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Linearized Analysis of a One-speed B. G. K. Model in the Case of Strong Condensation

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I. Introduction

The steady one-dimensional condensation of a vapor on a plane surface has been numerically investigated by many authors [1—4]. The methods by which the problem has been approached can be divided into two groups:

i) Solution of a set of moment equations derived from a suitable Ansatz for the distribution function of the streaming vapor [1, 2].

ii) Numerical solution of the Boltzmann equation [3] or B. G. K. model equation [4]. The various methods agree in predicting that if the downstream vapor velocity is subsonic only two variables among the downstream temperature T_∞, pressure P_∞, and Mach number M_∞ can be arbitrarily prescribed.

The situation is quite different when downstream supersonic motions are considered. In fact, the surprising outcome of the moment method by Oguchi and Hatakeyama [1] was that, in complete analogy with the subsonic case, a relationship among T_∞, P_∞ and M_∞ is to exist in order to obtain a steady solution. However, it is to be expected that in the case of supersonic motions, the disturbances originated at the surface cannot affect the equilibrium solution at infinity. Accordingy, T_∞, P_∞, and M_∞ should be three independent parameters. The numerical investigations described in Refs 3 and 4 provided the evidence that other solutions exist in addition to Oguchi and Hatakeyama's. In Ref. 4 steady solutions of the strong condensation problem are found within a region of three-dimensional P_∞-T_∞-M_∞ space.

In this paper an analytical approach to the problem is attempted following the method proposed in Refs 5 and 6 to study a similar problem arising in the theory of evaporation. The method is based on the linearization of the solution of a one-dimensional B. G. K. model around the downstream equilibrium Maxwellian distribution function. The solution of the linearized equation will be a poor approximatian of the solution of the full non-linear equation. Neverthless, it will retain some of the properties of the latter as far as the number of free parameters is concerned.

II. Basic equations

The problem to be examined is the description of the motion of a monatomic gas that occupies the half-space $x>0$ and moves towards a plane surface located at $x=0$ with downstream velocity U_∞. It is assumed that the temperature and pressure at infinity are T_∞ and P_∞ respectively. The mathematical analysis of the problem is greatly simplified assuming that the fluid motion is described by the following one-dimensional B. G. K. model [5, 6]:

(1.1) $$\xi \frac{\partial f}{\partial x} = \nu \left[M(x, \xi) - f(x, \xi)\right]$$

(1.2) $$M(x, \xi) = \frac{n}{\sqrt{2\pi RT}} \exp\left\{-\frac{(\xi - V)^2}{2RT}\right\}.$$

In Eq. (1) $f(x, \xi)$ is the distribution function of the single component of the molecular velocity ξ. The density, temperature and bulk velocity of the local Maxwellian $M(x, \xi)$ are to be calculated from the density, temperature and bulk velocity of $f(x, \xi)$. Since it is assumed that the molecules reaching the surface from the vapor phase are re-emitted according to the wall Maxwellian distribution function

(2) $$f_w = \frac{n_w}{\sqrt{2\pi RT_w}} \exp\left\{-\frac{\xi^2}{2RT_w}\right\},$$

the boundary conditions for $f(x, \xi)$ are

(3.1) $$f(0, \xi) = f_w \quad \xi > 0$$

(3.2) $$\lim_{x \to \infty} f(x, \xi) = f_\infty = \frac{n_\infty}{\sqrt{2\pi RT_\infty}} \exp\left\{-\frac{(\xi - U_\infty)^2}{2RT_\infty}\right\}.$$

The following normalized form of Eq. (1)

(4) $$(c + S_\infty) \frac{\partial f}{\partial y} = M(x, c) - f(x, c)$$

is obtained setting

$$c = \frac{\xi - U_\infty}{\sqrt{2RT_\infty}} \qquad y = \frac{x\nu}{\sqrt{2RT_\infty}} \qquad S_\infty = \frac{U_\infty}{\sqrt{2RT_\infty}}.$$

The solution of Eq. (4) is now written as

(5) $$f(y, c) = f_\infty(c)\left[1 + h(y, c)\right].$$

Substituting the expression above into Eq. (4) and keeping only first order terms, the following linear integro-differential equation is obtained:

(6) $$(c + S_\infty) \frac{\partial h}{\partial y} + h = \langle 1, h\rangle + 2c\langle c, h\rangle + 2\left(c^2 - \frac{1}{2}\right)\left\langle c^2 - \frac{1}{2}, h\right\rangle.$$

In Eq. (6) the inner product $\langle \Phi \Psi \rangle$ is defined as

(7) $$\langle \Phi, \Psi \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\mu^2} \Phi(\mu) \Psi(\mu) \, d\mu.$$

Following case [7], solutions of Eq. (6) are now sought in the form

(8) $$h(y, c) = \Phi(\eta \mid c)\, e^{-\frac{y}{\eta + s_\infty}}.$$

Substituting the expression above into Eq. (6) and taking into account the conservation equations, the following equation is obtained:

(9) $$(\eta - c)\, \Phi(\eta \mid c) = \frac{1}{\sqrt{\pi}} (\eta + S_\infty)\, q(c)\, e^{-\eta^2},$$

where

(10) $$q(c) = 1 - 2 S_\infty c + 2\left(S_\infty^2 - \frac{1}{2}\right)\left(c^2 - \frac{1}{2}\right).$$

The function Φ has been normalized according to

(11) $$\int_{-\infty}^{+\infty} \Phi(\eta \mid c)\, e^{-c^2} dc = e^{-\eta^2}.$$

The solution of Eq. (9) is the ordinary function

(12) $$\Phi(\eta \mid c) = \frac{1}{\sqrt{\pi}} \frac{(\eta + S_\infty)\, q(c)\, e^{-\eta^2}}{(\eta - c)}$$

if η is not a real number. In this case, owing to the normalization condition (11) the speed ratio S_∞ and the separation constant η are related by the following dispersion equation:

(13) $$\Omega(\eta, S_\infty) = 1 + (\eta + S_\infty)\, q(\eta)\, z(\eta) + 2\,(\eta + S_\infty)\left[\eta\left(S_\infty^2 - \frac{1}{2}\right) - S_\infty\right] = 0,$$

where $z(\eta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-t^2}}{t - \eta}\, dt.$

The function $z(\eta)$ coincides with the plasma dispersion function for $\mathrm{Im}\,(\eta) > 0$. When η is a real number, the solution of Eq. (9) is

(14) $$\psi(\eta \mid c) = \lambda(\eta)\, \delta(\eta - c) + \frac{1}{\sqrt{\pi}}\, (\eta + S_\infty)\, q(c)\, P\, \frac{1}{\eta - c}\, e^{-\eta^2},$$

and the normalization condition (11) is now used to obtain

(15) $$\lambda(\eta) = \frac{\Omega^+(\eta) + \Omega^-(\eta)}{2},$$

where Ω^+ and Ω^- are the limit values taken by the sectionally analytic function Ω when η tends to the real axis from the upper or lower half-plane respectively. The application of the principle of the argument shows that Eq. (3) has no zeroes for $S_\infty \in R$ Therefore, the result found in Ref. 6 for positive values of S_∞ is also valid in the case of condensation ($S_\infty < 0$).

III. Half-space completeness

The key step in the present solution method is the possibility of expand ing the $h(0, c)$ for ξ positive as

(16) $$h(0, c) = \int_{|S_\infty|}^{\infty} A(\eta)\, \Psi(\eta),\, c)\, d(\eta). \qquad c > |S_\infty|.$$

Taking into account the expression of ψ as given by Eq. (14), Eq. (16) is transformed into the following singular integral equation:

(17) $$h(0, c) = A(c)\lambda(c) + q(c)\, \frac{1}{\sqrt{\pi}} \int_{|S_\infty|}^{\infty} \frac{(\eta + S_\infty)\, e^{-\eta^2} A(\eta)}{\eta - c}\, d\eta.$$

Following Muskhelishvili [8], the solution of Eq. (17) is found in terms of the sectionally analytic function

(18) $$N(z) = \frac{1}{2i\pi} \int_{|S_\infty|}^{\infty} \frac{(\eta + S_\infty) e^{-\eta^2} A(\eta)}{\eta - z}\, d\eta,$$

which satisfies the following inhomogeneous Hilbert problem:

$$N^{+}\Omega^{+}-N^{-}\Omega^{-}=(c+S_{\infty})\,e^{-c^{2}}\,h(0,c). \tag{19}$$

The function $N(z)$ is then found as

$$N(z)=X(z)\,\frac{1}{2i\pi}\int_{|S_{\infty}|}^{\infty}\frac{(c+S_{\infty})e^{-c^{2}}}{\Omega^{-}(c)\,X^{-}(c)}\cdot\frac{h(0,c)}{c-z}\,dc, \tag{20}$$

where $X(z)$ is a bounded sectionally analytic function solution of the homogeneous Hilbert problem:

$$\Omega^{+}X^{+}-\Omega^{-}X^{-}=0 \tag{21}$$

It is easily shown that the fnnction $X(z)$ is given by the following expressions:

$$X(z)=(z+S_{\infty})\exp\left\{-\frac{1}{2i\pi}\int_{|S_{\infty}|}^{\infty}\log\frac{\Omega^{+}}{\Omega^{-}}\,\frac{1}{c-z}\,dc\right\}\qquad S_{\infty}<\left(\frac{3}{2}\right)^{\frac{1}{2}} \tag{22.1}$$

$$X(z)=\exp\left\{-\frac{1}{2i\pi}\int_{|S_{\infty}|}^{\infty}\log\frac{\Omega^{+}}{\Omega^{-}}\,\frac{1}{c-z}\,dc\qquad S_{\infty}>\left(\frac{3}{2}\right)^{\frac{1}{2}}\right. \tag{22.2}$$

In the subsonic case ($S_{\infty}<\sqrt{3/2}$), the behaviour of $X(z)$ for large values of z is such that $N(z)$ will not vanish for $z\to\infty$ unless the following condition is satisfied:

$$\int_{|S_{\infty}|}\frac{(c+S_{\infty})\,e^{-c^{2}}}{\Omega^{-}(c)\,X^{-}(c)}\,h(0,c)\,dc=0. \tag{23}$$

Since

$$h(0,c)=\frac{f_{w}(c)}{f_{\infty}(c)}-1,$$

Eq. (23) represents a constraint on the parameters T_{∞}/T_{w}, n_{∞}/n_{w}, S_{∞}. In the supersonic case $N(z)$ automatically behaves in the correct way. Therefore, no costraint is imposed to the downstream parameters from the present linear analysis.

IV. Conclusions

The results obtained from the linearized analysis of the one-dimensional B. G. K. kinetic model can be summarized as follows:

— The calculations confirm the numerical results obtained in the case of subsonic condensation. It is found that a steady solution exists only if the three driving parameters P_{∞}/P_{w}, T_{∞}/T_{w} and S_{∞} satisfy the condition expressed by Eq. (23). Of course, it is to be expected that the results obtained from Eq. (23) will not compare favourably with those presented in Refs 1—4 because of the intrinsic inadequacy of the model and of the linearization of the problem.

— The linearized analysis of supersonic condensation has shown that the boundary conditions imposed to the solution of Eq. (6) can be satisfied for any combination of the parameters.

It is worth remembering that the numerical study described in Ref. 4 has shown that, in the case of supersonic condensation, steady solutions do not exist in certain region of the three-dimensional space of the parameters. The determination of the conditions on T_{∞}/T_{w}, P_{∞}/P_{w}, and S_{∞}, that seem to exist in this case, is apparently beyond the capabilities of the method presented in this paper.

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