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Elastic-plastic torsion problem of a cracked plate

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1. Introduction

Elastic — plastic analysis of near crack tip stress fields is of immediate relevance to the fracture mechanics. Progress has been made in some important two-dimensional problems of tensile loadings opening a crack in a hardening material (Rice and Rosengren, [1]; Hutchinson, [2]) with the aid of an analytical technique, based on asymptotic considerations and the usage of the well known energy line integral J, proposed by Rice [3]. Recently a certain path independent energy integral J_B has been suggested for the class of problems of pure bending of thin cracked plates and succesfully used in the frames of an analogous technique in solving the small scale yielding problem of pure cylindrical bending of an elastic-plastic cracked plate [4].

The present investigation concerns the plastic stress and strain concentration in the vicinity of the tip of a crack in an elastic-plastic plate with a power type hardening law of the material, subjected to pure torsion.

2. Path independent integral

Consider the equilibrium state of a thin plate containing an internal straight crack of finite length subjected to pure bending (or torsion). A Cartesian coordinate system Ox_1x_2 is introduced in the midplane of the plate so that the crack line lies in the Ox_1 axis.

The well known geometrically linear Kirchhoff plate bending theory is applied, i. e. the relations between the curvatures of the plate mean surface $k_{ij}(x_1, x_2)$ and the deflection function of the mean surface $w(x_1, x_2)$ are

$$k_{ij}(x_1, x_2) = -\frac{\partial^2 w(x_1, x_2)}{\partial x_i \partial x_j}. \tag{1}$$

Throughout the article indices i and j are assigned to values 1 and 2 (and 3 if specified explicitly). Index doubling implies, as usually, summation.

The bending moments $M_{ij}(x_1, x_2)$ and the transverse shear forces $N_i(x_1, x_2)$, defined according to Kirchhoff's plate bending theory, satisfy the equations of equilibrium

(2) $$\frac{\partial M_{ij}}{\partial x_j}=N_i, \quad \frac{\partial N_i}{\partial x_i}=0.$$

The material of the plate is isotropic and elastic which implies that the energy density $U(k_{ij})$ excists as a single-valued function in the curvature space and is defined as

(3) $$U(k_{ij})=\int_0^{k_{ij}} M_{mn}\,dk_{mn}.$$

The crack surfaces are traction free.

Let L be a simple curve in the plate midplane connecting two arbitrary points of the opposite surfaces of the crack and encircling the tip of the crack. n_i and t_i are the direction cosines of the outward normal and the tangent unit vectors of the path L. S denotes the arc length of the curve L.

Then as shown by Mihovsky [4] under the above made assumptions the line integral

(4) $$J_B=\int_L U(k_{ij})dx_2-\left[M_{ij}\,n_j\,k_{1i}+N_i\,n_i\,\frac{\partial w}{\partial x_1}+\frac{\partial}{\partial S}\left(M_{ij}\,n_j\,t_i\,\frac{\partial w}{\partial x_1}\right)\right]dS$$

has the same value for all paths L surrounding the crack tip.

Furthermore a detailed proof excists [5] of the validity of the relation

(5) $$J_B=-\frac{\partial P}{\partial L_c},$$

where P is the potential energy of the bended plate and L_c indicates the current crack length.

3. Basic assumptions

The plate under consideration is assumed to be infinite in both directions. The interval $[-L_c, L_c]$ along the Ox_1 axis is assumed to imitate the real crack. A system of external uniformly distributed purely torsional moments is applied to the plate at infinity (as $|x_2|\to\infty$) so that the only nontrivial components of the stress state are the moments $M_{12}(x_1, \pm\infty)=M$, where the constant M represents the linear density of the applied torsional moments.

The material of the plate is elastic-plastic. Let T_e and E_e be the equivalent shear stress and shear strain respectively defined by the deviatoric stress and strain components s_{ij} and e_{ij} as $2T_e^2=s_{ij}s_{ij}$ and $E_e^2=2e_{ij}\,e_{ij}$ (this is the only case in which the indices i and j assume values 1, 2 and 3). Besides being incompressible, the plate material is assumed to harden according to a law of the type used by Hutchinson [2], namely

(6) $$E_e=\frac{2(1+\nu)}{E}\,T_e+AE_e^0\left(\frac{T_e}{T_e^0}\right)^{\frac{1}{m}},$$

where $1/m\geq 1$ is the hardening exponent, A is a small non-dimensional parameter, E is Young's modulus, ν is Poisson's ratio (incompressibility implies $\nu=1/2$) and T_e^0 and E_e^0 are the yield stress and strain in shear, respectively.

Since this study deals only with the near crack tip zone of stress and strain concentration the elastic strains are neglected. This results in the following short form of the hardening law (6)

$$T_e = V E_e^m, \tag{7}$$

where V is a material constant defined as

$$V = T_e^0 (A E_c^0)^{-m}. \tag{8}$$

Note that incompressibility is not expected to be a poor approximation for moderate strain hardening as plastic strains are incompressible.

As long as the resulting near crack tip solution involves proportional flow and unloading is not supposed to occur the deformation theory of plasticity will be applied, i. e. the relations between the moments M_{ij} in the plate and the curvatures k_{ij} of the plate mean surface will be

$$M_{ij} = D k^{m-1} (k_{ij} + \tilde{k} \delta_{ij}), \tag{9}$$

where δ_{ij} is the Cronecker delta and

$$\tilde{k} = k_{ij} \delta_{ij}, \tag{10}$$

$$D = \frac{V h^{m+2}}{2(m+2)}, \tag{11}$$

$$k^2 = k_{11}^2 + k_{22}^2 + k_{12}^2 + k_{11} k_{22}. \tag{12}$$

The quantity h in (11) means the plate thickness.

The present analysis will be restricted to the case of small-scale yielding. In this case a special boundary layer formulation is known to be possible for defining the near crack tip elastic-plastic field (Rice, [6]). This implies that the actual boundary conditions at infinity may be replaced by a boundary layer type requirement for an asymptotic approach to the elastic stress distribution at large distances from the crack tip. It follows from the above and from the applicability of the deformation theory of plasticity (which is in fact a nonlinear elasticity) that the following equation is valid:

$$J_{Be} = J_{Bp}. \tag{13}$$

The quantities J_{Be} and J_{Bp} in (13) are the values of the integral J_B for the paths L_e and L_p of integration chosen in a way such that L_e is entirely in the elastic region at some distance from the crack tip and L_p lies within the near tip plastic zone.

Choosing a circular path of radius r for evaluating the vintegral J_B one is led to the conclusion that the product $M_{ij} k_{ij}$ of moments and curvatures results in an $1/r$ singularity at the crack tip as $r \to 0$. Within the range of this assumption and keeping in mind the possibility of an asymptotic power series expansion of the solution, it is assumed in addition that the following presentation of the moments M_{ij} and the curvatures k_{ij} in the vicinity of the crack tip is valid (as expressed in a polar coordinate system r, θ with its origin at the crack tip)

$$M_{ij} = r^{s_1} f_{ij}^{(1)}(\theta), \quad k_{ij} = r^{s_2} f_{ij}^{(2)}(\theta), \tag{14}$$

where the power exponents s_1 and s_2 and the sets of functions $f_{ij}^{(1)}(\theta)$ and $f_{ij}^{(2)}(\theta)$ are to be determined. Equations (14) actually represent only the dominant terms of the above mentioned expansions. These terms will be considered.

Finally it is assumed that the crack surfaces are not in contact with one another.

It should be noted that all the above made assumptions under which the problem will be considered are quite common in the fracture crack mechanics and their applicability is considered in detail by a number of authors such as Rice [6], Rice and Rosengren [1], Hutchinson [2], Mihovsky [5]. Because of this their analysis will not be discussed in detail in the present paper.

4. Crack tip singularities

With equations (1) and (9) in mind it is easy enough to prove that expressions (14) are nothing else but a result of the deflection function presentation in the form

$$w(r,\theta)=Kr^{s}F(\theta), \tag{15}$$

where K is a certain multiplication constant introduced for convenience. It is quite natural to interpret the right side of eq. (15) as a dominant term of the asymptotic expansion of the deflection function $w(r,\theta)$.

In view of eq. (15) the elastic-plastic torsion problem of the considered cracked plate is reduced to the problem of determining the constants K and s and the function $F(\theta)$. With the function $w(r,\theta)$ already determined the distribution of the moments $M_{ij}(r,\theta)$ and the curvatures $k_{ij}(r,\theta)$ could be obtained on the basis of equations (1) and (9).

Let L_p be a circle of a sufficiently small radius R (with a center at the crack tip), which lies within the region where eq. (15) is valid. Then eq. (4) yields

$$J_{Bp}=R^{1+s_1+s_2}\int_{-\pi}^{\pi}f(\theta)d\theta. \tag{16}$$

where $f(\theta)$ is a certain function depending on $F(\theta)$. According to the path independence of the integral J_B it should be required that

$$1+s_1+s_2=0,$$

which together with eqs (7) and (1) implies that

$$s=\frac{2m+1}{m+1},\quad s_1=-\frac{m}{m+1},\quad s_2=-\frac{1}{m+1}. \tag{17}$$

Inserting the moments $M_{ij}(r,\theta)$ (expressed with the aid of eqs (1), (9) and (15)) into the equations of equilibirum (2) leads to the following fourth-order non-linear differential equation for the function $F(\theta)$:

$$F^{\mathrm{IV}}+\frac{s(s+1}{2}F''+\frac{s-1}{Q+mH^2}\Big\{pH(H'^2+Q'')+p(H^2+Q)'[2H'+(2-s)F'] \\ +\frac{H^2+Q}{2}[(2s^3-4s^2+3s)F+(4-s)F'']+\frac{p(p-1)}{H^2+Q}H(H^2+Q)'^2\Big\}=0, \tag{18}$$

where

$$p=\frac{m-1}{2},\quad H=\frac{1}{s-1}\left[\frac{s(s+1)}{2}F+F''\right],\quad Q=\frac{3s^2}{4}F^2+F'^2. \tag{19}$$

The multiplication constant K is determined from eq. (13), where according to eqs (1), (9) and (15)

$$J_{Bp}=DK^{m+1}(s-1)^{m+1}I \tag{20}$$

with I defined as

$$I=\int_{-\pi}^{\pi}\left\{\frac{2}{m+1}(H^2+Q)^{\frac{m+1}{2}}\cos\theta+\frac{1}{s-1}(sF\cos\theta-F'\sin\theta)\left[(3-2s)\widetilde{M}_r\right.\right.$$
$$\left.\left.-\widetilde{M}_\theta+2\widetilde{M}'_{r\theta}\right]\right\}d\theta \tag{21}$$

and the "moments" $\widetilde{M}_{ij}(\theta)$ defined according to the relations

$$M_{ij}=-\widetilde{M}_{ij}r^{-\frac{m}{m+1}}DK^m(s-1)^m. \tag{22}$$

The value J_{B_e} can be calculated as

$$J_{B_e}=\frac{K_s^2h(1+\nu)}{3E(3+\nu)},$$

or for the case of an incompressible material ($\nu=1/2$)

$$J_{Be}=\frac{hK_s^2}{7E}, \tag{23}$$

where $K_s=6Mh^{-2}\sqrt{\pi L_c}$ is the stress intensity factor for the similar pure elastic problem.

Eq. (18) is solved over the interval $[0,\pi]$. The boundary conditions which are to be imposed must provide a solution of eq. (18) corresponding to a deformation asymmetric to the crack line. This asymmetry requires that both $w(r,\theta)$ and $k_{\theta\theta}(r,\theta)$ should vanish at $\theta=0$, which (according to eqs (1) and (15)) is to say that

$$F(0)=F''(0)=0. \tag{24}$$

It should be noted that the multiplication constant K introduced in eq. (15) allows the value of $F'(0)$ to be chosen arbitrarily, for instance

$$F'(0)=1. \tag{25}$$

The traction free crack surfaces require that the terms M_θ and $N_\theta+\partial M_{r\theta}/\partial r$ should vanish at $\theta=\pi$. It is not difficult to check that these two boundary conditions imply one another in the frames of the assumptions made above. A more detailed analysis (Mihovsky, [5]) shows that it is sufficient to use the condition

$$H(\pi)=0, \tag{26}$$

where H is determined according to (19).

As a result of the above the problem of evaluating the function $F(\theta)$ appears to be a boundary value problem for the fourth-order differential equation (18) with boundary conditions (24), (25) and (26).

5. Numerical computations and results

For the numerical solution of the boundary value problem for the differential eq. (18) Hamming's predictor-corrector method (Ralston, [7]) of integration was used. Since this method is not self starting a fourth-order Runge-Kutta procedure was used for the first three steps of integration, thus providing the starting values for Hamming's method of integration. Because of using the Runge — Kutta method the boundary value problem was reduced to an initial value problem, i. e. starting values of $F'''(0)$ were selected such that together with equations (24) and (25) a shooting procedure should result in

$$H(\pi)=0,$$

thus satisfying equation (26).

The integration in (21) was performed by using a simple trapezoidal rule.

In the course of the numerical computations the quantities $F(\theta)$, $\widetilde{M}_{ij}(\theta)$ and $R(\theta)$ were evaluated for a large set of values m_k of the hardening exponent m $(0<m_k\leq 1)$, where the function $R(\theta)$ was determined as

$$R(\theta)=VJ_{Be}^{-1}h\,[hK(s-1)\,(H^2+Q)^{\frac{1}{2}}]m+1. \tag{27}$$

It is not difficult to prove that if $T_e=T_e^0$ the function $R(\theta)$ determines the shape of the boundary of the near crack tip plastic zone on the plate surfaces.

Fig. 1 shows the dependence of the function $F(\theta)$ on the hardening exponent m of the material of the plate. It can be observed that with the values of the exponent m decreasing, a noticeable smoothing of $F(\theta)$ or of the deflection function $w(r,\theta)$ (see eq. (15)) takes place. This smoothing effect is analogous to the one observed in the case of pure bending of a cracked plate (Mihovsky, [4]). This effect is quite natural and can be explained by the reduction of the plate stiffness due to the plastic stage of the material. It is further noted that the function $F(\theta)$ is symmetric about the point $\theta=\pi/2$ (and respectively $\theta=-\pi/2$) and assumes an almost sine-shaped form for small values of m, which approximately corresponds to the case of an elastic-perfectly-plastic material.

An analogous case of smoothing is also observed in the distribution of the "moments"$\widetilde{M}_{ij}(\theta)$ as shown in Fig. 2*a*, *b*, *c*, *d*. It should be noted that with the values of m decreasing, the stress state exhibits a well pronounced steady approach to the state of stress drawn in Fig. 2*d*. The latter is defined by asymmetric $\widetilde{M}_r(\theta)$ and $\widetilde{M}_{r\theta}(\theta)$ and symmetric $\widetilde{M}_\theta(\theta)$ distribution with respect to $\theta=\pm\pi/2$. Since the values of $\widetilde{M}_r$ and $\widetilde{M}_{r\theta}$ are close to zero in the regions surrounding the axes $\theta=\pm\pi/2$, the stress state in these regions can be interpreted as pure bending. This stress state can be accepted as an approximate solution of the elastic-perfectly-plastic problem, since the further decrease of the values of m does not affect it significantly (computations have been performed for m decreasing to 0.0005).

In Fig. 3 the dependence of the function $R(\theta)$ (eq. (27)) on the hardening exponent m is shown, i. e. the correspondence between the boundary of the plastic zone, appear-

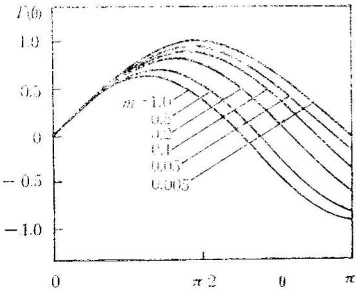


Fig. 1. Function $F(\theta)$ dependence on strain hardening exponent m

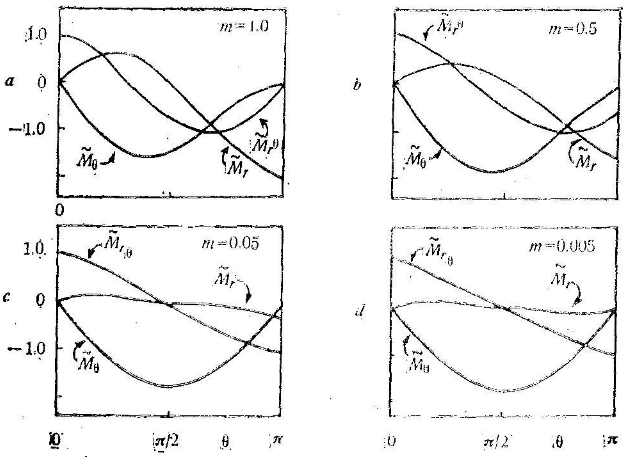
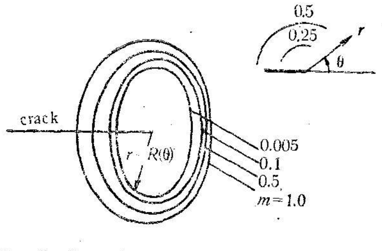


Fig. 2 (*a*), (*b*), (*c*) and (*d*). θ — variation of "moments" $\tilde{M}_{ij}$ at crack tip for different values of strain hardening exponent m

ing at the crack tip and the different values of m is exhibited. It is well observed that the decreasing of m is actually a process, which implies a permanent almost isotropic contraction of the plastic zone. According to the computations the plastic zone is symmetric with respect to $\theta=\pm\pi/2$ for values of m equal to 0.005 and smaller. With this in mind, together with the fact that the plastic zone is also symmetric with respect to the crack line it should be considered that for small values of m the elastic-plastic boundary has the shape of a flattened circle with a centre located at the crack tip and with a ratio between the smallest and the largest values of its radius approximately equal to 0.6. This result could also be usefull when the corresponding elastic-perfectly-plastic problem is solved.



Eig. 3. Location and shape of the near crack tip elastic-plastic boundary

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