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Effect of Thermal Conductivity on Rayleigh-Taylor Instability of Two Superposed Fluids

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1. Introduction

The problem of the hydromagnetic stability of conducting fluids of variable density is of considerable importance in astrophysics. Lord Rayleigh [1] first initated the study of the hydromagnetic stability of variable density. Having developed the equations for a horizontally stratified, non-rotating, inviscid, incompressible fluid, he discussed two special, density variations corresponding to (a) two superposed fluids and (b) a continuously stratified fluid. Talwar [2] investigated the influence of a horizontal magnetic on the character of the equilibrium of two superposed fluids.

In all above cases, the density of the fluid is kept constant. Theoretical treatments of fluid motion taking density stratification into account are greatly simplified when variations of density are so weak that the Boussinesq approximation can be used. Thus, when g greatly exceeds all the acceleration terms $\left(\frac{\partial u}{\partial t}\text{ etc}\right)$ on the left side of the momentum equation, variations can be neglected in all but the buoyancy term pg. Differential heating produces temperature variations from place to place in the fluid and owing to thermal expansion they give rise to density variations. If the heated incompressible fluid has a volume coefficient of thermal expansion, the equation of state is

$$p=p_0(1-\Theta T), \tag{1}$$

where p_0 is the density at $T=0$ and a heat transfer equation such as

$$\frac{DT}{Dt}=K\nabla^2 T+q, \tag{2}$$

where K is the coefficient of thermal conductivity and q is the rate of internal heating per unit volume. When $q=0$, so that the change in density of a moving fluid elements is due solely to thermal induction, we have by equations (1) and (2)

$$\frac{Dp}{Dt}=K\nabla^2 p.$$

It is, therefore, the motivation of this study to examine the effects of thermal conductivity on the stability of the plane interface separating two incompressible superposed fluids of uniform densities.

2. Perturbation Equations

We consider the motion of an incompressible infinitely conducting fluid in the presence of a uniform horizontal magnetic field. Retaining only the first order terms in perturbation quantities, the relevant linearized perturbation equations in the presence of a uniform gravitational force $g=(0,\ 0,\ -g)$ are then

(3) $$\rho \frac{\partial u}{\partial t} = -\nabla \delta p + g\delta\rho + \frac{1}{4\pi}(\nabla \times \mathbf{h}) \times \mathbf{H},$$

(4) $$\left[\frac{\partial}{\partial t} - K(D^2-k^2)\right]\delta\rho + (\mathbf{u}\,.\,\nabla)\rho = 0,$$

(5) $$\frac{\partial \mathbf{h}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{H}),$$

(6) $$\nabla\,.\,\mathbf{u} = 0,$$

and

(7) $$\nabla\,.\,\mathbf{h} = 0,$$

where $\mathbf{u}(u,\ v,\ w)$, $\mathbf{h}(h_x,\ h_y,\ h_z)$, $\delta\rho$ and δp are the perturbations, respectively in velocity, magnetic field, density ρ and pressure p. K is the coefficient of thermal conductivity. In this investigation, we have considered the change in density of a moving fluid element is due solely to thermal conduction.

Analysing in terms of normal modes we assume that the perturbed quantities have space $(x,\ y,\ z)$ and time (t) dependence of the form

(8) $$f(x,\ y,\ z,\ t) = f(z)\exp(ik_x x + ik_y y + nt),$$

where $f(z)$ is some function of z; n is the growth rate of the harmonic disturbance and k_x, k_y are the horizontal wave number $(k^2 = k_x^2 + k_y^2)$. For solutions having this dependence x, y and t equations (3)—(7) become

(9) $$\rho n u - \frac{\mathbf{H}}{4\pi}(Dh_x - ik_x h_z) = -ik_x \delta p,$$

(10) $$\rho n v - \frac{\mathbf{H}}{4\pi}(Dh_y - ik_y h_z) = -ik_y \delta p,$$

(11) $$\rho n w = -D\delta p - g\delta\rho,$$

(12) $$[n - K(D^2-k^2)]\delta\rho + w(D\rho) = 0,$$

(13) $$nh_z = \mathbf{H} Dw,$$

(14) $$ik_x u + ik_y v + Dw = 0,$$

(15) $$ik_x h_x + ik_y h_y + Dh_z = 0,$$

where D stands for d/dz.

3. Boundary Conditions

If the fluid is confined between two rigid boundaries, say $z=0$ and $z=d$ then the boundary conditions are

(16) $w=0$ and $Dw=0$ on the boundary surfaces.

It is also necessary to specify which quantities should be continuous and which bounded at an interface between two fluids. The first condition is that the normal component of the velocity w must be continuous across an interface between two fluids. The continuity of Dw follows from equation (14) and the continuity of u and v, where the first condition also ensures the vanishing of the horizontal components of the velocity. Thus, the conditions at an interface w and Dw must be continuous. The last of these conditions requires in particular, that must be continuous at an interface.

To satisfy the requirement of continuity of pressure, we obtain the following equation after eliminating δp between equations (9)—(15)

$$(17) \qquad [n-K(D^2-k^2)]\left[D(\rho Dw)+\frac{k_x^2H^2}{4\pi n^2}(D^2-k^2)w-\rho wk^2\right]=\frac{gk^2}{n^2}(D\rho)w.$$

Integration of (17) across the interface gives in view of continuity of w

$$(18) \qquad (n+Kk^2)\left[\Delta_s(\rho Dw)+\frac{k_x^2H^2}{4\pi n^2}\Delta_s(Dw)\right]-K\left[\Delta_s D^2(\rho Dw)+\frac{k_x^2H^2}{4\pi n^2}\Delta_s(D^2-k^2)Dw-\rho k^2\Delta_s(Dw)\right]=-\frac{gk^2}{n^2}\Delta_s(\rho)w_s,$$

where $\Delta_s(\ldots)$ denotes the jump of a quantity experience in crossing the surface of discontinuity and w_s is the common normal component of velocity.

4. Two Superposed Fluids of Uniform Densities

We consider the case of two semi-infinity extending fluids of constant densities ρ_1 and ρ_2 separated by a horizontal boundary $z=0$, the magnetic field being uniform throughout in each region where K is also constant. We rewrite equation (18) in the form

$$(19) \qquad [n-K(D^2-k^2)][D^2-q^2]w=0.$$

The general solution (19) is a linear combination of $e^{\pm kz}$ and $e^{\pm qz}$, where

$$(20) \qquad q=\sqrt{\frac{n}{K}+k^2}.$$

Since w must vanish when $z\to-\infty$ (in the lower fluid) and $z\to+\infty$ (in the upper fluid), we can write the solutions, of the two regions, as

$$(21) \qquad w_1=A_1e^{kz}+B_1e^{qz},$$

$$(22) \qquad w_2=A_2e^{-kz}+B_2e^{-qz},$$

where A_1, A_2, B_1 and B_2 are constants. The above solutions must satisfy certain boundary conditions. These conditions require w, Dw and D^2w must be continuous at an interface between two fluids. A further condition follows from equation (18) by integrating it across the interface, $z=0$, and we obtain

$$(23) \qquad [n-K(D^2-k^2)]\left[\rho_2Dw_2-\rho_1Dw_1+\frac{k_x^2H^2}{4\pi n^2}(Dw_2-Dw_1)\right]=-\frac{gk^2}{n}(\rho_2-\rho_1)w_0.$$

The continuity of w, Dw, D^2w across the interface gives

$$(24) \qquad A_1+B_1=A_2+B_2,$$

$$(25) \qquad kA_1+qB_1=-kA_2-qB_2, \quad \text{and}$$

$$(26) \qquad k^2A_1+q^2B_1=k^2A_2+q^2B_2.$$

The remaining condition (23) now require

$$-\left(\alpha_1+\frac{k_x^2V^2}{n^2}\right)A_1-\left(\alpha_2+\frac{k_x^2V^2}{n^2}\right)A_2+\frac{gk}{2n^2}(A_1+A_2+B_1+B_2)=0, \tag{27}$$

(28) where $V^2=\frac{H^2}{4\pi\rho_0}$ is the Alfven velocity in the medium.

Eliminating the constants A_1, A_2, B_1 and B_2 from (24)—(27), we obtain

$$\begin{vmatrix} 1 & 1 & -1 & 1 \\ k & q & k & q \\ k^2 & q^2 & -k^2 & -q^2 \\ \frac{1}{2}R-\left(\alpha_1+\frac{k_x^2V^2}{n^2}\right)\frac{1}{2}R & \frac{1}{2}R & \frac{1}{2}R-\left(\alpha_2+\frac{k_x^2V^2}{n^2}\right) & \frac{1}{2R} \end{vmatrix}=0, \tag{29}$$

where (30) $\alpha_1=\frac{\rho_1}{\rho_1+\rho_2}$, $\alpha_2=\frac{\rho_2}{\rho_1+\rho_2}$,

and (31) $V^2=\frac{H^2}{4\pi(\rho_1+\rho_2)}$, $R=\frac{gk^2}{n^2}(\alpha_1-\alpha_2)$.

Evaluating the determinent (31), we obtain the following dispersion relation between n and k:

$$\begin{aligned} &n^5+n^4k^2K+2n^3[2k_x^2+gk(\alpha_1-\alpha_2)]+2n^2k^2K[2k_x^2+gk(\alpha_1-\alpha_2)] \\ &+n[2k_x^2+gk(\alpha_1-\alpha_2)]^2+4k^2k_x^2Kv^2[k_x^2V^2+gk(\alpha_1-\alpha_2)]=0. \end{aligned} \tag{32}$$

When $\alpha_1>\alpha_2$, we find by applying Hurwitz' criterion to equation (32), that as all the roots of n are either real and negative or there are complex roots with negative parts. The system is therefore stable in each case. Hence the potentially stable configuration, therefore, remains stable whether the effects of thermal conductivity is included or not.

For the potentially unstable arrangment $\alpha_2>\alpha_1$, the system is unstable in the hydrodynamic case for all wave numbers k. In the present case, again by applying Hurwitz' criterion to equation (32), when $\alpha_2>\alpha_1$, that the system is stable for all wave numbers which satisfy the inequality

$$kV^2\cos\Theta>g(\alpha_2-\alpha_1), \tag{33}$$

where Θ is the angle between k and H.

Also the system, in the present case, is unstable, if

$$kV^2\cos\Theta<g(\alpha_2-\alpha_1). \tag{34}$$

We now examine the behaviour of growth rate with respect to thermal conductivity analytically. Since $\alpha_2>\alpha_1$ and $g(\alpha_2-\alpha_1)>kV^2\cos\Theta$, equation (32) has one real positive root. Let n_0 denotes the positive root. Then equation (32) holds true for n_0, substituted in place of n. To study the behaviour of growth rate with respect to K, we examine the nature of $\frac{dn_0}{dK}$ from this resulting equation in n_0. It follows from equation (32) that

$$\frac{dn_0}{dK}=\frac{n_0[n_0^2-gK(\alpha_2-\alpha_1)+2k_x^2V^2]}{2n_0K(n_0+Kk^2)+[n_0^2-gk(\alpha_2-\alpha_1)+2k_x^2V^2]}. \tag{35}$$

Table 1

Table of growth rate (positive real value of n) against wave number k, for $K = 1, 2, 3$, taking $V = \frac{1}{2}$ and $\alpha_2 = 0.8$, $\alpha_1 = 0.3$

Value of growth rate n				
$V = 0.5$, $\alpha_2 = 0.8$, $\alpha_1 = 0.3$				
SN	k	$K = 1$	$K = 2$	$K = 3$
1	0.0	0.0000	0.0000	0.0000
2	0.1	0.2315	0.2432	0.2489
3	0.2	0.3335	0.3572	0.3812
4	0.25	0.3541	0.3912	0.4025
5	0.3	0.4102	0.4334	0.4456
6	0.4	0.4723	0.4957	0.5224
7	0.5	0.5123	0.5435	0.5618
8	0.6	0.5534	0.5876	0.5987
9	0.8	0.6056	0.6301	0.6502
10	1.0	0.6282	0.6606	0.6753
11	1.2	0.6136	0.6501	0.6638
12	1.5	0.5025	0.5512	0.5712
13	1.6	0.4912	0.5308	0.5418
14	1.8	0.3604	0.3904	0.4017
15	1.9	0.1115	0.1717	0.1820
16	2.0	0.0000	0.0000	0.0000

It is evident from equation (35) that $\frac{dn_0}{dK}$ is always positive, it can never be negative. Thus the growth rate increases in K. Numerical calculations where performed to locate the roots of n from equation (32) to ascertain the influence of thermal conductivity on the growth rate of the unstable configuration. It would be convenient to discuss equation (32) in its general form in a non-dimensional form by measuring n, K and V in terms of $\sqrt{g}$. For $\alpha_1=0{,}3$, $\alpha_2=0{,}8$ (potential unstable configuration) these calculations are presented in Table 1, where we have given the growth rate (positive real value of n) against wave number for different values of K, taking $V=\frac{1}{2}$ and $\Theta=0°$. It is clearly seen from the table as thermal conductivity increases, n (growth rate) also increases for same k, thereby exhibiting destabilizing character of thermal conductivity.

We may thus conclude that the influence of thermal conduction on the Rayleigh-Taylor instability of two superposed conducting fluid is destabilizing.

References

1. Rayleigh, Lord, Proc. Lond. Math. Soc. 1883, No 14, p. 170.
2. Talwar, S. P., Fluid Mech., 1960, No 9, p. 581.

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