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Free Convective Heat Transfer from the Outer Surface of an Unsteadily Moving Vertical Circular Cylinder

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1. Introduction

Free convection flows past vertical plates and cylinders have been a subject of research for a long time due to their applications in technological fields [1]. In plane flow, self-similar solutions of the equations for steady laminar free convection were discovered by Pohlhausen [2] in 1930 for a semi-infinite vertical plate at constant temperature, and by Zel'dovich [3] in 1937 for an infinite horizontal line source of heat. Ostrach [4] presented the first numerical results for free convection past a vertical flat plate for isothermal case. Sparrow and Gregg [5] have solved the equations of free convection for a non-isothermal vertical flat plate for two families of wall temperature for various values of the Prandtl number (Pr) using a similarly transformation involving Grashof number (Gr). Takhar [6] presented a numerical solution for free convection from a semi-infinite plate which is uniformly heated up to a length t from the base and insulated for the rest of its length.

In axisymmetric flow self-similar steady solutions were found by Yih [7] and others for a point source of heat, and by Millsaps and Pohlhausen [8] for the field outside of a vertical circular cylinder with linearly increasing surface temperature. Other full solutions corresponding to cones, kettle shapes and parabolic nosed bodies are given in [9].

The unsteady free convective flows are also important. Three recent contributions to the subject are those of Elliott L. [10], Pop I. [11] and Merkin J. [12].

The body may be assumed to be both non-porous and porous. In the case of a porous, infinite plate free convective flows were studied by Nanda and Sharma [13].

In all these investigations, the bodies were assumed to be stationary. If they are assumed to move steadily or unsteadily in the vertical or downward direction, then how is the flow of air or water affected by the free convection currents? This is the motivation of the papers by Soundalgekar V. M. [14, 15] and by Soundalgekar and Cupta S. K. [16]. The purpose of this paper is to investigate the laminar free convective flow past an unsteadily moving vertical circular cylinder. The velocity of the cylinder and wall temperature are supposed to vary as some power of the time. The determination of the Nusselt number is the main point of interest in this investigation.

2. Formulation of the problem

It is well-known that in heat transfer through viscous fluid flows the Prandtl number representing the connection between the viscous and temperature boundary layer plays a very important role. In order to obtain analytical approximate solution, in this paper we shall restrict ourselves to the analysis of such fluids for which $Pr=1$.

For extreme values of the Prandtl number ($Pr \to 0$, $Pr \to \infty$) the boundary layer of free convection approaches a singular character so that a direct regular perturbation technique cannot be applied for obtaining insight in free convection under such conditions.

We assume a large Reynolds number and a short time of investigation, so that

$$\frac{1}{\nu' \mathrm{Re}} = \frac{1}{S} \quad \varepsilon \ll 1$$

where ν' is a constant of order unity and ε is a small parameter. The Reynolds number Re and the Strouhal number S are calculated by a characteristic length a — radius of the cylinder, a characteristic velocity U_0 and a characteristic time τ_0.

The great success of the investigations into flow of viscous fluids along solid surfaces has to be contributed primarily to the concept of the boundary layer as proposed by Prandtl. In free convection this idea has persisted that the condition

$$\mathrm{Gr} = \frac{g\beta a^3 \Delta T_0}{\nu^2} \gg 1$$

has to be satisfied if the solutions are to be drawn from boundary layer equations. Here g is acceleration due to gravity, β is coefficient of thermal expansion

$$\Delta T_0 = T_{w_0} - T_0$$

where T_{w_0} is the wall temperature at $t=0$ and ν — kinematic viscosity. It has been shown in [17] that for boundary layer problems of free convection the Grashof number varies from 10^4 to 10^9. We suppose $\mathrm{Gr} = \beta \mathrm{R}_e^2 S$, where β is a constant of order unity.

Besides the equation of state there are four equations governing the phenomenon of free convection in unsteady state viz., the equation of continuity

$$(1) \qquad \frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw)$$

the momentum equations

$$(2) \qquad \frac{\partial w}{\partial t} + \varepsilon \left[u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right] = \nu' \varepsilon^2 \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right] + \beta \theta$$

$$(3) \qquad \frac{\partial u}{\partial t} + \varepsilon \left[u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right] = \nu' \varepsilon^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{g z^2} - \frac{u}{r^2} \right]$$

and the energy equation

$$(4) \qquad \frac{\partial \theta}{\partial t} + \varepsilon \left[u \frac{\partial \theta}{\partial r} + w \frac{\partial \theta}{\partial z} \right] = \nu' \varepsilon^2 \left[\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{g z^2} \right].$$

Here we use cylindrical coordinate system (r, φ, z), $\theta = \frac{T - T_\infty}{\Delta T_0}$, u and w are r and z components of velocity. Now we suppose that

$$\theta_w(t)=\frac{T_w(t)-T_\infty}{\Delta T_0}=t^\alpha$$

where $\alpha\geq 0$. The initial and boundary conditions for the velocity and temperature are

(5) $$u=v=0,\quad \theta=0 \text{ at } t=0 \text{ and } r\geq 1$$

(6) $$u=0,\quad w=t^{\alpha+1},\quad \theta=\theta_w(t) \text{ at } r=1 \text{ and } t>0$$

(7) $$u\to 0,\ w\to 0,\ \theta\to 0 \text{ as } r\to\infty \text{ and } t>0.$$

We may satisfy the continuity equation (1) by writing

(8) $$u=-\frac{1}{r}\frac{\partial\psi}{\partial z},\qquad w=\frac{1}{r}\frac{\partial\psi}{\partial r}.$$

The problem (1—7) is solved by means of regular parameter expansion. We assume

(9) $$\begin{aligned} &r=1+\varepsilon\zeta \\ &\theta=\theta_1+\varepsilon\theta_2+\varepsilon^2\theta_3+\cdots \qquad && u=\varepsilon u_1+\varepsilon^2 u_2+\cdots \\ &w=w_1+\varepsilon w_2+\varepsilon^2 w_3+\cdots && \psi=\varepsilon\psi_1+\varepsilon\psi_2+\cdots \end{aligned}$$

Substituting (9) into equations (2—4) and comparing like powers of ε we find: the first order equations

(10) $$\frac{\partial\theta_1}{\partial t}-\nu'\frac{\partial^2\theta_1}{\partial\zeta^2}=0$$

(11) $$\frac{\partial w_1}{\partial t}-\nu'\frac{\partial^2 w_1}{\partial\zeta^2}=\beta\theta_1$$

the second order equations

(12) $$\frac{\partial\theta_2}{\partial t}-\nu'\frac{\partial^2\theta_2}{\partial\zeta^2}=\nu'\frac{\partial\theta_1}{\partial\zeta}-u_1\frac{\partial\theta_1}{\partial\zeta}-w_1\frac{\partial\theta_1}{\partial z}$$

(13) $$\frac{\partial w_2}{\partial t}-\nu'\frac{\partial^2 w_1}{\partial\zeta^2}=\nu'\frac{\partial w_1}{\partial\zeta}+\beta\theta_2-u_1\frac{\partial w_1}{\partial\zeta}-w_1\frac{\partial w_1}{\partial z}$$

(14) $$\frac{\partial u_1}{\partial t}-\nu'\frac{\partial^2 u_1}{\partial\zeta^2}=0$$

and the third order equations

(15) $$\frac{\partial\theta_3}{\partial t}-\nu'\frac{\partial^2\theta_3}{\partial\zeta^2}=\nu'\frac{\partial\theta_2}{\partial\zeta}-\nu'\zeta\frac{\partial\theta_1}{\partial\zeta}-u_2\frac{\partial\theta_1}{\partial\zeta}-u_1\frac{\partial\theta_2}{\partial\zeta}$$

(16) $$\frac{\partial w_3}{\partial t}-\nu'\frac{\partial^2 w_3}{\partial\zeta^2}=\nu'\frac{\partial w_2}{\partial\zeta}-\nu'\zeta\frac{\partial w_1}{\partial\zeta}+\beta\theta_3-u_2\frac{\partial w_1}{\partial\zeta}-u_1\frac{\partial w_2}{\partial\zeta}$$

(17) $$\frac{\partial u_2}{\partial t}-\nu'\frac{\partial^2 u_2}{\partial\zeta^2}=\nu'\frac{\partial u_1}{\partial\zeta}-u_1\frac{\partial u_1}{\partial\zeta}-w_1\frac{\partial u_1}{\partial z}$$

5. Construction of the solution

The solutions of (10) and (11), satisfying the conditions

$$\theta_1=t^\alpha,\quad w_1=t^{\alpha+1}\quad \text{at}\quad r=1\quad \text{and}\quad t\geq 0$$

$$\theta_1\to 0,\quad w_1\to 0\quad \text{as}\quad r\to\infty\quad \text{and}\quad t\geq 0$$

are:

(18) $$\theta_1 = 2^{2a}\Gamma(a+1)g_a(r_1)t^a$$

(19) $$w_1 = [(1-\beta)2^{2a+2}\Gamma(a+2)g_{a+1}(r_1) + \beta 2^{2a}\Gamma(a+1)g_a(r_1)]t^{a+1}$$

where

(20) $$g_a(r_1) = \frac{2}{\sqrt{\pi}\Gamma(2a+1)}\int_{r_1}^{\infty}(\gamma - r_1)^{2a}e^{-\gamma^2}d\gamma, \quad r_1 = \frac{2}{2\sqrt{\nu' t}}\cdot$$

It is easy to find that the solution of the equations (14) and (17) satisfying the conditions (5—7) are $u_1 = 0$ and $u_2 = 0$.

In order to solve equations (12) and (13) we substitute (18) and (19) and obtain

(21) $$\frac{\partial\theta_2}{\partial t} - \nu'\frac{\partial^2\theta_2}{\partial\zeta^2} = -2^{2a-1}\sqrt{\nu'}\Gamma(a+1)g_{a-\frac{1}{2}}(r_1)t^{a-\frac{1}{2}}$$

(22) $$\frac{\partial w_2}{\partial t} - \nu'\frac{\partial^2 w_2}{\partial\zeta^2} = \left\{\left[\beta\sqrt{\nu'}2^{2a+1}\frac{\Gamma(a+1)\Gamma\left(a+\frac{3}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)} - \sqrt{\nu'}(1-\beta)2^{2a+1}\Gamma(a+2)\right]g_{a+\frac{1}{2}}\right.$$
$$\left. - 2\beta\sqrt{\nu'}2^{2a-1}\Gamma(a+1)g_{a-\frac{1}{2}}(r_1)\right\}t^{a+\frac{1}{2}}$$

Using conditions $\theta_2(0) = w_2(0) = 0$ and $\theta_2(\infty) = w(\infty) = 0$ we find

(23) $$\theta_2 = \left[2^{2a+1}\sqrt{\nu'}\frac{\Gamma(a+1)\Gamma\left(a+\frac{3}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)}g_{a+\frac{1}{2}}(r_1) - 2^{2a-1}\sqrt{\nu'}\Gamma(a+1)g_{a-\frac{1}{2}}(r_1)\right]t^{a+\frac{1}{2}}$$

(24) $$w_2 = [Cg_{a+\frac{3}{2}}(r_1) + Mg_{a+\frac{1}{2}}(r_1) + Ng_{a-\frac{1}{2}}(r_1)g_{a-\frac{1}{2}}(r_1)]t^{a+\frac{3}{2}}$$

where

$$C = -\beta\sqrt{\nu'}2^{2a+3}\frac{\Gamma(a+1)\Gamma\left(a+\frac{5}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)} + \sqrt{\nu'}(1-\beta)2^{2a+3}\frac{\Gamma(a+2)\Gamma\left(a+\frac{5}{2}\right)}{\Gamma\left(a+\frac{3}{2}\right)} +$$
$$+\beta\sqrt{\nu'}2^{2a+3}\frac{\Gamma(a+1)\Gamma\left(a+\frac{5}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)}; \quad N = -\beta\sqrt{\nu'}2^{2a-1}\Gamma(a+1);$$

$$M = \beta\sqrt{\nu'}2^{2a+1}\frac{\Gamma(a+1)\Gamma\left(a+\frac{3}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)} - \sqrt{\nu'}(1-\beta)2^{2a+1}\Gamma(a+2).$$

Substituting (18—20) and (23) into (15) we obtain the following equation for θ_3:

$$\frac{\partial^2\theta_3}{\partial r_1^2}+2r_1\frac{\partial\theta_3}{\partial r_1}-4t\frac{\partial\theta_3}{\partial t}=-4\left\{3\,.\,2^{2a-2}\nu'\Gamma(a+1)g_{a-1}(r_1)\right.$$

(25)

$$\left.-\left[2^{2a}\nu'\frac{\Gamma(a+1)\Gamma\left(a+\frac{3}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)}+2^{2a+1}a\nu'\Gamma(a+1)\right]g_a(r_1)\right\}t^{a+1}.$$

The solution of this equation that subjects the boundary conditions $\theta_3(0)=$ $=\theta_3(\infty)=0$ is

$$\theta_3=[Ag_{a+1}(r_1)+C_1g_{a-1}(r_1)+C_2g_a(r_1)]t^{a+1}$$

where

$$A=-\frac{3\,.\,2^{2a+1}\nu'\Gamma(a+1)\Gamma(a+2)}{\Gamma(a)}+\frac{2^{2a+2}\nu'\Gamma(a+1)\Gamma\left(a+\frac{3}{2}\right)\Gamma(a+2)}{\Gamma\left(a+\frac{1}{2}\right)}+2^{2a+3}\nu'a\Gamma(a+1)\Gamma(a+2)$$

$$C_1=3\,.\,2^{2a-3}\nu'\Gamma(a+1)$$

$$C_2=-2^{2a}\nu'\frac{\Gamma(a+1)\Gamma\left(a+\frac{3}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)}-2^{2a+1}a\nu'\Gamma(a+1).$$

For w_3 we have the equation

$$\frac{\partial^2 w_3}{\partial r_1^2}+2r_1\frac{\partial w_3}{\partial r_1}-4t\frac{\partial w_3}{\partial t}=-4[Pg_{a+1}(r_1)+Qg_{a-1}(r_1)+Rg_a(r_1)]t^{a+2}$$

where $$P=A-\nu'(1-\beta)2^{2a+3}a\Gamma(a+2)-\frac{C}{2\sqrt{\nu'}}$$

$$Q=C_1+\beta\nu'2^{2a-1}\Gamma(a+1)-\frac{N}{2\sqrt{\nu'}}$$

$$R=C_2+\nu'(1-\beta)2^{2a+1}\Gamma(a+2)-\beta\nu'2^{2a}\Gamma(a+1)-\frac{M}{2\sqrt{\nu'}}\cdot$$

Hence

$$w_3=\left[Bg_{a+2}(r_1)+Pg_{a+1}(r_1)+\frac{Q}{3}\cdot g_{a-1}(r_1)+\frac{R}{2}\cdot g_a(r_1)\right]t^{a+2}$$

where

$$B=-\frac{4P\,.\,\Gamma(a+3)}{\Gamma(a+2)}-\frac{64Q\,.\,\Gamma(a+3)}{3\Gamma(a)}-\frac{8R\,.\,\Gamma(a+3)}{\Gamma(a+1)}\cdot$$

Now, we can check that w_3 satisfies the conditions $w_3(0)=w_3(\infty)=0$. Fourier law gives an expression for local heat transfer

$$q=-k\left(\frac{\partial T}{\partial r'}\right)_{r'=a}$$

where k is the thermal conductivity. The local Nusselt number is given by

$$N_u = -\frac{a}{AT_0}\left(\frac{\partial T}{\partial r'}\right)_{r'=a} = -\left(\frac{\partial \theta}{\partial r}\right)_{r=1}$$

After some algebra we obtain

$$\left(\frac{\partial \theta}{\partial r}\right)_{r=1} = \sqrt{\mathrm{Re}S}\,\frac{\Gamma(a+1)}{\Gamma\left(a+\frac{1}{2}\right)}\,t^{a-\frac{1}{2}} + \left[a+1-\left(a+\frac{1}{2}\right)\Gamma\left(a+\frac{1}{2}\right)\right]t^{a} - \frac{1}{\sqrt{\mathrm{Re}S}}\cdot\frac{L}{\nu'}\cdot t^{a+\frac{1}{2}}$$

where

$$\frac{L}{\nu'} = \frac{A}{2^{2a+2}\Gamma\left(a+\frac{3}{2}\right)} + \frac{C_1}{2^{2a-2}\Gamma\left(a-\frac{1}{2}\right)} + \frac{C_2}{2^{2a}\Gamma\left(a+\frac{1}{2}\right)}.$$

The method of regular asymptotic expansion is systematic and we shall not go into higher orders in this paper.

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