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Problems of Optimal Design of Axially Symmetrical Shells in Membrane State

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1. Introduction

Elimination of the bending state in shells, if it is possible with respect to mode of loading and support, constitutes a certain step towards optimization. Since it leads to the uniform stress distribution in any cross-section of a shell and under the additional condition of equalized reduced stress at every point of middle surface, to the shell of uniform strength (fully stressed shell). A detailed survey on shell optimization is given by M. Zyczkowski [5].

The first papers assumed the simplest form of equalization of stresses

(1) $$\sigma_{\Phi}=\sigma_{\Theta}=\sigma_0,$$

which is called here "the condition of uniform strength in the narrower sense". In many cases the use of the condition (1) is an unjustified simplification of the problem and this condition should be replaced by the more general

(2) $$\sigma_{\text{red}}=f(\sigma_{ij})=\sigma_0,$$

where σ_{red} denotes reduced stress according to assumed hypothesis, σ_0 — the allowable stress; (2) is called here "the condition of the uniform strength in the broader sense". It should be noted that whereas the condition (1) and two equilibrium equations determine uniquely for example an axially symmetrical shell then the broader condition (2) is an insufficient condition of optimality because it leaves one function "free". The remaining "degree of freedom" permits to apply an additional variational optimization.

M. Milanković [3] was the first (in 1908) who designs optimal axially symmetrical shell in the membrane state loaded by the own weight and hydrostatic pressure. He applied the condition (1). Drop-like shells, spherical shells [2] were designed using the same condition. On the other hand, the condition of the type (2), namely the Tresca-Guest condition was applied by H. Ziegler [4]. He looked for the optimal change of wall thickness of the spherical shell loaded by its own weight. P. Csonka [1] determined the optimal wall thickness of the hyperboloid shaped shell.

In the present paper the more general problems are discussed, namely two problems of minimum weight optimization of axially symmetrical shells loaded by body forces (own weight, rotation) and external loading (pressure) under the broader condition (2). The first deals with the optimal change of wall thickness and the optimal form of the middle surface which assure the membrane state of the shell under arbitrary axially symmetrical loading. The second one is concerned with choosing of external loadings which transform the optimal shell of the uniform strength in broader sense into the shell of equal stresses, $\sigma_\Phi = \sigma_\Theta = \sigma_0$.

2. Formulation of the problem

Consider an axially symmetrical shell loaded by internal pressure $p(r)$, own weight and rotating with a constant angular velocity ω. The shell is supported at a circular rim of a given radius "a", Fig. 1.

We look for the optimal change of the wall thickness and the optimal form of the middle surface which assure the membrane state of the shell. In the problem mentioned as the second one we additionally look for such distribution of pressure $p(r)$ which transforms the shell of the uniform strength in the broader sense into the optimal shell of equal stresses. The design objective is the minimal volume of the shell.

We formulate the problem of optimization of the shell as the classical problem of calculus of variations. Thus, we are looking for the minimal value of the functional — dimensionless volume of the shell

$$G_0 = \frac{V}{2\pi a^3} = v = \int_0^1 \frac{\varrho h}{\cos \Phi}\, d\varrho = \min \tag{3}$$

with subsidiary conditions in the form of the two equilibrium equations

$$G_1 = h[s_\Phi \varrho \dot{\Phi} \cos \Phi + s_\Theta \sin \Phi - \varrho(\bar{\gamma} \cos \Phi + w \varrho \sin \Phi)] - \varrho \bar{p}(\varrho) = 0, \tag{4}$$

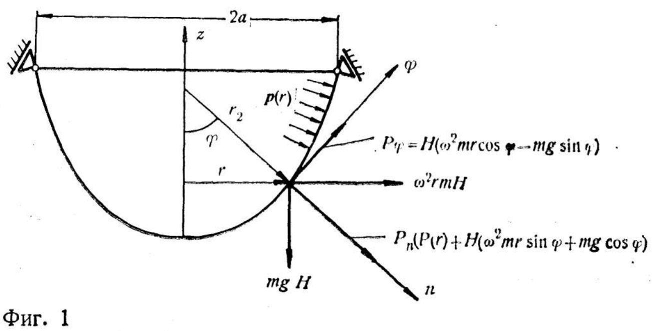
$$G_2 = h(s_\Phi - s_\Theta) + \varrho(\dot{h} s_\Phi + h \dot{s}_\Phi) + h\varrho(w\varrho - \bar{\gamma} \operatorname{tg} \Phi) = 0, \tag{5}$$

where $(\dot{\ }) = \frac{d}{d\varrho}$,

and with the HMH condition of uniform strength

$$G_3 = s_\Phi^2 + s_\Theta^2 - s_\Phi s_\Theta - 1 = 0, \tag{6}$$

where the following dimensionless quantities $\varrho = \frac{v}{a}$, $h = \frac{H}{a}$, $\bar{p}(\varrho) = \frac{p(r)}{\sigma_0}$, $\bar{\gamma} = \frac{mga}{\sigma_0}$, $w = \frac{\omega^2 m a^2}{\sigma_0}$, $s_\Phi = \frac{\sigma_\Phi}{\sigma_0}$, $s_\Theta = \frac{\sigma_\Theta}{\sigma_0}$ denote independent variable, wall thickness, pres-



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sure, specific weight of the shell material, angular velocity squared, meridional and circumferential stresses respectively.

The condition (6) can be replaced by other formulations. For example, the assumption of the uniform strength in the form $\sigma_{red} = \max(s_\Phi, s_\Theta) = 1$ leads to optimal shells with respect to the time of brittle creep rupture according to the Kochanov theory.

The equations (3), (4), (5) were derived in the more convenient (ϱ, Φ) coordinate system which allows, in comparison with the usual (ϱ, z) coordinate system $(\dot{z} = \operatorname{tg} \Phi)$, to reduce the second order equations to the equations of the first order.

In order to solve the problems we shall apply the Lagrangian multiplier method. Thus, the new functional can be written in the form

$$J^* = \int_0^1 \left(G_0 + \sum_{i=1}^{3} \lambda_i G_i \right) d\varrho = \int_0^1 F^* d\varrho, \tag{7}$$

where $\lambda_1(\varrho)$ — Lagrangian multipliers, and the appropriate Euler — Lagrange equations have the form

$$\frac{\partial F^*}{\partial y_i} - \frac{d}{d\varrho} \frac{\partial F^*}{\partial \dot{y}_i} = C_i, \tag{8}$$

where $y_1 = \Phi, h, s_\Phi, s_\Theta$ and $C_1 = 0$ for h, s_Φ, s_Θ, whereas C_i may be different from zero for $y_i = \Phi$ since Φ depends on the derivative of z and then (8) should be regarded as the first integral. However, it can be shown (using suitable transversality condition) that for the considered functional even this constant C_i is equal to zero hence all $C_i = 0$.

After application of equations (8) to (7) and elimination λ_3 we obtain the following system of equations:

$$h\varrho \sin \Phi + \lambda_1 \{ h[(s_\Theta - s_\Phi) \cos \Phi - \varrho(w\varrho \cos \Phi - \bar{\gamma} \sin \Phi)] - (\dot{s}_\Phi h + s_\Phi \dot{h}) \varrho \cos \Phi \} \cos^2 \Phi - \dot{\lambda}_1 h \varrho s_\Phi \cos^3 \Phi - \lambda_2 \bar{\gamma} h \varrho = 0, \tag{9}$$

$$\frac{\varrho}{\cos \Phi} + \lambda_1 [s_\Phi \varrho \dot{\Phi} \cos \Phi + s_\Theta \sin \Phi - \varrho(\bar{\gamma} \cos \Phi + w\varrho \sin \Phi)] + \lambda_2 [\varrho(w\varrho - \bar{\gamma} \operatorname{tg} \Phi) - s_\Theta] - \dot{\lambda}_2 \varrho s_\Phi = 0, \tag{10}$$

$$\lambda_1 [\varrho \dot{\Phi} \cos \Phi (2s_\Theta - s_\Phi) - \sin \Phi (2s_\Phi - s_\Theta)] + \lambda_2 (2s_\Phi - s_\Theta) - \dot{\lambda}_2 \varrho (2s_\Phi - s_\Theta) = 0, \tag{11}$$

which are the basic relations in both cases under consideration.

3. Shell of the uniform strength in the broader sense

For a given value and distribution of the pressure $\bar{p}(\varrho)$ the equations (9), (10), (11) together with the equilibrium equations (4), (5) and the condition of the uniform strength (6) constitute the sufficient system which evaluates the unknown functions of the shape: $\Phi = \Phi(\varrho)$, $h = h(\varrho)$; distribution of the stresses $s_\Phi = s_\Phi(\varrho)$, $s_\Theta = s_\Theta(\varrho)$ and $\lambda_1(\varrho)$, $\lambda_2(\varrho)$. The shell defined in this way will be the structure of the uniform strength in the broader sense which has the least weight.

The simplest example of optimization in this way refers to the shell subjected only to the constant internal pressure, $\bar{p}(\varrho)=p_0$ $(w=\bar{\gamma}=0)$. Then the derived system of equations can be reduced to one differential equation of the second order

$$\ddot{x}=\frac{2}{3}\frac{3x^2+\varrho^2\dot{x}^2-3\varrho x\dot{x}}{\varrho^2 x^2(1-x^2)}(3x^3-2\varrho\dot{x}-\varrho\dot{x}x^2+3x)+\frac{\dot{x}(2\varrho\dot{x}-3x)(3x-\varrho\dot{x})}{3\varrho x^2}, \tag{12}$$

where $x=\sin\Phi$. The relation $\sin\Phi=A\varrho$ which describes the shape of a sphere does not satisfy the equation (12) and hence it turns out that a spherical shell is not the optimal structure.

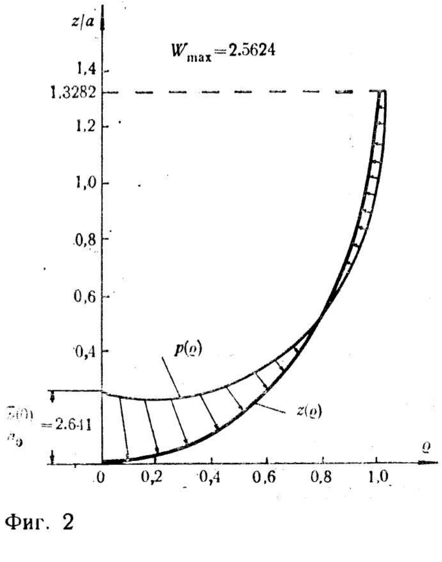
The equation (12) shows singularity at the point $\varrho=0$. To investigate this singularity we expand the function $x=x(\varrho)$ into a power series in the vicinity of $\varrho=0$. It turns out, that a simple ordinary expansion holds

$$x=\sum_{i=1,3,\dots}^{\infty} a_i\varrho^i. \tag{13}$$

Substituting (13) into (12) we can express the coefficients a_i, $i=3,5,\dots$ in terms of the coefficient a_1, which remains arbitrary (subject to optimization). The procedure of integration was as follows. For $0\le\varrho\le 0.01$ the series (13) was used. Then (12) was integrated numerically, using the Runge-Kutta method for various values of the parameter a_1. Finally a_1 was evaluated from the optimality condition $V=\min$.

4. Shell of uniform strength in the narrower sense as the optimal shell

As it was mentioned previously the shell of uniform strength in the narrower sense is determined uniquely from the equilibrium equations and the strength criterion but, in general, they are not optimal. On the other hand, the condition of the uniform strength in the broader sense is insufficient condition of optimality and should be completed by the Euler-Lagrange equation.



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An interesting theoretical problem is to find when an optimal shell reduces to a shell of uniform strength in narrower sense (1). This reduction takes places for a certain external loading; hence we are going now to evaluate the corresponding function $\bar{p}(\varrho)$. In this case $\bar{p}(\varrho)$ is an additional unknown function but the distribution of the stresses are assumed in the form

$$s_\Phi=1,\quad s_\Theta=1. \tag{14}$$

Substituting (14) into (4), (5), (9), (10), (11) we obtain the system of equations which determine unknown functions: $\Phi(\varrho)$, $h(\varrho)$, $\bar{p}(\varrho)$ and $\lambda_1(\varrho)$, $\lambda_2(\varrho)$.

After elimination of λ_1 and λ_2 we obtain the differential equation of the second order with respect to Φ which determines the middle surface of the optimal shell in this particular case. In view of the fact that equation obtained is very long, it will not be quoted here in its general form.

As an example we consider the rotating shell ($w \neq 0$, $\bar{\gamma}=0$), where its own weight is neglected. In this case the optimal shape is described by

$$\dot{\Phi}=\frac{w\varrho^2-2}{\varrho}\operatorname{tg}\Phi+2\frac{\varrho w\cos\Phi}{\sin\Phi} \tag{15}$$

and the change of wall thickness is given by

$$h=h_0\exp\left(-\frac{w\varrho^2}{2}\right), \tag{16}$$

where h_0 — wall thickness for $\varrho=0$. The relation which determines of pressure $\bar{p}(\varrho)$ has the form

$$\frac{\bar{p}(\varrho)}{h_0}=\exp\left(-\frac{w\varrho^2}{2}\right)\left(2w\frac{\varrho\cos^2\Phi}{\sin\Phi}-\frac{\sin\Phi}{\varrho}\right). \tag{17}$$

The equation (15) shows singularity at the point $\varrho=0$. To investigate this singularity we expand the function $\Phi=\Phi(\varrho)$ into power series in the vicinity of $\varrho=0$

$$\Phi=\sum_{i=1,3,5,\ldots}^{\infty} c_i\varrho^i. \tag{18}$$

The equation (15) was integrated numerically using the Runge-Kutta method. It turns out that angular velocity w cannot be arbitrarily large. The maximal value of w was evaluated from the condition

$$\Phi(1)=\pi/2 \tag{19}$$

and equals $w_{\max}=2.5624$.

The optimal shape of the shell and distribution of pressure $\bar{p}(\varrho)$ are given in Fig 2.

References

1. Csonka, P. Acta Techn. Acad. Sci. Hung., **44** (1963), 1—2,215—221.
2. Flügge, W. Stresses in shells, Springer-Verlag, New York Inc., 1967.
3. Milanković, M. Rad. Jugoslovenske Akademije, Zagreb, **175** (1908), 140—152.
4. Ziegler, H. Ing. — Archiv., **26** (1958), 5, 378—382.
5. Zyczkowski, M. IPPT PAN—PWN, Warszawa, 1977.