

A Method for Solving Systems of Linear Partial Differential Equations in Mechanics

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The basic disadvantages of all analytical methods, employed for the solution of systems of linear partial differential equations, lie in the significant difficulties that one could come across, attempting to satisfy the boundary conditions. The existing methods are applicable for simple boundary surfaces, defined in corresponding coordinate systems. What is proposed is a method of obtaining a great number of particular solutions, involving a lot of free parameters and constants of integration. These particular solutions are the basis for formal building of a general solution and can satisfy boundary conditions on complicated surfaces or contours.

One of the basic methods for solving linear partial differential equations appears to be the Fourier method. It is based on applying Cartesian or curvilinear coordinate systems with subsequent separation of variables. However, in the Fourier method as well, the basic difficulty lies in the satisfaction of given boundary conditions.

For better understanding of the main idea in the method proposed, let us consider the simple case of a system of Q second order linear partial differential equations with constant coefficients, involving functions of two arguments x and y :

$$(1) \quad \sum_{p=1}^Q L_{pq}(u_p) = 0, \quad (q=1, 2, \dots, Q),$$

where

$$L_{pq}(u_p) = a_{20}^{pq} \frac{\partial^2 u_p}{\partial x^2} + a_{02}^{pq} \frac{\partial^2 u_p}{\partial y^2} + a_{11}^{pq} \frac{\partial^2 u_p}{\partial x \partial y} + a_{10}^{pq} \frac{\partial u_p}{\partial x} + a_{01}^{pq} \frac{\partial u_p}{\partial y} + a_{00}^{pq} u_p = 0$$

According to the method proposed in [1] and [2], we look for particular solutions $\{u_p\}_{p=1}^Q$ of the type

$$(2) \quad u_p(x, y) = \sum_{i=0}^{m-1} X^{(i)}(x) \cdot Y_{pi}(y),$$

where m is an arbitrary integer number, chosen according to our requirements. Function $X(x)$ in (2) is the general integral of the following ordinary differential equation:

$$(3) \quad X^{(m)}(x) = \sum_{i=0}^{m-1} c_i X^{(i)}(x),$$

where $c_i = \text{const}$ (parameters).

Let us substitute (2) and (3) in (1) and consider coefficients preceding $X^{(i)}(x)$ ($i=0, 1, \dots, m-1$) equal to zero. Thus we obtain a system of $m \cdot Q$ linear ordinary differential equations with constant coefficients for $m \cdot Q$ unknown functions $Y_{pi}(y)$:

$$(4) \quad \begin{aligned} & \sum_{p=1}^Q \{ a_{00}^{pq} Y_{p0} + a_{01}^{pq} Y'_{p0} + a_{02}^{pq} Y''_{p0} + a_{10}^{pq} c_0 Y_{p(m-1)} + a_{20}^{pq} c_0 [Y_{p(m-2)} + c_{m-1} Y_{p(m-1)}] \\ & \quad + a_{11}^{pq} c_0 Y'_{p(m-1)} \} = 0, \\ & \sum_{p=1}^Q \{ a_{00}^{pq} Y_{p1} + a_{01}^{pq} Y'_{p1} + a_{02}^{pq} Y''_{p1} + a_{10}^{pq} (c_1 Y_{p(m-1)} + Y_{p0}) + a_{20}^{pq} [c_1 Y_{p(m-2)} \\ & \quad + (c_{m-1} c_1 + c_0) Y_{p(m-1)} + a_{11}^{pq} (c_1 Y'_{p(m-1)} + Y'_{p0})] \} = 0, \\ & \sum_{p=1}^Q \{ a_{00}^{pq} Y_{pi} + a_{01}^{pq} Y'_{pi} + a_{02}^{pq} Y''_{pi} + a_{10}^{pq} (Y_{p(i-1)} + c_i Y_{p(m-1)} + a_{20}^{pq} [Y_{p(i-2)} \\ & \quad + (c_{i-1} + c_i c_{m-1}) Y_{p(m-1)} + c_i Y'_{p(m-2)}] + a_{11}^{pq} (Y'_{p(i-1)} + c_i Y'_{p(m-1)})) \} = 0, \\ & \quad (i=2, 3, \dots, m-1); \quad (q=1, 2, \dots, Q). \end{aligned}$$

The solutions $Y_{pi}(y)$ of the above system could be easily obtained and substituted in (2). Solution of (3) could be easily obtained as well. However, for a special choice of c_i ($i=0, 1, \dots, m-1$), solution of (3) could be obtained, expressed in terms of hyperbolic or trigonometrical functions. For the special case $c_i=0$, the expression for $X(x)$ is a polynomial.

Regarding functions of more arguments, the already described operation must be repeated several times, aiming at a gradual reduction of the number of arguments, until there will remain only one argument. Hence, a system of ordinary differential equations, considering the remaining argument, could be obtained.

For example, for arguments x, y and z we look for an unknown function of the type:

$$(5) \quad u_p(x, y, z) = \sum_{i=0}^{m-1} X^{(i)}(x) \cdot \omega_{pi}(y, z), \quad (p=1, 2, \dots, Q),$$

where $X(x)$ is the solution of (2) and $\omega_{pi}(y, z)$ could be obtained from

$$(6) \quad \omega_{pi}(y, z) = \sum_{j=0}^{n-1} Y^{(j)}(y) \cdot Z_{pij}(z),$$

where function $Y(y)$ is the solution of

$$(7) \quad Y^{(n)}(y) = \sum_{j=0}^{m-1} b_j Y^{(j)}(y),$$

$b_j = \text{const}$. (parameters).

Thus, a system of $m.n.Q$ number of linear ordinary differential equations for $Z_{pij}(z)$ is obtained. Its characteristic determinant is of $m.n.Q$ order and has $P.Q$ number of roots which are $m.n$ times multiplex (P is the order of the partial differential equation). If one's aim is to determine those roots and to find out solutions for functions $Z_{pij}(z)$, it is not necessary to perform all laborious operations needed to obtain a system of ordinary differential equations and to solve a corresponding characteristic determinant of quite a high order.

The following simple and convenient rule can be defined. Roots of the characteristic determinant of the system of ordinary differential equations of $m.n.Q$ order for $Z_{pij}(z)$, are equal to the roots of a "basic characteristic determinant" only of Q order, which is to be taken $m.n$ times. "Basic characteristic determinant" of the system of partial differential equations is the one, obtained by using the following formal substitution:

$$(8) \quad \begin{aligned} \frac{\partial u_p}{\partial x} &\rightarrow a_i, \\ \frac{\partial u_p}{\partial y} &\rightarrow \beta_j, \quad \frac{\partial^{u+v+\lambda} u_p}{\partial x^u \partial y^v \partial z^\lambda} \rightarrow a_i^u \beta_j^v r^\lambda, \\ \frac{\partial u_p}{\partial z} &\rightarrow r, \end{aligned} \quad \begin{aligned} (p=1, 2, \dots, Q), \quad (i=0, 1, \dots, m-1), \\ (j=0, 1, \dots, n-1). \end{aligned}$$

The corresponding characteristic determinant is:

$$(9) \quad \begin{vmatrix} L_{11}(a_i, \beta_j, r) & L_{21}(a_i, \beta_j, r) & \dots & L_{Q1}(a_i, \beta_j, r) \\ L_{12}(a_i, \beta_j, r) & L_{22}(a_i, \beta_j, r) & \dots & L_{Q2}(a_i, \beta_j, r) \\ \dots & \dots & \dots & \dots \\ L_{1Q}(a_i, \beta_j, r) & L_{2Q}(a_i, \beta_j, r) & \dots & L_{QQ}(a_i, \beta_j, r) \end{vmatrix} = 0$$

If we consider the important special case when

$$(10) \quad X^{(m)}(x) = - \sum_{i=1}^{m/2} \binom{i}{m/2} X^{(m-2i)}(x) \cdot a^{2i},$$

and

$$(11) \quad Y^{(n)}(y) = - \sum_{j=1}^{n/2} \binom{j}{n/2} Y^{(n-2j)}(y) \cdot \beta^{2j},$$

the basic characteristic determinant is obtained formally by a substitution

$$(12) \quad \begin{aligned} \frac{\partial u_p}{\partial x} &\rightarrow i a, \\ \frac{\partial u_p}{\partial y} &\rightarrow i \beta, \quad \frac{\partial^{u+v+\lambda} u_p}{\partial x^u \partial y^v \partial z^\lambda} \rightarrow (i a)^u (i \beta)^v r^\lambda = (i)^{u+v} a^u \beta^v r^\lambda, \\ \frac{\partial u_p}{\partial z} &\rightarrow r, \end{aligned} \quad (p=1, 2, \dots, Q)$$

in the given system of partial differential equations. The roots of the basic characteristic determinant are to be taken $m.n$ times multiplex.

Let us consider now the application of the method to the anisotropic elasticity. Lamé equations in the case of total anisotropy with 21 moduli of anisotropy are as follows:

$$\begin{aligned}
& A_{11} \frac{\partial^2 u}{\partial x^2} + A_{66} \frac{\partial^2 u}{\partial y^2} + A_{55} \frac{\partial^2 u}{\partial z^2} + 2A_{16} \frac{\partial^2 u}{\partial x \partial y} + 2A_{56} \frac{\partial^2 u}{\partial y \partial z} + 2A_{15} \frac{\partial^2 u}{\partial z \partial x} \\
& + A_{16} \frac{\partial^2 v}{\partial x^2} + A_{26} \frac{\partial^2 v}{\partial y^2} + A_{45} \frac{\partial^2 v}{\partial z^2} + (A_{12} + A_{66}) \frac{\partial^2 v}{\partial x \partial y} + (A_{46} + A_{25}) \frac{\partial^2 v}{\partial y \partial z} + (A_{14} + A_{56}) \frac{\partial^2 v}{\partial z \partial x} \\
& + A_{15} \frac{\partial^2 w}{\partial x^2} + A_{46} \frac{\partial^2 w}{\partial y^2} + A_{25} \frac{\partial^2 w}{\partial z^2} + (A_{14} + A_{56}) \frac{\partial^2 w}{\partial x \partial y} + (A_{36} + A_{45}) \frac{\partial^2 w}{\partial y \partial z} + (A_{13} + A_{55}) \frac{\partial^2 w}{\partial z \partial x} = 0 \\
(13) \quad & A_{16} \frac{\partial^2 u}{\partial x^2} + A_{26} \frac{\partial^2 u}{\partial y^2} + A_{45} \frac{\partial^2 u}{\partial z^2} + (A_{12} + A_{66}) \frac{\partial^2 u}{\partial x \partial y} + (A_{25} + A_{46}) \frac{\partial^2 u}{\partial y \partial z} \\
& + (A_{14} + A_{56}) \frac{\partial^2 u}{\partial z \partial x} + A_{66} \frac{\partial^2 v}{\partial x^2} + A_{22} \frac{\partial^2 v}{\partial y^2} + A_{44} \frac{\partial^2 v}{\partial z^2} + 2A_{26} \frac{\partial^2 v}{\partial x \partial y} + 2A_{24} \frac{\partial^2 v}{\partial y \partial z} \\
& + 2A_{46} \frac{\partial^2 v}{\partial z \partial x} + A_{56} \frac{\partial^2 w}{\partial x^2} + A_{24} \frac{\partial^2 w}{\partial y^2} + A_{34} \frac{\partial^2 w}{\partial z^2} + (A_{46} + A_{25}) \frac{\partial^2 w}{\partial x \partial y} + (A_{23} + A_{44}) \frac{\partial^2 w}{\partial y \partial z} \\
& + (A_{36} + A_{45}) \frac{\partial^2 w}{\partial z \partial x} = 0 \\
& A_{15} \frac{\partial^2 u}{\partial x^2} + A_{46} \frac{\partial^2 u}{\partial y^2} + A_{35} \frac{\partial^2 u}{\partial z^2} + (A_{56} + A_{14}) \frac{\partial^2 u}{\partial x \partial y} + (A_{45} + A_{36}) \frac{\partial^2 u}{\partial y \partial z} \\
& + (A_{55} + A_{13}) \frac{\partial^2 u}{\partial z \partial x} \\
& + A_{56} \frac{\partial^2 v}{\partial x^2} + A_{24} \frac{\partial^2 v}{\partial y^2} + A_{34} \frac{\partial^2 v}{\partial z^2} + (A_{25} + A_{46}) \frac{\partial^2 v}{\partial x \partial y} + (A_{23} + A_{44}) \frac{\partial^2 v}{\partial y \partial z} \\
& + (A_{45} + A_{36}) \frac{\partial^2 v}{\partial z \partial x} \\
& + A_{55} \frac{\partial^2 w}{\partial x^2} + A_{44} \frac{\partial^2 w}{\partial y^2} + A_{33} \frac{\partial^2 w}{\partial z^2} + 2A_{45} \frac{\partial^2 w}{\partial x \partial y} + 2A_{34} \frac{\partial^2 w}{\partial y \partial z} + 2A_{35} \frac{\partial^2 w}{\partial z \partial x} = 0.
\end{aligned}$$

We look for particular solutions of (13) according to (5), (6), (10) and (11). Using (12), we obtain the basic characteristic determinant

$$(14) \quad \begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix} = 0 \rightarrow \begin{matrix} r_k = \dots & (k=1, 2, \dots, 6) \\ m.n \text{ times multiplex roots,} \end{matrix}$$

where

$$\begin{aligned}
B_{11} &= A_{55}r^2 + i(2A_{15}\alpha + 2A_{56}\beta)r - (A_{11}\alpha^2 + A_{66}\beta^2 + 2A_{16}\alpha\beta), \\
B_{22} &= A_{44}r^2 + i(2A_{46}\alpha + 2A_{24}\beta)r - (A_{66}\alpha^2 + A_{22}\beta^2 + 2A_{26}\alpha\beta), \\
B_{33} &= A_{33}r^2 + i(2A_{35}\alpha + 2A_{34}\beta)r - (A_{55}\alpha^2 + A_{44}\beta^2 + 2A_{45}\alpha\beta), \\
(15) \quad B_{12} &= B_{21} = A_{45}r^2 + i[(A_{14} + A_{56})\alpha + (A_{25} + A_{46})\beta]r - [A_{16}\alpha^2 + A_{26}\beta^2 + (A_{12} + A_{66})\alpha\beta], \\
B_{13} &= B_{31} = A_{35}r^2 + i[(A_{13} + A_{55})\alpha + (A_{36} + A_{45})\beta]r - [A_{15}\alpha^2 + A_{46}\beta^2 + (A_{14} + A_{56})\alpha\beta], \\
B_{23} &= B_{32} = A_{34}r^2 + i[(A_{36} + A_{45})\alpha + (A_{23} + A_{44})\beta]r - [A_{56}\alpha^2 + A_{24}\beta^2 + (A_{25} + A_{46})\alpha\beta].
\end{aligned}$$

Obtaining roots of the characteristic determinant (14), according to the already described procedure, one can easily obtain expressions for particular integrals

of the system (13). They well depend on two parameters α and β , m constants of integration for $X(x)$ according to (10), n constants of integration for $Y(y)$ according to (11) and m, n, P, Q constants of integration for $Z_{pij}(z)$. All constants of integration are functions of α and β . The final general integrals of (13) are sums or integral of these particular solutions as in the Fourier method. Constants of integration are to be obtained, using the problem boundary conditions.

In the case of orthotropic bodies (which is very common in practice) the basic characteristic determinant is of the form :

$$(16) \quad \begin{vmatrix} A_{55}r^2 - \alpha^2 A_{11} - \beta^2 A_{66} & -\alpha\beta(A_{12} + A_{66}) & i(A_{13} + A_{55})\alpha r \\ -\alpha\beta(A_{12} + A_{66}) & A_{44}r^2 - \alpha^2 A_{66} - \beta^2 A_{22} & i(A_{23} + A_{44})\beta r \\ i(A_{55} + A_{13})\alpha r & i(A_{23} + A_{44})\beta r & A_{33}r^2 - \alpha^2 A_{55} - \beta^2 A_{44} \end{vmatrix} = 0.$$

Let us finally consider application of the method to the case of micropolar continuum. Lamé equations for the Cosserat continuum, according to Nowacki [4], are of the form :

$$(17) \quad \begin{aligned} & (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + (\mu + \nu) \frac{\partial^2 u}{\partial y^2} + (\mu + \nu) \frac{\partial^2 u}{\partial z^2} + (\lambda + \mu - \nu) \frac{\partial^2 v}{\partial x \partial y} + (\lambda + \mu - \nu) \frac{\partial^2 w}{\partial z \partial x} \\ & + 2\nu \frac{\partial W}{\partial y} - 2\nu \frac{\partial V}{\partial z} = 0 \\ & (\lambda + 2\mu) \frac{\partial^2 v}{\partial y^2} + (\mu + \nu) \frac{\partial^2 v}{\partial z^2} + (\mu + \nu) \frac{\partial^2 v}{\partial x^2} + (\lambda + \mu - \nu) \frac{\partial^2 w}{\partial y \partial z} + (\lambda + \mu - \nu) \frac{\partial^2 u}{\partial x \partial y} \\ & + 2\nu \frac{\partial U}{\partial z} - 2\nu \frac{\partial W}{\partial x} = 0 \\ & (\lambda + 2\mu) \frac{\partial^2 w}{\partial z^2} + (\mu + \nu) \frac{\partial^2 w}{\partial x^2} + (\mu + \nu) \frac{\partial^2 w}{\partial y^2} + (\lambda + \mu - \nu) \frac{\partial^2 u}{\partial z \partial x} + (\lambda + \mu - \nu) \frac{\partial^2 v}{\partial y \partial z} \\ & + 2\nu \frac{\partial V}{\partial x} - 2\nu \frac{\partial U}{\partial y} = 0 \\ & (\delta + 2\gamma) \frac{\partial^2 U}{\partial x^2} + (\gamma + \varepsilon) \frac{\partial^2 U}{\partial y^2} + (\gamma + \varepsilon) \frac{\partial^2 U}{\partial z^2} + (\delta + \gamma - \varepsilon) \frac{\partial^2 V}{\partial x \partial y} + (\delta + \gamma - \varepsilon) \frac{\partial^2 W}{\partial z \partial x} \\ & + 2\nu \frac{\partial w}{\partial y} - 2\nu \frac{\partial v}{\partial z} = 0 \\ & (\delta + 2\gamma) \frac{\partial^2 V}{\partial y^2} + (\gamma + \varepsilon) \frac{\partial^2 V}{\partial z^2} + (\gamma + \varepsilon) \frac{\partial^2 V}{\partial x^2} + (\delta + \gamma - \varepsilon) \frac{\partial^2 W}{\partial y \partial z} + (\delta + \gamma - \varepsilon) \frac{\partial^2 U}{\partial x \partial y} \\ & + 2\nu \frac{\partial u}{\partial z} - 2\nu \frac{\partial w}{\partial x} = 0 \\ & (\delta + 2\gamma) \frac{\partial^2 W}{\partial z^2} + (\gamma + \varepsilon) \frac{\partial^2 W}{\partial x^2} + (\gamma + \varepsilon) \frac{\partial^2 W}{\partial y^2} + (\delta + \gamma - \varepsilon) \frac{\partial^2 U}{\partial z \partial x} + (\delta + \gamma - \varepsilon) \frac{\partial^2 V}{\partial y \partial z} \\ & + 2\nu \frac{\partial v}{\partial x} - 2\nu \frac{\partial u}{\partial y} = 0 \end{aligned}$$

The corresponding basic characteristic determinant is of the form :

$$\begin{array}{c}
 \begin{array}{ccccccc}
 \frac{(\mu+\nu)r^2}{(\lambda+2\mu)a^2} & -(\lambda+\mu-\nu)a\beta & i(\lambda+\mu-\nu)ar & 0 & -2v\delta & 2iv\beta & \\
 \frac{(\mu+\nu)\beta^2}{(\mu+\nu)\beta^2} & & & & & & \\
 \hline
 -(\lambda+\mu-\nu)a\beta & \frac{(\mu+\nu)r^2}{-(\mu+\nu)a^2} & i(\lambda+\mu-\nu)\beta r & 2vr & 0 & -2iva & \\
 & \frac{(\lambda+2\mu)\beta^2}{(\lambda+2\mu)\beta^2} & & & & & \\
 \hline
 i(\lambda+\mu-\nu)ar & i(\lambda+\mu-\nu)r & \frac{(\lambda+2\mu)r^2}{-(\mu+\nu)a^2} & -2iv\beta & 2iva & 0 & \\
 & & \frac{(\mu+\nu)\beta^2}{(\mu+\nu)\beta^2} & & & & \\
 \hline
 0 & & & \frac{(\gamma+\varepsilon)r^2}{-(\delta+2\gamma)a^2} & -(\delta+\gamma-\varepsilon)a\beta & i(\delta+\gamma-\varepsilon)ar & \\
 & & & -(\gamma+\varepsilon)\beta^2-4v & & & \\
 \hline
 2vr & 0 & -2i & -(\delta+\gamma-\varepsilon)a\beta & \frac{(\gamma+\varepsilon)r^2}{-(\gamma+\varepsilon)a^2} & i(\delta+\gamma-\varepsilon)\beta r & \\
 & & & & -(\delta+2\gamma)\beta^2-4v & & \\
 \hline
 -2iv\beta & 2iva & 0 & i(\delta+\gamma-\varepsilon)ar & i(\delta+\gamma-\varepsilon)\beta r & \frac{(\delta+2\gamma)r^2}{-(\gamma+\varepsilon)a^2} & \\
 & & & & & -(\gamma+\varepsilon)\beta^2-4v & \\
 \hline
 \end{array}
 \end{array}
 =0$$

(18)

Its roots are necessary for building particular solutions for the unknown functions, namely displacement and rotations $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$, $U(x, y, z)$, $V(x, y, z)$, $W(x, y, z)$. After some transformations in (18), one obtains

$$\begin{aligned}
 & \begin{vmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{vmatrix} = 0 \rightarrow \begin{matrix} r_k = \dots & (k=1, 2, \dots, 12) \\ m.n \text{ times multiplex roots} \end{matrix} \\
 & D_{11} = (r^2 - \beta^2) [\omega_3(r^2 - \alpha^2 - \beta^2) + 1] - (\omega_1 \alpha^2 + \omega_0)(r^2 - \alpha^2 - \beta^2), \\
 & D_{22} = (r^2 - \alpha^2) [\omega_3(r^2 - \alpha^2 - \beta^2) + 1] - (\omega_1 \beta^2 + \omega_0)(r^2 - \alpha^2 - \beta^2), \\
 & D_{33} = -(\alpha^2 + \beta^2) [\omega_3(r^2 - \alpha^2 - \beta^2) + 1] - (\omega_1 r^2 + \omega_0)(r^2 - \alpha^2 - \beta^2), \\
 (19) \quad & D_{12} = D_{21} = -[\omega_2(r^2 - \alpha^2 - \beta^2) - 1] \alpha \beta, \\
 & D_{13} = D_{31} = [\omega_2(r^2 - \alpha^2 - \beta^2) - 1] i \alpha r, \\
 & D_{23} = D_{32} = [\omega_2(r^2 - \alpha^2 - \beta^2) - 1] i \beta r,
 \end{aligned}$$

where

$$\begin{aligned}
 \omega_0 &= \frac{\lambda + 2\mu}{\lambda + \mu}, & \omega_1 &= \frac{\delta + 2\gamma}{4\nu} \cdot \frac{\lambda + 2\mu}{\lambda + \mu}, \\
 \omega_2 &= \frac{(\delta + \gamma - \varepsilon)}{4\nu} \cdot \frac{\lambda + 2\mu}{\lambda + \mu} + \frac{\lambda + \mu - \nu}{4\nu} \cdot \frac{\gamma + \varepsilon}{\lambda + \mu}, \\
 \omega_3 &= \frac{\mu + \nu}{4\nu} \cdot \frac{\gamma + \varepsilon}{\lambda + \mu}
 \end{aligned}$$

In conclusion we can say that what is presented is a method for reducing linear partial differential equations to ordinary linear differential equations, increasing the number of functions. There are no principal difficulties for a formal application not only to problems of mathematical elasticity theory, but also to solutions of an equation or a system of equations which are linear and involve partial derivatives of any rank, whereas coefficients are constant or depend only on one argument. It stands clear that for obtaining complete solutions of practical problems, modern computers will be of great and effective use.

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References

1. Коларов Д., Метод интегрирования уравнений плоской задачи теории упругости, ТПМ кн. 2, 1982.
2. Коларов Д., Интегрирование уравнений пространственной граничной задачи теории упругости, ТПМ, кн. 2, 1982.
3. Лехницкий С. Г., Теория упругости анизотропного тела, М., 1950.
4. Новацкий В., Теория упругости, Мир, М., 1975.