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On the Ideas of Mass and Force in a Model of Continuum Media

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In classic mechanics of continuum media one makes the assumption that in each moment, the system of forces which acts on the continuum media, may be represented as a sum of two systems of forces, which are usually called system of mass forces and system of surface forces.

In this work we'll show that in the framework of the axiomes given in (2), the system of forces, which acts on the continuum media is represented as a sum of mass forces and forces acting on the surface of the body or on a surface in it or in a point in the body (we use "or" not in an excluding sense).

In the above-mentioned assumption, in this case, comes out proovable, as a private case of a more general result. For that purpose we'll use the well-known theorem of the prolongation of an additive function, defined on a family of sets, to a measure, and the Dunford-Pettis theorem for the representation of a vector measure.

I. We'll use the model given in ((2), part I, § 3, ex. 2), and for lucidity we'll give it here. Let X be a locally compact space (it sufficient an Euclidean space with $dim\ X=3$ for the purposes of mechanics). We'll consider an Universe Ω, where the bodies are the closures of the open kernels of the sets in i. e. $\Omega=\{A: A\subset X\ \&\ A=\overline{\mathring{A}}\}$. As in (2) $\bar{\Omega}=\Omega\cup\{0,\infty\}$ where 0 and ∞ are objects defined the next way:

$$0\subset B\forall B\in\bar{\Omega}\qquad B\subset 0\Rightarrow B=0\qquad \forall B\notin\bar{\Omega}$$

$$B\subset\infty\forall B\in\bar{\Omega}\qquad \infty\subset B\Rightarrow B=\infty\qquad \forall B\in\bar{\Omega}.$$

Let the next operations be defined among the bodies of (we call these operations unity and section accordingly):

$$A\vee B=\overline{\mathring{A}\cup\mathring{B}},\qquad A\wedge B=\overline{\mathring{A}\cap\mathring{B}}.$$

There is a nonnegative function given in $\bar{\Omega}$ by the next axiomes:

$(PM_{\mathrm{I}})M\subset N=\alpha(M)\leq\alpha(N)$,

$(PM_{\mathrm{II}})\alpha(M\vee N)\leq\alpha(M)+\alpha(N)$,

$(PM_{\mathrm{III}})M\wedge N=0\Rightarrow\alpha(M\vee N)=\alpha(M)+\alpha(N)$.

In the framework of the chosen model one can make the more concrete assumption:

$(PM_{IV})\forall\varepsilon>0\ \forall N\in\bar{\Omega}\exists K, U: K\subset N\subset U$ where K is a compact set, U—an open set in X, such that for each $M\in\bar{\Omega}$ from $K\subset M\subset U$ follows that $|\alpha(M)-\alpha(N)|<\varepsilon$. Assuming that (PM_{IV}) and from $\alpha(\infty)=\infty$ one may conclude that it is possible to consider only limited bodies in $\bar{\Omega}$. From the definition of a body there follows:

$$(PC_I)A,\ B\in\bar{\Omega}\Rightarrow A\vee B\in\bar{\Omega}\quad and\quad A\wedge B\in\bar{\Omega}$$

and yet:

$(PC_{II})\forall(K, U)$ where K is a compact set and U—an open set in X and $K\subset U$ then there exists an $M\in\bar{\Omega}$, such that $K\subset M\subset U$.

Theorem 5 from ((I), ch. IV, § 4) is appliable in the situation, mentioned above, after some slight corrections, connected with the application of the operations $\wedge$ and $\vee$, instead of the common ones $\cap$ and $\cup$, which it is prooved about. We'll consider those corrections closely in the appendix at the end of the work. So, using th. 5 we construct a measure μ by means of the additive, nonnegative function α. From now on we'll use it instead of μ, cause it posesses a richer mathematical sense than α.

II. Following the axiomes from ((2), part I, § 5) about an additive vector function $(\vec{f})$ and using assumptions analogous to those in part I, we may receive a vector measure $\vec{f}$. Taking in mind that bodies with μ-measure zero may have a non-zero $\vec{f}$-measure, and using Dunford-Pettis theorem about vector measures ((3) ch. IV § 5, cor. 2), we obtain:

$$\vec{f}=\vec{p}\mu+\vec{\nu},$$

where the measure $\vec{\nu}$ is μ-singular, i. e. it is defined on sets with a μ-measure zero.

That representation of the force $\vec{f}$ means that it is a sum of mass forces and forces acting on the surface of the body or on a part of it or on a surface in the body or in a point on or in the body (we have mentioned only those μ-singular sets, which are most natural of mechanical point of view), i. e.

$$\vec{f}=\vec{p}\mu+\vec{\nu}_{\Sigma}+\vec{\nu}_{s}+\vec{\nu}_{p}$$

where Σ is the surface of the body, S—a surface in it, P — a point on or in the body. Let's consider the next example: a body with surface S and with surface in it Σ, exposed to an electric input. If we put it in an electrostatic field, the above-mentioned result gives:

$$\vec{f}=\vec{p}\mu+\vec{\nu}_{\Sigma}+\vec{\nu}_{s}.$$

This example shows that the obtained representation is appliable in nonstandard occasions, which are not usually concerned in classic mechanics also. So the necessity of assuming that in each moment the force acting on the body is a sum of mass and surface forces falls away.

We can obtain the same result next way also. Let's leave (PM_{III}) unchanged for the scalar additive function of the model: (PM_{II}) for $M \wedge N = 0 \Rightarrow \alpha(M \vee N) = \alpha(M) + \alpha(N)$. Let's replace the same condition for the vector additive function with:

$$(PM_{III}) \quad \text{for} \quad M \cap N = \emptyset \Rightarrow \vec{f}(M \vee N) = \vec{f}(M) + \vec{f}(N).$$

So we obtain again:

$$\vec{f} = \vec{p}\mu + \vec{\nu}$$

without assuming that a body with a zero μ-measure has a zero $\vec{f}$-measure.

Appendix

Consider theorem 5((I), ch. IV, § 4). It is the next: Let Φ be a set of subsets from a locally compact space and let (PC_I) and (PC_{II}) hold in it. Let α be a finite numerical function defined on Φ, which satisfies (PM_I), (PM_{II}) and (PM_{III}). The necessary and sufficient condition for existing of a positive measure μ on X, such that the sets from Φ be μ-measurable and $\mu(M) = \alpha(M)$ for each $M \in \Phi$, is the function α to satisfy the condition (PM_{IV}). If that is so, there is only one μ and for each compact set K, $\mu(k) = \inf_{M \in \Phi,\ K \subset M} \alpha(M)$ hands, and for each open set U, $\mu^*(U) = \sup_{M \in \Phi,\ M \subset U} \alpha(M)$ holds.
The above-mentioned conditions of th. 5 are:
(PC_I) $A, B \in \Phi$ then $A \cup B \in \Phi$ and $A \cap B \in \Phi$
$(PC_{II}) \forall (K, U)$ where K is a compact, U-open and $K \subset U$, then such that $\exists M \in \Phi \Rightarrow K \subset M \subset U$.
For each $M, N \in \Phi$,
$(PM_I) M \subset N \Rightarrow \alpha(M) \leq \alpha(N)$,
$(PM_{II}) \alpha(M \cup N) \leq \alpha(M) + \alpha(N)$,
$(PM_{III}) M \cap N = \Phi \Rightarrow \alpha(M \cup N) = \alpha(M) + \alpha(N)$,
$(PM_{IV}) \forall \varepsilon > 0 \forall M \in \Phi \exists K, U : K \subset M \subset U$,
where K is compact, and U—open in X, such that for each $N \in \Phi$, such that $K < N < U$, $|\alpha(M) - \alpha(N)| < \varepsilon$ holds. The denotations in the model and in theorem 5 are the same. This way it comes clear that (PC_{II}), (PM_I) and (PM_{II}) are the same for the model and in th. 5 and from (PM_{III}) in the model there follows (PM_{III}) in th. 5 and the distinction is in (PM_{IV}) and (PC_I). Our further goal is to proove that theorem 5 remains true with a small change in the proof. We'll use (PC_I) from the model of the Universe. The proof of requires a change: $(PM_{IV}) \Rightarrow (PM'_{IV})$
$\varepsilon > 0$ $M \in \Phi$ from $(PM_{IV}) \Rightarrow \exists K, U : K \subset M \subset U$
$\forall P \in \Phi$ and $K \subset P \subset U \Rightarrow |\alpha(P) - \alpha(M)| < \varepsilon$.
Let $N \in \Phi$ and $K \subset N$; let V be an open set. From the local compactness of X the following includings $K \subset V$ and $\overline{V} \subset U$ hold.
Lemma I. Let $A \in \Phi$ and U is an open set and $B \subset A$ and $B \subset U$.

Then $B \subset A \wedge U$ where formally $\overline{\overset{\circ}{A} \cap U}$ we've denoted with $A \wedge U$.

Proof: $x \in B \Rightarrow x \in A = \overset{\circ}{\overline{A}}$ and $x \in U$ $x \in V$-neighbourhood $x \in U \cap V \Rightarrow \exists y \in \overset{\circ}{A} : y \in V \wedge U y \in U$

and $y\in V\Rightarrow y\in U\cap A$ and $j\in V$ for $\forall$ neighbourhood V of $x\exists y\in U\cap A$, $y\in V\Rightarrow x\in U\cap \overset{\circ}{A}=U\wedge A$. Then from Lemma I there follows: $K\subset N\wedge V\in\Phi$

$$|\alpha(N\wedge V)-\alpha(M)|<\varepsilon\Rightarrow\alpha(M)-\varepsilon\leq\alpha(N\wedge V)\leq\alpha(N).$$

A change is required at the end of 3°:
Let K_1 and K_2 are compact sets and $K_1\cap K_2=\Phi$, then:

$$\bar{\alpha}(K_1\subset K_2)\geq\bar{\alpha}_1(K_1)+\bar{\alpha}(K_2).$$

Proof:
$I=1, 2$ $K_i\subset U_i$ $U_1\cap U_2=\emptyset$ and let V_i-open
$K_i\subset V_i\subset\bar{V}_i\subset U_i$; $P\in\Phi$, such that $P\supset K_1\cup K_2$
$P_1=V_1\wedge P\subset\bar{V}_1\subset U_1$; $P_1\in\Phi$
$P_2=V_2\wedge P\subset\bar{V}_2\subset U_2$; $P_2\in\Phi$
$P_1\cup P_2\in\Phi$, $P_1\cap P_2=\Phi$ and $P_1\cup P_2\subset P\Rightarrow$
$\alpha(P)\geq\alpha(P_1\cup P_2)\geq\alpha(P_1)+\alpha(P_2)\geq\bar{\alpha}(K_1)+\bar{\alpha}(K_2)$
$\therefore\bar{\alpha}(K_1\cup K_2)\geq\bar{\alpha}(K_1)+\bar{\alpha}(K_2)$.
In the last chain of inequalities we have used that $K_i\subset P_i$. Really $x\in K_i$, $x\in O$—a neighbourhood, but $O\cap V_i\ni x$ is a neighbourhood of x also; $\exists y\in\overset{\circ}{P}: y\in O\cap V_i$ $\Rightarrow\forall x\in O\exists y\in\overset{\circ}{P}\cap V_i$ and $y\in O\Rightarrow x\in\overset{\circ}{P}\cap V_i=P_i$.

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