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On the Second Law of Thermodynamics for a Singular Surface*

P. Marinov

1. Introduction

In the recent years considerable attention has been given to thermodynamic properties of singular surfaces in continuous medium. Green and Naghdy [1] developed a rather general thermodynamic theory in which surface sources were involved. Recently Wilmanski [3, 4] presented a new form of Kotchene's condition in which the surface sources were taken into account. This form of Kotchine's condition was used to obtain Clausius-Duhem inequality for a singular surface.

In the present paper a somewhat more different point of view is adopted, although the approach taken is similar to that of the earlier papers [3, 4]. Using the formulation on Meixner's theory wherein the local Clausius-Duhem inequality is replaced by the so-called fundamental inequality a new local thermodynamic inequality for a singular surface is obtained.

The definition of singular surface, the generalized balance equation for a continuous medium and Kotchine's condition are given in the second section. In the third section we derive a new local thermodynamic inequality for a singular surface. The fourth section is devoted to the particular case of the material at rest before being reched by a singular surface.

2. Preliminaries

In [4] the following generalized form of balance equation is given

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{v} \varphi \, \mathrm{d}v = \oint_{\mathrm{d}v} (\mu+\mu^{*}) ds + \int_{v} (\gamma+\lambda^{*}) dv, \tag{2.1}$$

where the surface sources are denoted with star.

The eq. (2.1) is assumed to hold for any subvolume v and for any field quantity φ.

Definition of Singular Surface. Let a surface σ, oriented by a unit vector **n**, be a part of the region $\partial v^{-} \cap \partial v^{+}$, Fig. 1. Let the field φ, described by the balance equation (2.1), is smooth in v^{-} and v^{+}, i. e. it is continuously di-

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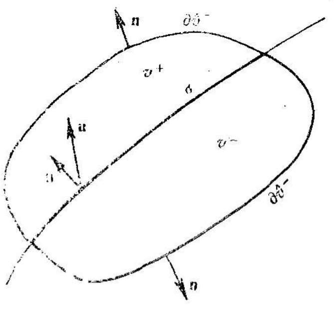

Fig. 1

fferentiable in v^- and v^+, it approaches limits φ^- and φ^+ for every point $\mathbf{x} \in \sigma$, both limits are differentiable on any path on σ, and grad φ approaches limits (grad $\varphi)^-$, (grad $\varphi)^+$ on σ. For the fields λ, λ^* continuous in $v^- \cup v^+$ and μ, μ^*, $\dot{\mathbf{x}}$ approaching limits μ^-, μ^+, μ^{*-}, μ^{*+}, $\dot{\mathbf{x}}^-$, $\dot{\mathbf{x}}^+$ on σ we say, that the surface σ is singular with respect to the balance equation (2.1), if at least one of the listed quantities has a jump on σ, i. e. the limits of this quantity on σ are not equal [4].

We consider a medium, in which a singular surface σ is moving with a velocity $\mathbf{u}(\mathbf{x})$, $\mathbf{x} \in \sigma$, and the balance eq. (2.1) is fulfilled v.

It is deduced in [4]

$$[\varphi(\dot{\mathbf{x}}_n - u_n)] - [\mu] = [\mu^*], \tag{2.2}$$

which is called the generalized Kotchine's condition for a singular surface.

The following form of Clausius-Duhem inequality for a singular surface is obtained [4]

$$[\varrho\eta(\dot{\mathbf{x}}_n - u_n)] - [h] \geq 0 \quad \text{almost everywhere s on } \sigma, \tag{2.3}$$

where: ϱ — mass density, η — entropy density and h — entropy flux.

It is interesting to note that the surface sources of entropy h^* vanish in eq. (2.3). The standard notation of Christoffel is used

$$[\varrho\eta(\dot{x}_n - u_n)] := \varrho^+\eta^+(\dot{x}_n^+ - u_n) - \varrho^-\eta^-(\dot{x}_n^- - u_n)$$

$$[h] := h^+ - h^-.$$

3. A Local Thermodynamic Inequality for a Singular Surface

The base of our investigation is the fundamental inequality given by Meixner [2]. Starting from equilibrium state at $t = -\infty$, we have

$$\int_{-\infty}^{\tau} \{\varrho(\Lambda_{st}(\varepsilon) - V)\dot{\varepsilon} + \nabla\Lambda \cdot \mathbf{q}\}\, \mathrm{d}t \geq 0 \qquad \text{for all } \tau \text{ and } v. \tag{3.1}$$

The following assumptions are introduced:

A 1. There exists a thermodynamic temperature function $\Lambda = \Lambda(\varepsilon, \dot{\varepsilon})$ [6] with the properties

$$\Lambda|_E = \frac{1}{\theta_{st}} \tag{3.2}$$

and an energy flux $\mathbf{q} = -\mathbf{k}(\varepsilon; \dot{\varepsilon})$, where θ_{st} is the Kelvin temperature and ε — an independent variable (energy density),

A 2. There exists only entropy function in equilibrium state

$$\eta_{st}(\varepsilon) := \eta(\varepsilon, 0) \geq \eta(\varepsilon, \dot{\varepsilon}), \tag{3.3}$$

where the relation between $\Lambda_{st}(\varepsilon)$ and $\eta_{st}(\varepsilon)$ is given by

$$\Lambda_{st}(\varepsilon)=\frac{\mathrm{d}\,\eta_{st}(\varepsilon)}{\mathrm{d}\,\varepsilon}. \tag{3.4}$$

Bearing in mind the relation (3.4) and the first law of thermodynamics [2].

$$\dot{\varepsilon}+\nabla\,.\,\mathbf{q}=0$$

we can rewrite the inequality (3.1) in the form

$$\int\limits_{-\infty}^{\tau}\{\varrho\dot{\eta}_{st}(\varepsilon)+\nabla\,.\,(\Lambda\mathbf{q})\}\,\mathrm{d}t\geq 0 \qquad \text{for all } \tau \text{ and } v, \tag{3.1$_1$}$$

or

$$\int\limits_{v}\{\varrho\dot{\eta}_{st}(\varepsilon)+\nabla\,.\,(\Lambda\mathbf{q})\}\,\mathrm{d}\,v\geq 0, \tag{3.1$_2$}$$

since the process is fast.

The local formula of inequality $(3.1)_2$ yields

$$\varrho\dot{\eta}_{st}(\varepsilon)+\nabla\cdot(\Lambda\mathbf{q})\geq 0. \tag{3.5}$$

Comparing eq. $(3.1)_2$ with eq. (2.1) we conclude that

$$\varphi\equiv\varrho\eta_{st}(\varepsilon) \quad \text{and} \quad -(\mu+\mu^{*})\equiv\Lambda\mathbf{q}\,.\,\mathbf{n}, \tag{3.6}$$

ince the vector **q** is pointed inwards.

There are two possibilities to obtain the local thermodynamic inequality for a singular surface.

P 1. We use the generalized Kotchine's condition (2.2), relations (3.6) and the second law of thermodynamics (3.1).

P 2. We apply directly the definition of singular surface to the inequality $(3.1)_2$ and afterwards we take a limit $s(\sigma)\to 0$.

We choose the second possibility. At first we rewrite the inequality $(3.1)_2$ in the form

$$\int\limits_{v}\varrho\dot{\eta}_{st}(\varepsilon)\,\mathrm{d}v+\oint\limits_{\delta\,v}(\Lambda\mathbf{q})\,.\,\mathbf{n}\,\mathrm{d}\,s\geq 0. \tag{3.1$_3$}$$

The first term of eq. $(3.1)_3$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int\limits_{v}\varrho\eta_{st}(\varepsilon)\mathrm{d}\,v=\int\limits_{v^{-}}\varrho\frac{\partial\,\eta_{st}(\varepsilon)}{\partial\,t}\,\mathrm{d}\,v+\int\limits_{v^{+}}\varrho\frac{\partial\,\eta_{st}(\varepsilon)}{\partial\,t}\,\mathrm{d}\,v$$
$$+\oint\limits_{\partial v^{-}}\varrho\eta_{st}(\varepsilon)V_{n}\,\mathrm{d}\,s+\oint\limits_{\partial v^{+}}\varrho\eta_{st}(\varepsilon)V_{n}\,\mathrm{d}\,s, \tag{3.7}$$

where V_n is the normal component of a velocity of an arbitrary point on either ∂v^{-} or ∂v^{+}. If the point is lying on ∂v, then $\mathbf{V}=\dot{\mathbf{x}}$ and $\mathbf{V}_{n}=\dot{\mathbf{x}}$, while for σ we have $\mathbf{V}=\mathbf{u}$. Separating the common part of ∂v^{-} and ∂v^{+}, we obtain from (3.7)

$$\text{(3.8)}\qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{v}\varrho\eta_{st}(\varepsilon)\,\mathrm{d}v=\int_{v^{-}}\varrho\frac{\partial\eta_{st}(\varepsilon)}{\partial t}\,\mathrm{d}v+\int_{v^{+}}\varrho\frac{\partial\eta_{st}(\varepsilon)}{\partial t}\,\mathrm{d}v$$

$$-\int_{\partial\widehat{v}^{-}}\varrho\eta_{st}(\varepsilon)\dot{x}_{n}\,\mathrm{d}\varepsilon+\int_{\partial\widehat{v}^{+}}\varrho\eta_{st}(\varepsilon)\dot{x}_{n}\,\mathrm{d}s+\int_{\sigma}\varrho\eta_{st}^{-}(\varepsilon)u_{n}\,\mathrm{d}s-\int_{\sigma}\varrho\eta_{st}^{+}(\varepsilon)u_{n}\,\mathrm{d}s,$$

where we have taken into account the change of orientation of the surface $v^{+}\cap\sigma$.

The second term of eq. $(3.1)_3$ can be written in the form

$$\text{(3.9)}\qquad \oint_{\partial v}(\Lambda\mathbf{q})\cdot\mathbf{n}\,\mathrm{d}s=-\int_{\partial\widehat{v}^{-}}(\Lambda\mathbf{q})\cdot\mathbf{n}\,\mathrm{d}s+\int_{\partial\widehat{v}^{+}}(\Lambda\mathbf{q})\cdot\mathbf{n}\,\mathrm{d}s$$

$$+\int_{\sigma}(\Lambda\mathbf{q})^{-}\cdot\mathbf{n}\,\mathrm{d}s-\int_{\sigma}(\Lambda\mathbf{q})^{+}\cdot\mathbf{n}\,\mathrm{d}s,$$

where we have taken into account the change of orientation of surface $v^{+}\cap\sigma$ However, the third and fourth terms are zero in relation (3.9) since $\mathbf{q}$ is the energy flux vector through the surface $\partial v=\partial\widehat{v}^{+}\cup\partial\widehat{v}^{-}$.

Substitution of eqs. (3.8) and (3.9) into eq. $(3.1)_3$ yields

$$\text{(3.10)}\qquad \int_{v^{-}}\varrho\frac{\partial\eta_{st}(\varepsilon)}{\partial t}\,\mathrm{d}v+\int_{v^{+}}\varrho\frac{\partial\eta_{st}(\varepsilon)}{\partial t}\,\mathrm{d}v-\int_{\partial\widehat{v}^{-}}\varrho\eta_{st}(\varepsilon)\dot{x}_{n}\,\mathrm{d}s$$

$$+\int_{\partial\widehat{v}^{+}}\varrho\eta_{st}(\varepsilon)\dot{x}_{n}\,\mathrm{d}s+\int_{\sigma}\varrho\eta_{st}^{-}(\varepsilon)u_{n}\,\mathrm{d}s-\int_{\sigma}\varrho\eta_{st}^{+}(\varepsilon)u_{n}\,\mathrm{d}s$$

$$-\int_{\partial\widehat{v}^{-}}(\Lambda\mathbf{q})\cdot\mathbf{n}\,\mathrm{d}s+\int_{\partial\widehat{v}^{+}}(\Lambda\mathbf{q})\cdot\mathbf{n}\,\mathrm{d}s\geq 0.$$

Under our assumptions the inequality (3.10) must hold for any v. Then we are allowed to take a limit $v\to 0$ in eq. (3.10) and we shrink down $\partial\widehat{v}^{-}$ and $\partial\widehat{v}^{+}$ to σ. Hence $v^{-}\to 0$, $v^{+}\to 0$, preserving σ. Bearing in mind the definition of the singular surface, we obtain

$$\text{(3.11)}\qquad -\int_{\sigma}\varrho\eta_{st}^{-}(\varepsilon)\dot{x}_{n}^{-}\,\mathrm{d}s+\int_{\sigma}\varrho\eta_{st}^{+}(\varepsilon)\dot{x}_{n}^{+}\,\mathrm{d}s+\int_{\sigma}\varrho\eta_{st}^{-}(\varepsilon)u_{n}\,\mathrm{d}s$$

$$-\int_{\sigma}\varrho\eta_{st}^{+}(\varepsilon)u_{n}\,\mathrm{d}s-\int_{\sigma}(\Lambda\mathbf{q})^{-}\cdot\mathbf{n}\,\mathrm{d}s+\int_{\sigma}(\Lambda\mathbf{q})^{+}\cdot\mathbf{n}\,\mathrm{d}s\geq 0,$$

where $(\Lambda\mathbf{q})^{-}$ is the limit of $(\Lambda\mathbf{q})$ from v^{-} and $(\Lambda\mathbf{q})^{+}$ — from v^{+}.

Using the standard notation of Christoffel

$$[\varrho\eta_{st}(\varepsilon)(\dot{x}_{n}-u_{n})]:=\varrho\eta_{st}^{+}(\varepsilon)(\dot{x}_{n}^{+}-u_{n})-\varrho\eta_{st}^{-}(\varepsilon)(\dot{x}_{n}^{-}-u_{n}),$$

$$[\Lambda\mathbf{q}]\cdot\mathbf{n}:=\{(\Lambda\mathbf{q})^{+}-(\Lambda\mathbf{q})^{-}\}\cdot\mathbf{n},$$

then the inequality (3.11) can be written in the form

$$\int_{\sigma} \{[\varrho\eta_{st}(\varepsilon)(\dot{x}_n - u_n)] + [\Lambda \mathbf{q} \cdot \mathbf{n}]\}\, \mathrm{d}\, s \geq 0. \tag{3.12}$$

The function Λ may be accepted as a continuous scalar function for all values of ε, $\dot{\varepsilon}$, i. e. $[\Lambda] = 0$. Then the inequality (3.12) yields

$$\int_{\sigma} \{[\varrho\eta_{st}(\varepsilon)(\dot{x}_n - u_n)] + \Lambda[\mathbf{q} \cdot \mathbf{n}]\}\, \mathrm{d}\, s \geq 0. \tag{3.13}$$

Assuming the continuity of the integrand on σ, we can take a limit from the relation (3.13)

$$\lim_{S(\sigma) \to 0} \frac{1}{S(\sigma)} \int_{\sigma} \{[\varrho\eta_{st}(\varepsilon)(\dot{x}_n - u_n)] + \Lambda[\mathbf{q} \cdot \mathbf{n}]\}\, \mathrm{d}\, s \geq 0,$$

where $S(\sigma)$ is the area of surface σ. Hence the relation (3.13) is equivalent to the following local formula

$$[\varrho\eta_{st}(\varepsilon)(\dot{x}_n - u_n)] + \Lambda[\mathbf{q} \cdot \mathbf{n}] \geq 0 \text{ almost everywhere } S \text{ on } \sigma. \tag{3.14}$$

The inequality (3.14) is the main result of our investigation. In comparison with inequality (2.3) it has some distincts:

i. The inequality (3.14) does not contain the entropy function η and entropy flux h,

ii. The inequality (3.14) contains the entropy function in equilibrium state η_{st} and thermodynamic temperature function Λ. Also it contains the energy flux vector $\mathbf{q}$.

4. Material at rest

We consider as [4] a material which has been at rest before being reached by the surface σ, i. e. $\dot{\mathbf{x}} \equiv 0$ at any point of v^+. In this particular case the inequality (3.14) takes the following form

$$[\varrho\eta_{st}(\varepsilon)]u_n + \varrho^- \eta_{st}^-(\varepsilon)\dot{x}_n^- - \Lambda[\mathbf{q} \cdot \mathbf{n}] \leq 0. \tag{4.1}$$

If we assume that $[\mathbf{q}] = 0$ then the following inequality yields

$$[\varrho\eta_{st}(\varepsilon)]u_n + \varrho^- \eta_{st}^-(\varepsilon)\, \dot{x}_n \leq 0. \tag{4.2}$$

Bearing in mind the balance of mass — (6.4) from [4]

$$u_n = \frac{\varrho^-}{[\varrho]}\, \dot{x}_n^-$$

we can write the inequality (4.2) in the form

$$\varrho^-[\eta_{st}(\varepsilon)]\dot{x}_n \geq 0. \tag{4.3}$$

But in equlibrium state $[\eta_{st}(\varepsilon)] > 0$ and therefore the inequality (4.3) takes the form

$$[\varrho]\dot{x}_n^- \geq 0. \tag{4.4}$$

The result (4.4) has been obtained previously by Wilmanski on other basis.

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