find that the successive approximation scheme fails for $r = O(|M|^{-1})$. Thus we must develop our solution not only in the inner region where $r = O(1)$, but also in an outer region in which $Mr = O(1)$. The inner and outer solutions must match in the overlapped region. We concentrate first on the solution in the inner region.

We expect χ_2 to contain a term independent of τ in addition to the oscillatory one:

$$\chi_2(r, \mu, |M|, \tau) = \chi_2^{(s)}(r, \mu, |M|) + \chi_2^{(u)}(r, \mu, |M|, \tau) \tag{18}$$

where the superscript (s) denotes steady and (u) unsteady.

The reason that a steady streaming may be induced by a periodic motion of the liquid drop is the fact that the perturbation equations $O(\varepsilon)$ contain forcing terms $\infty \cos^2 \tau = \frac{1}{2}(1 + \cos 2\tau)$.

Since $|M| \ll 1$ we write

$$\chi_2 = \Phi_{20} + |M|\Phi_{21} + |M|^2\Phi_{22} + \cdots \tag{19}$$

The equations for the unsteady and steady parts of χ_2 are

$$\frac{1}{|M|^2} D^4 \chi_2^{(u)} - \frac{\partial}{\partial \tau}(D^2 \chi_2^{(u)}) = \frac{1}{2r^2}\left[\frac{\partial(\chi_1, D^2\chi_1)}{\partial(r, \mu)} + 2D^2\chi_1 \cdot L\chi_1\right]^{(u)} \tag{20a}$$

$$\frac{1}{|M|^2} D^4 \chi_2^{(s)} = \frac{1}{2r^2}\left[\frac{\partial(\chi_1, D^2\chi_1)}{\partial(r, \mu)} + 2D^2\chi_1 \cdot L\chi_1\right]^{(s)}, \tag{20b}$$

where

$$L = \frac{\mu}{(1-\mu^2)} \cdot \frac{\partial}{\partial r} + \frac{1}{r} \cdot \frac{\partial}{\partial \mu}.$$

The terms on the right of (20) are evaluated using χ_1 from (16).

If we substitute (18) into (20) and equate coefficients of like powers of M we have

$$D^4\Phi_{20}=0;\quad D^4\Phi_{21}=0.$$

It may be verified a posteriori, using matching of the inner and outer solutions, that

$$\Phi_{20}=0,\quad \Phi_{21}=0$$

are the only possible solutions.

According to (19) we write

$$\Phi_{22}=\Phi_{22}^{(0)}(r,\mu)+\Phi_{22}^{(2)}(r,\mu)e^{2i\tau}. \tag{21}$$

The functions $\Phi_{22}^{(j)}$ $(j=0, 2)$ must satisfy the equation

$$D^4\Phi_{22}^{(j)}=-\frac{3\mu(1-\mu^2)}{8}\cdot\frac{(3k+2)}{(k+1)}\cdot\frac{1}{r^3}\left[2r-\frac{(3k+2)}{(k+1)}+\frac{k}{(k+1)}\cdot\frac{1}{r^2}\right]. \tag{22}$$

The solution of this equation has the form

$$\Phi_{22}^{(j)}=\left\{a_1^j+a_2^jr^3+a_3^jr^5+a_4^jr^{-2}\right.$$
$$\left.+\frac{(3k+2)}{16(k+1)}\left[r^2-\frac{(3k+2)}{2(k+1)}r-\frac{k}{2(k+1)}\cdot\frac{1}{r}\right]\right\}Q_2(\mu), \tag{23}$$

where a_i^j $(i=1, 2, 3, 4;\ j=0, 2)$ are constants,

$$Q_n(\mu)=\int_{-1}^{1}P_n(x)dx$$

and P_n is the Legendre polynomial of degree n.

In seeking an expansion that holds for large r, it is necessary to find transformation of variables which will allow expansions well behaved as $r\to\infty$. The proper coordinate transformation for this problem is $\varrho=rM$. If this transformation is introduced into (3) the result is

$$u=\frac{M^2}{\varrho^2\sin\theta}\cdot\frac{\partial\chi}{\partial\theta},\quad v=-\frac{M^2}{\varrho\sin\theta}\cdot\frac{\partial\chi}{\partial\varrho}.$$

Since u and v should not vanish as $M\to 0$, the variable $\psi=M^2\chi$ is introduced to remove the dependence of $|M|$ from the velocities. The equation for the stream function ψ in the outer region is

$$\frac{\partial}{\partial\tau}(Д^2\psi)+\frac{R_e}{|M|^2}\left[\frac{|M|}{\varrho^2}\cdot\frac{\partial(\psi, Д^2\psi)}{\partial(\varrho,\mu)}+\frac{2}{\varrho^2}|M|\cdot Д^2\psi\cdot L\psi\right]=Д^4\psi, \tag{24}$$

where

$$Д^2=\frac{\partial^2}{\partial\varrho^2}+\frac{(1-\mu^2)}{\varrho^2}\cdot\frac{\partial^2}{\partial\mu^2};\quad L=\frac{\mu}{(1-\mu^2)}\cdot\frac{\partial}{\partial\varrho}+\frac{1}{\varrho}\cdot\frac{\partial}{\partial\mu}.$$

The first-order solution ψ_1 as $M\to 0$ is obtained to be

$$\psi_1=\frac{(1-\mu^2)}{2}\varrho^2e^{i\tau}+\frac{|M|(3k+2)}{2(k+1)}(1-\mu^2)\left[\left(1-\frac{1}{\varrho}\right)e^{-\varrho}+\frac{1}{\varrho}\right]e^{i\left(\tau+\frac{\pi}{4}\right)}+\cdots$$

We suppose that

(25) $$\psi_2 = F_{20} + |M| \cdot F_{21} + |M|^2 \cdot F_{22} + O(|M|^3)$$

and

$$F_{22} = F_{22}^{(0)}(\varrho, \mu) + F_{22}^{(2)}(\varrho, \mu) e^{2i\tau}$$

As in equation (18) one can show that $F_{20} = F_{21} = 0$.
For $F_{22}^{(j)}$, $j = 0, 2$ we have

(26) $$Д^4 F_{22}^{(j)} - jД^2 F_{22}^{(j)} = \frac{(3k+2)}{2(k+1)} Q_2(\mu) e^{-\varrho}\left(1 + \frac{3}{\varrho} + \frac{3}{\varrho^2}\right).$$

The resulting solutions for $F_{22}^{(0)}$ and $F_{22}^{(2)}$ are

$$F_{22}^{(0)} = \left[A_{20} + \frac{B_{20}}{\varrho^2} + \frac{(3k+2)}{2(k+1)}\left(1 + \frac{3}{\varrho} + \frac{3}{\varrho^2}\right)e^{-\varrho}\right] Q_2(\mu),$$

$$F_{22}^{(2)} = \left[\frac{B_{22}}{\varrho^2} + A_{22}\left(1 + \frac{3}{\sqrt{2}\varrho} + \frac{3}{2\varrho^2}\right)e^{-\sqrt{2}\cdot\varrho}\right.$$

$$\left. - \frac{(3k+2)}{2(k+1)}\left(1 + \frac{3}{\varrho} + \frac{3}{\varrho^2}\right)e^{-\varrho}\right] Q_2(\mu).$$

The matching procedure is based on the idea that χ and ψ are different forms of the same function.
By means of the matching condition

$$\lim_{r\to\infty}[\chi_1 + \varepsilon\chi_2 + \cdots] = \lim_{\varrho\to 0}\left[\frac{1}{M^2}\cdot\psi_1 + \frac{\varepsilon}{M^2}\psi_2 + \cdots\right]$$

we obtain

$$a_i^{(j)} = 0, \quad (j = 0, 2; \quad i = 1, 2, 3, 4);$$

$$A_{20} = \frac{(3k+2)}{4(k+1)}; \quad A_{22} = \frac{(3k+2)}{2(k+1)}; \quad B_{20} = -\frac{3}{2}\cdot\frac{(3k+2)}{(k+1)}; \quad B_{22} = \frac{3}{4}\cdot\frac{(3k+2)}{(k+1)}.$$

Therefore

(27) $$\psi = -\varrho^2 Q_1(\mu) e^{i\tau} - |M|\frac{(3k+2)}{(k+1)}\left[\left(+\frac{1}{\varrho}\right)e^{-\varrho} - \frac{1}{\varrho}\right] Q_1(\mu) e^{(\tau+\pi/4)} + R_e\frac{(3k+2)}{2(k+1)}$$
$$\times\left[\frac{1}{2} - \frac{3}{\varrho^2} + \left(1 + \frac{3}{\varrho} + \frac{3}{\varrho^2}\right)e^{-\varrho}\right] Q_2(\mu) + R_e\frac{(3k+2)}{2(k+1)}\left[\frac{3}{2\varrho^2} + \left(1 + \frac{3}{\varrho\sqrt{2}} + \frac{3}{2\varrho^2}\right)\right.$$
$$\left. \times e^{-\sqrt{2}\varrho} + \left(1 - \frac{3}{\varrho} + \frac{3}{\varrho^2}\right)e^{-\varrho}\right] Q_2(\mu) e^{2i\tau} + \cdots$$

(28) $$\chi = -\frac{Q_1(\mu)}{2}\left[2r^2 - \frac{(3k+2)}{(k+1)}r + \frac{k}{(k+1)}\cdot\frac{1}{r}\right]e^{i\tau} - \frac{|M|Q_1(\mu)}{3}\cdot\frac{(3k=2)}{(k+1)^2}\left[-\frac{(3k+2)}{2}r + \frac{k}{2r}\right.$$
$$\left. + (k+1)r^2\right] e^{i\left(\tau+\frac{\pi}{4}\right)} + \frac{R_e}{16}\cdot\frac{(3k+2)}{(k+1)}\left\{\frac{k(5k+4)}{10(k+1)^2} + \frac{k(5k+6)}{10(k+1)^2}\cdot\frac{1}{r^2}\right.$$
$$\left. + r^2 - \frac{(3k+2)}{2(k+1)}r - \frac{k}{2(k+1)}\cdot\frac{1}{r}\right\} Q_2(\mu)(1 + e^{2i\tau}) + \cdots$$

Similarly, for the region inside the drop, we seek $\hat{\chi}_2$ in the form

$$\hat{\chi}_2 = \hat{\Phi}_{20} + |M|\hat{\Phi}_{21} + |M|^2\hat{\Phi}_{22} + \cdots$$

where

$$\tilde{\Phi}_{22} = \hat{\Phi}_{22}^{(0)} + \hat{\Phi}_{22}^{(2)} e^{2i\tau}.$$

Straightforward but tedious computations, which will not be reported here, yield

$$\hat{\Phi}_{20} = \hat{\Phi}_{21} = 0; \quad \hat{\Phi}_{22}^{(0)} = \hat{\Phi}_{22}^{(2)} = Q_2(\mu) \frac{(4k+5)(3k+2)}{160(k+1)^3} (r^5 - r^3).$$

Hence for the $\hat{\chi}$ we have

$$\hat{\chi} = \frac{Q_1(\mu)(r^2-r^4)}{2(k+1)} e^{i\tau} + \frac{M\,|Q_1(\mu)(3k+2)(r^2-r^4)}{6(k+1)^2} e^{i\left(\tau+\frac{\pi}{4}\right)} \tag{29}$$

$$+ R_e \cdot Q_2(\mu)(1+e^{2i\tau}) \frac{(4k+5)(3k+2)}{160(k+1)^3} (r^5 - r^3) + \cdots$$

It is of interest to calculate the drag on the drop. We have

$$F_D = \frac{\text{drag}}{a^2 \varrho U_\infty^2} = \frac{2\pi}{R_e} \int_0^\pi [p_{rr}\,|_{r=1} \cos\theta - p_{r\theta}\,|_{r=1} \sin\theta] \sin\theta d\theta, \tag{30}$$

where

$$p_{rr} = -p + 2\frac{\partial u}{\partial r}, \tag{31}$$

$$p_{r\theta} = \frac{1}{r} \cdot \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}.$$

By manipulating the radial and tangential momentum equations, we can derive an expression for the pressure

$$p = p_0 - \frac{(2k+2)}{2r^2(k+1)} \cos\theta \left(1 + \frac{(3k+2)}{3(k+1)} \cdot M\right) e^{i\tau}. \tag{32}$$

Using (30), (31) and (32), we get

$$F_D = \frac{2\pi}{R_e} \cdot \frac{(3k+2)}{(k+1)} \left(1 + \frac{(3k+2)}{3(k+1)} \cdot M\right) e^{i\tau}. \tag{33}$$

This expression is (in some sense) a generalization of the Taylor & Acrivos's formula (10a) for the case of an oscillating fluid drop immersed in another fluid.

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