БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ
ТЕОРЕТИЧНА И ПРИЛОЖНА МЕХАНИКА
София · 1979, год. X, № 2

BULGARIAN ACADEMY OF SCIENCES
THEORETICAL AND APPLIED MECHANICS
Sofia · 1979, Year X, No. 2

The Equations of Caustics for Plate Problems in Bending by Using Complex Potentials

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Introduction

The method of caustics was developed by Theocaris [1], initially for the estimation of stress intensity factors at crack tips in plane isotropic elastic media under generalized plane stress conditions. The equation of the initial curve of the caustic on the specimen was at first determined by zeroing an appropriate Jacobian determinant [1] expressed in real form. Also the equation of the caustic itself was initially expressed in real form. In reference [2], Theocaris proposed the expression of these equations by using the complex potential Z(z) of Westergaard, so popular in plane elasticity. This function is equal to the double of the complex potential $\Phi(z)$ of Muskhelishvili. The expression of the equations of caustics through the use of these complex potentials permitted the direct and free from undue calculations application of the method of caustics to a series of engineering problems, most of which are reviewed in reference [3].

On the other hand, the method of caustics was recently applied to isotropic elastic thin plate problems [4—6]. In such problems the classical or the Reissner bending theories (the second being more accurate than the first) are generally used [7—9]. In both these methods the deflection w(x, y) of the points of the plate can be expressed in terms of two complex potentials $\Phi(z)$ and $\chi(z)$, exactly as happened with Airy's function in plane elasticity [7—9]. In this paper we will express the equations of the initial curve of the caustic and the caustic itself in terms of these potentials, hoping that this will lead to an easier application of the method of caustics to bent plate problems. For simplicity, the plate will be assumed under pure bending conditions, free from any normal loading distribution.

Derivation of the Equations

In accordance with the developments of references [4—5], the correspondence between the points (x, y) of the plane of a thin isotropic elastic plate and the corresponding points (u, v) of the screen, when illuminating the plate by a light beam, is established by [4, 5]

$$(1)\qquad u=\lambda\left(x+C\frac{\partial w}{\partial x}\right),\quad v=\lambda\left(y+C\frac{\partial w}{\partial y}\right),$$

where $w=w(x, y)$ denotes the deflection of the points of the plate due to bending, λ is the magnification ratio of the optical set-up and C is the overall optical constant of the experiment. Furthermore, in order that a caustic be formed on the screen, it is necessary that [4, 5]

$$J=\frac{\partial(u, v)}{\partial(x, y)}\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}=0. \tag{2}$$

By taking into account eqs. (1), eq. (2) can be written as

$$\begin{vmatrix} 1+C\frac{\partial^2 w}{\partial x^2} & C\frac{\partial^2 w}{\partial x \partial y} \\ C\frac{\partial^2 w}{\partial x \partial y} & 1+C\frac{\partial^2 w}{\partial y^2} \end{vmatrix}=0. \tag{3}$$

The points (x, y) of the plate for which eq. (3) is satisfied (if such points exist) form the initial curve of the caustic on the plate. The corresponding points on the screen form the caustic itself and are easily determined by using eqs. (1).

Now, we take into account the fact that both in the classical and in the more advanced Reissner theory the deflection $w(x, y)$ of the potnts of the plate can be expressed (in the absence of normal loading of the plate) in terms of two complex potentials $\Phi(z)$ and $\chi(z)$ [7—9], where $z=x+iy$, that is

$$w(x, y)=\mathrm{Re}[\bar{z}\varphi(z)+\chi(z)]=\frac{1}{2}[\bar{z}\varphi(z)+z\bar{\varphi}(\bar{z})+\chi(z)+\bar{\chi}(\bar{z})]. \tag{4}$$

By taking further into account that

$$z=x+iy, \quad \bar{z}=x-iy, \quad \frac{\partial}{\partial x}=\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial y}=i\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z}\right) \tag{5}$$

we find that

$$2\frac{\partial w}{\partial z}=\frac{\partial w}{\partial x}-i\frac{\partial w}{\partial y}=\bar{z}\varphi'(z)+\bar{\varphi}(\bar{z})+\chi'(z). \tag{6}$$

Thus eqs. (1) can be written as

$$u+iv=\lambda\{z+C[z\bar{\varphi}'(\bar{z})+\varphi(z)+\bar{\chi}'(\bar{z})]\}. \tag{7}$$

This equation will determine the form of the caustic on the screen after the points $z=x+iy$ of its initial curve on the plate are determined.

Hence, we have to take into account eq. (3). By using eqs. (4, 5), we can easily find after some algebra that the equation of the initial curve of the caustic (3) can be written in complex form as

$$|1+2C\,\mathrm{Re}\,\varphi'(z)|=C\,|\bar{z}\varphi''(z)+\chi''(z)|. \tag{8}$$

Equations (8) and (7) can completely determine the shape of the caustic if such a curve is formed on the screen. Evidently, in several cases eq. (8) may have no roots.

An Application

As an application of eqs. (8) and (7), we consider the problem of an equilateral triangular isotropic elastic plate of height 3c and flexural rigidity D loaded by bending moments of intensity M uniformly distributed along its boundary. The deflection $w(x, y)$ in this case is given by [9]

$$w(x, y) = \frac{M}{12cD}(x^3 - 3cx^2 - 3xy^2 - 3cy^2 + 4c^3), \quad M > 0. \tag{9}$$

By inserting this expression into eqs. (1) and (2), we find

$$\begin{aligned} u &= \lambda\left[x + \frac{CM}{4cD}(x^2 - 2cx - y^2)\right], \\ v &= \lambda\left[y - \frac{CM}{2cD}(x + c)y\right], \end{aligned} \tag{10}$$

and

$$\left|1 - \frac{CM}{2D}\right| = \frac{|C|M}{2D}\frac{r}{c}, \quad r = (x^2 + y^2)^{\frac{1}{2}}. \tag{11}$$

From eq. (11) it is clear that the initial curve of the caustic on the plate is a circle of radius

$$r = c\left|\frac{2D}{CM} - 1\right|. \tag{12}$$

The same results can be obtained more easily if eqs. (8) and (7) are used together with the expressions

$$\varphi(z) = -\frac{Mz}{4D}, \qquad \chi(z) = \frac{M}{12D}\left(\frac{z^3}{c} + 4c^2\right) \tag{13}$$

for the complex potentials $\varphi(z)$ and $\chi(z)$ [9].

Some Remarks

The use of complex potentials for the solution of bent plate problems is well-known long ago [7—9]. Besides the theoretical results contained in these references, a series of stress concentration around holes, where caustics are generally created, have been treated by this method (by assuming the validity of the classical theory of bending of thin plates) in ref. [9]. Recently, Tamate [10] has considered crack problems in the theory of bending of thin plates on the basis of the theory of Reissner and using complex potentials. The results of Tamate are interesting not only because the method of caustics is particularly appropriate for the study of crack problems, but also because he formulated these problems through complex functions by using the concept of dislocation distributions. Such a formulation can be proved quite effective for the treatment of almost all general plate bending problems, similarly to what has happened in plane elasticity problems. The method of caustics seems to be one of the most powerful experimental techniques to be used together with theoretical results (e. g. for the estimation of stress intensity or stress concentration factors).

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Submitted 28 October 1978