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Critical discussion on the stability of the plane and circular Poiseulle flows

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The flows created from the pressure gradient in conduits are the most frequently studied motions of the viscous liquid. The steady streaming in pipes of infinite length and circular cross section is the most famous among them and it is usually refered as Circular Poiseulle Flow (CPF), because Poiseulle [1] suggested the parabolic velocity profile of that flow. Respectively, the steady streaming between two parallel plates which is a limiting case of the flow in rectangular channel adopted the name "Plane Poiseulle Flow" (PPF), because of its parabolic velocity profile similar to that of CPF. Theoretically, the parabolic velocity profile for both flows was confirmed by Stokes [2] who solved the Navier-Stokes equations for the case.

When the turbulence phenomena was recognized as the most important problem of the modern hydrodynamics, the Poiseulle flows were the first to undergo theoretical and experimental examination for stability and transition to turbulence due to their outstanding simplicity. Darsy [3] inverted the attention to the turbulent regimes in the channels, but only Reynolds [4] consistently examined the transition phenomena. He discovered that the type of the motion depends solely on the value of the nondimensional parameter (now called Reynolds number) $Re=dU^m/\nu$, where U^m was the mean-profile velocity, d—diameter of the tube, ν—kinematic coefficient of viscosity. The flow tended to keep the laminar regime when Re was less than 2000, while for $Re>$ 2020 the turbulence took place in his experiments. The result was essentially confirmed by many other experimental works [5, 6, 7].

The first theoretical attempt was again by Reynolds [8] employing an energy method in the case of PPF which was more amenable to theoretical calculations. The critical value obtained was 517 and thus he introduced PPF as a stability case far before it was experimentally investigated in [9]. On the basis of so-called hydraulic-diameter analogy, Reynolds rendered that result to 1034 for CPF, which lied well bellow the experimental one. An improvement of the agreement for energy method was attempted in [10], but instead the gap widened to 117 for PPF and 180 for CPF.

Another theoretical approach took origin from Sommerfeld [11] who derived the boundary value problem for disturbances directly from the Navier-Stokes equations. The predominant part of the succeeding works, among which [12—17], ruled out that PPF is stable to infinitesimal disturbances at all Reynolds numbers. The same conclusion was reached in [18—20] for CPF. Only

Heisenberg [21] suggested instability, but his point gained popularity when Lin [22] clarified some obscure points and calculated the critical Reynolds number of PPF as $Re_{cr}=10\,667$. After that the Lin's result has been many time verified and specified [23—29]. Finally, it could be assumed to be approximately 7600. This value exceeded the experimental one [9] by more than three times and has been till now thought of as an evidency of the brush contradiction between the theory and experiment This conclusion, however, does not appear to be true. The present work is managed to show that the drastic difference is due to the ill-chosen definition of the Reynolds number, employing the mean-profile velocity. The latter is a function of the flow and abruptly changes at the point of transition to turbulence. Picking the Reynolds number, based on the frictional velocity (and therefore on the pressure gradient), one can discover that the difference between the linear theory prediction and the experimental result is less than 23%. In this connection, it is interesting to note, that even the non-linear theories [26—29], which was thought to have brought the comparison significantly closer, in fact predict the critical value in between the same 20%.

It is to be mentioned here that the linear stability of CPF has been recently extended to the case of disturbances without axial symmetry [30, 31]. This has given a strong support to the maveric experimental works [32—35] in which the critical Reynolds number has reached the high of 51 000. Only the theory admitting finite-amplitude disturbances succeeded to prove instability of CPF and to establish as well the dependency of Re_{cr} on the amplitude of the disturbances.

For both CPF and PPF an increase of the amplitude of disturbances appeared to lower the critical value of Re. This raised the question of how low the value could become. In the present work it is suggested that the lower critical value is governed by the principle of least dissipation for the one-dimensional turbulent flows [38].

1. List of the Reynolds number definitions

The principal Reynolds number of present work is the frictional one $Re^*=v^*a/\nu$, based on the frictional velocity $v^*=\sqrt{\tau_W/\varrho}$, where τ_W is the drag force acting on the channel (tube) wall and a—the channel half-width or tube radius. The most important advantage of the frictional velocity is that it depends on the applied pressure gradient, but is not affected by the type of the flow. In the most of the theoretical works, a Reynolds number based on the maximal velocity U^0 and the depth (diameter) $d=2a$ has been used, namely $Re^0=U^0 2a/\nu$. It is obvious that for laminar flows one has $Re^0=Re^{*2}$. At the time, in virtually all of the experimental works the original version [4] has been adopted, i. e. $Re=2aU^m/\nu$, where

$$U^m=\frac{1}{2a}\int_{-a}^{a}\bar{u}_x(y)by$$

is the mean-profile velocity. In the case of laminar flow it is easy to show that $Re=\frac{2}{3}Re^{*2}$ for PPF and $Re=\frac{1}{2}Re^{*2}$ for CPF. In the turbulent case, however, U^m is not known priory and the connection between Re and Re* can be derived only if the frictional coefficient λ is measured. Since by definition $\lambda=2a\left|\frac{\partial p}{\partial x}\right|/0.5\varrho U^m$, then $Re = \frac{8}{\sqrt{\lambda}} Re^*$ for PPF and $Re = \frac{16}{\sqrt{\lambda}} Re^*$ for CPF, respectively. Fortunately, the data concerned with λ has been presented in every experimental work, which has allowed us to connect Re and Re^* and, as though, to render the results in terms of Re^*.

2. The transition of the plane poiseulle flow

It has been widely thought till now that the transition phenomena is concerned with two critical values of Reinolds number called lower and upper critical values. However, the more carefull examination of the existing information requires a revision of this opinion and an introduction of third critical value. The three critical magnitudes can be specified as follows. The lowest value: Re_1^* is the highest Reynolds number for which the resistance coefficient λ rigorously obeys the laminar law. Between Re_1^* and Re_2^* lies the region in which a steady, though slight, departure from the laminar law exists. In the region between Re_2^* and Re_3^* the flow changes dramatically and obtains after Re_3^* the fully developed shape. Originally this was the classification in [9] where the three values were specified as 36.53 ($Re=890$), 48(1400) and 83.36(2400). A similar conclusion was reached in [37] with $Re_2=1300$ and $Re=1800$, but Re_1 was not discerned explicitly.

Each of these three values can be matched with a responsible theoretical value. Thus Re_1^* is the number below which any distortion of the flow decays regardless to its amplitude. For very intensive disturbances, of course, the flow has to be considered only far from the point of disturbing to let the latter to decay. The value Re_1^* can be predicted employing the principle of least dissipation suggested recently in author's work [38]. This principle requires that only the flow with the least total dissipation takes place. Since the total dissipation of the steady mean Poiseulle flow is equal to the total production of turbulence, at the time when the latter is equal to the flux, one can predict the transition value of Reynolds number merely comparing the fluxes of the laminar turbulent flows. At first, such a comparison was done, perhaps, by Lin, 1952 (see for reference [39]) on the basis of a hypothesis that the transition occurs when the dissipation of the laminar flow becomes equal to the projected production of the fully developed turbulent flow at the same frictional Reynolds number Re^*. The obtained value was $Re^*_1=35.7$ (Re=890) which was in good agreement with the value of 36.53 of [9] and 39.2 (1025) of [39].

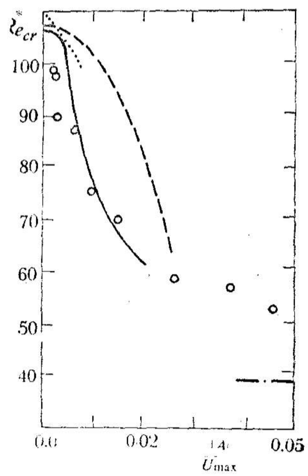

Fig. 1 The critical Reynolds number as a function of the treshold amplitude A: o o o experiments of Karnitz, Potter and Smith (1974); — - — principle of least dissipation; - - - - theory of Reynolds and Potter (1967); . . . . . theory of Pekiris and Scholler (1969); —— theory of Itoh (1974)

Here is to be emphasized that the breaking of the flow with a parabolic profile in the vicinity of Re_1^* does not mean an establishing of a turbulent flow. In fact, there exists a region between Re_1^* and Re_2^* in which the disturbances are not yet able to grow sharply, but are not supressed either. This is

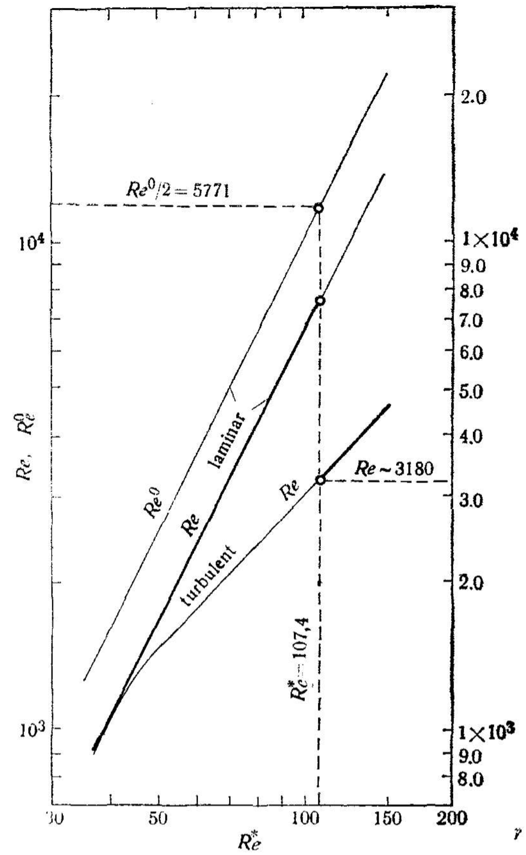

Fig. 2 Connection between the different Reynolds numbers in the region of transition: Re^* is the frictional number; Re^0 is based on the maximal velocity; Re is based on mean-profile velocity

the flow which is solely laminar with certain steady secondary motions provided by the wall roughness superimposed on the main parabolic flow. This flow is laminar and stable, but deviates slightly from the parabolic flow.

The second critical Reynolds number is generally regarded as approximately 48 ($Re=1400$) [9, 37, 40] and most of the authors think of it as the lower critical value in the two-value scheme. Indeed, it is more worth of the name "critical" than Re^*_1, because after passing the value Re^*_2 the instabillity really occurs, though for very high-amplituded disruptions. Generally speaking Re^*_2 must be regarded as the lowest Reynolds number at which a disturbance with an infinite amplitude grows and breaks down the laminar regime. As Re^* increases further even the moderate-tempered disturbances obtain ability to grow. The threshold amplitude decreases with the increasing of Re^* and approaches zero when Re^* reaches Re^*_3. This is shown on fig. 1, where the theoretical results [27, 28, 29] and the experiments [41] are compiled. It should be mentioned that the theoretical curves on fig. 1 have been obtained after taking the minimal Reynolds number at given wave number of the disturbance. Experiments [41] are of outstanding importance, since they prove that reducing the level of disturbances can increase the critical Reynolds number almost up to predicted from the linear theory value $Re_3^*=107.5$. Nishioka et al [42] exceeded even that value and reached $Re_{cr}^*=126.5$, but it was, probably, due to the relative shortness of their channel which had not allowed the disturbances to grow significantly before leaving the channel.

The major feature in the region between Re_2 and Re_3 is the intermittency of turbulent "plugs" and laminar flow in time. This effect was investigated for channels in [37] and in [43] for tubes. The intermittency is a sign that the fully developed turbulent regime has not been yet established. Thus one comes to the physical meaning of Re_3^* which appears to be the limit after which the intermittency disappears leaving a fully developed turbulent flow to be the only realizable regime. This limit has to be nothing but the predicted from the linear theory critical value, i. e. when the smallest disturbances grow and contribute to the fully developed picture. Along with ill choice of Reynolds number based on mean-profile velocity, the intermittency was the second major reason for the apparent disagreement between the theoretical and experimental conclusions. Most of the authors tried to stretch the linear theory fitting the value Re_2^*. Recognizing the value of Re_3^* as the counterpart of the linear theory and rendering it in the right terms of frictional velocity one can find that the agreement of the theory with the experiments (even with the old ones) is surprisingly good.

In the end, fig. 2 shows the general picture of the transition to turbulence of PPF.

3. The circular poiseulle flow.

Although essentially similar to PPF, the CPF offers some differences. There exists no linear instability, on one hand, (theoretically $Re_3^*=\infty$) and the values of Re_1^* and Re_2^* differ from the ones of PPF, on the other hand. Thus for CPF $Re_1^*=41.1$ (845) and $Re_2^*=66.9$ (2020).

Though the infinite magnitude of Re_3^* suggests that the intermittency is extended up to infinity, the conclusion of Rotta [43] that the intermittency ceases after $Re_3^*=100$ (2840) is not really contradictory with the theory, since the intermittency is expected to vanish asymptotically with the increase of Re^* going beyond the accuracy of the experiments. Most of the investigators reached Re=2800 as the frontier of the fully developed turbulent flow. Only Schiller went up to 12 000 [34] Leite [35] — to about 20 000 and Ekman [33] — up to 51 000.

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Literature cited

1. Poiseulle, J. L. M., Compte rend. hebd. des séances de l'Acad. des sciences, 11 (1840) pp. 961—967, 1041—1048.
2. Stokes, Sir G. G., Trans. Cambr. Phil. Soc., 8 (1845) pp. 287—319.
3. Darcy, H., Mem. Acad. Sci., 15 (1858) 141—403.
4. Reynolds, O., Trans. Roy. Soc. London, 174 A (1883) pp. 935—982.
5. Saph, A. V., E. W. Schoder, Trans. Am. Soc. Civil. Eng., 51 (1903) pp. 253—312.

6. G r i e n d l y, J. H., A. H. G i b s o n, Proc. Roy. Soc. London, 80 A (1908) pp. 114—139.
7. S t a n t o n, T. E., J. R. P a n n e l, Trans. Roy. Soc. London, 214 A (1914) pp. 199—224.
8. R e y n o l d s, O. Trans. Roy. Soc. London, 186 A (1886) pp. 123—164.
9. D a v i e s, S. J. and White, C. M., Proc. Roy. Soc. London, 119 A (1928) pp. 92—107.
10. O r r, W. M c F., Proc. Roy. Irish Acad., 27 (1907) 69—138.
11. S o m e r f e l d, A., Atti Congr. Intern. Math. Rome, vol. 3 (1908) pp. 116—124.
12. H o p f, L., Ann. Physik, 4 (1914) pp. 1—60.
13. S o l b e r g, H., Int. Congress Appl. Mech., Delft (1924) pp. 387—394.
14. N o e t h e r, F., ZAMM, 1 (1921) pp. 125—138.
15. G o l d s t e i n, S., Proc. Cambr. Phil. Soc., 32 (1936) pp. 40—54.
16. P e k e r i s, C. L., J. Aero. Sci., 5 (1938) pp. 237—246.
17. S y n g e, J. L., Semicent. Address. Am. Math. Soc., vol. 2 (1938) pp. 227—269.
18. S e x l, T h., Ann. Physik, 83, ser. 4 (1927) pp. 835—848, and 84, ser. 4 (1927) pp. 807—822.
19. P e k e r i s, C. L., Proc. Nat. Acad. Sci. USA, 30 (1944) pp. 316—325.
20. P r e t s c h, J., ZAMM, 21 (1941) pp. 204—217.
21. H e i s e n b e r g, W., Ann. Physik, 74, ser. 4 (1924) pp. 567—627.
22. L i n, C. C., Proc. Nat. Acad. Sci. USA, 30 (1944) pp. 316—325.
23. M e k s y n, D., Proc. Roy. Soc. London, 185 A (1946) pp. 391—409.
24. T h o m a s, L. H., Phys. Rev., 91, ser. 2 (1953) pp. 780—783.
25. S h e n, S. F., J. Aero. Sci., 21 (1954) pp. 62—64.
26. M e k s y n, D., J. T. S t u a r t, Proc. Roy. Soc. London, 208 A (1951) pp. 517—526.
27. P e k e r i s, C. L., B. S c h o l l e r, J. Fluid Mech., 39 (1969) 629—639.
28. R e y n o l d s, W. C., M. C. P o t t e r, J. Fluid Mech., 27 (1967) pp. 465—492.
29. I t o h, N., Trans. Japan Soc. Aero. Space Sci., 17, No. 38 (1974) pp. 160—174.
30. L e s s e n, M., S. G. S a d l e r, T. Y. L i n, Phys. Fluids, 11 (1968) pp. 1404—1409.
31. S a l v e n, H., F. W. C o t t o n, C. H. G r o s c h, J. Fluid Mech., 98 (1979) pp. 273—284.
32. B a r n e s, H. T., E. G. C o k e r, Proc. Roy. Soc. London, 74 A (1905) pp. 341—358.
33. E k m a n, V. W., Ark. Mat. Astronom. Fys., 6, No. 12 (1910) pp. 1—16.
34. S c h i l l e r, L., Forschungsarb. Ing.-Wessen., 248 (1922).
35. L e i t e, R. J. J. Fluid Mech., 5 (1959) pp. 81—96.
36. D a v e y, A., H. P. F. N g u y e n, J. Fluid Mech., 45 (1971) 701—720.
37. P a t e l, V. C., M. R. H e a d, J. Fluid Mech., 38 (1969) pp. 181—201.
38. K h r i s t o v, K h. I., Sov. Phys. Dokl., 24 (4) (1979) pp. 252—253.
39. N a r a y a n, M. A. B., T. N a r a y a n a, ZAMP, 18 (1967) pp. 642—650.
40. K a o, T. W., C. P a r k, J. Fluid Mech., 43 (1970) pp. 145—164.
41. K a r n i t z, M. A., M. C. P o t t e r, M. C. S m i t h, Trans. ASME, J. of Fluid Eng., 96 (1974) pp. 384—388.
42. N i s h i o k a, M., S. I i d a, Y. I c h i k a w a, Fluid Mech., 72 (1975) pp. 731—751.
43. R o t t a, J., Ingenieur-Archiv, 24 (1956) pp. 258—281.