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SOLID MECHANICS

ANALYSIS OF THE NONLINEAR ELASTIC RESPONSE OF RUBBER MEMBRANE WITH EMBEDDED CIRCULAR RIGID INCLUSION*

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Abstract. Rubber membranes exhibit a particular nonlinear elastic behaviour known as hyper elasticity. Analysis has been proposed by utilizing the modified strain energy function from Gao's constitutive model, in order to reveal the mechanical property of rubber membrane containing circular rigid inclusion. Rubber membrane is taken into incompressible materials under axisymmetric stretch, based on finite deformations theory. Stress distribution of different constitutive parameters has been analyzed by deducing the basic governing equation. The effects on membrane deformation by different parameters and the failure reasons of rubber membrane have been discussed, which provides reasonable reference for the design of rubber membrane.

Key words: The finite deformation, rigid inclusion, rubber like materials, constitutive equation.

0. Introduction

The hyper elastic materials are extensively applied to many fields such as automobile, aerospace and biotechnology [1]. The rubber membrane is often used in sealing equipment of aerospace and space engineering. In electrical

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engineering, rubber membrane is made as insulating layer of the electrical appliance. Therefore, it is very important to put research on the security and the reliability of the engineering rubber membrane.

In the evaluation of the security and the reliability, from the mechanics perspective, the first problem to be solved is to select the reasonable and practical strain energy density function, that describes the property of mechanics of rubber like material. Many constitutive models have been constructed describing the physical characteristics of rubber like materials, based on multi-discipline and multi-field, such as continuum mechanics, thermodynamics, phenomenological theory of the phase transition, statistic physics [2]. In 1948, Rivlin put forward the strain energy function model to the isotropic hyper elastic materials [3]:

$$W = \sum_{i,j=0}^{\infty} C_{ij}(I_1 - 3)^i (I_2 - 3)^j, \tag{1}$$

in which C_{ij} stands for material constant; I_1 and I_2 are the first and second invariants of the left Cuachy-Green deformation tensor, respectively. The constitutive model is complicated, just because there are many physical parameters in it. To be simplified, the first of the Rivlin model can be used and it is the Neo-Hookean material, which can be expressed as follow:

$$W\left(I_1\right) = \frac{1}{2} nkT\left(I_1 - 3\right), \tag{2}$$

while, taking the linear combination of the Rivlin model, we can get the Mooney-Rivlin material, which can be considered as one of the simplest forms of the invariant-based models:

$$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3). \tag{3}$$

The above two models lay a solid foundation for the new constitutive model, because they are easy in form and easy to verify by experiments; what's more, they are extensively applied to the engineering and have accumulated large quantities of theoretical and experimental achievements of research. These two models can fit the material mechanics features of the incompressible rubber materials with small and medium deformation, but can't precisely describe the mechanics features of the rubber like materials, that become hardened or describe the "sharp lifting" characteristic [5] of the stress-strain curve in the large deformation.

To solve these problems, people construct logarithmic type and power law type constitutive models [4–5]. Gao put forward a strain energy function from the perspectives of tensile and compression of the materials in 1997 and applied it to the material-fracture research [6]. However, under incompressible conditions, strain energy function cannot be simplified to Neo-Hookean material, or to Mooney-Rivlin material, which influences the experimental basis of the strain energy function application.

In practical production, rubber membrane will generate rigid inclusion in blowing process, because there is remaining catalyst and impurity in the raw material or generate holes, when the membrane is blown unevenly. The structure will be destroyed due to larger deformation and stress concentration, where rigid inclusion occurs with the effect of external applied load, which has attracted the attention of engineering field. With Mooney strain energy function, Yang. W. H. studied the stress concentration of the circular membrane near the hole [8]. Haughton came up the theoretical solution of the elastic membrane containing the hole [9]. Oscar Lopez-Pamies [10, 11] have a research of the elastic response of the rigid inclusions in rubber. Guo Z. Y. [12] studied the incompressible particle-reinforced neo-Hookean composites. With the appearance of new materials, it is very important to research on the large deformation of the membrane containing rigid inclusion.

Based on Gao's constitutive model, a modified model from Gao has been proposed to describe the incompressible rubber like materials in [7], which has been used to analyze the basic deformation of polymer materials. That is, when $n = 1$ and $\alpha = 0$, the new model transforms to Neo-Hookean model. However, when $n = 1$ and $\alpha = 1$, the new model transforms to Mooney-Rivlin model. With the given model, this paper explores the finite deformation of hyper elastic material membrane containing rigid inclusion. The paper makes an analysis and calculation of the membrane finite deformation, by deducing the basic governing equation for solving the problem. Figures out the stress distribution of different constitutive parameters and discusses the effects on membrane deformation by different parameters and the reasons for the membrane failure. The results shows that the greater the constitutive parameter n, the greater the ability for the material to resist the pressure, which provides reference for the design of rubber like material.

1. Discussion on the constitutive model of rubber like materials

Gao proposed the following strain energy function [6] in 1997:

$$W = A(I_1^n + I_{-1}^n). \tag{4}$$

Based on the elastic finite deformation theory, the Cuachy stress tensor can be expressed as follows:

$$\boldsymbol{\sigma} = 2AnJ^{-1}(I_1^{n-1}\boldsymbol{B} - I_{-1}^{n-1}\boldsymbol{B}^{-1}). \tag{5}$$

In which, $\boldsymbol{B} = \boldsymbol{F}\cdot\boldsymbol{F}^T$ stands for left Cuachy-Green deformation tensor, $\boldsymbol{F}$ is deformation gradient tensor, I_1, I_2 and I_3 are the three invariants of $\boldsymbol{B}$, J represents the Jacobin of the transformation, which can be calculated by the determinant of the deformation gradient $\boldsymbol{F}$, A and n are material parameters and $I_{-1} = I_2/I_3$. The calculation indicates, that the greater the tensile strain, the larger the I_1; the greater the compressive strain, the larger the I_{-1}. The two complement each other, which can describe the finite deformation features of the materials.

From eq. (4), we can see that strain energy function can't meet the conditions of the strain energy function $W = 0$, apparently when$I_1 = I_2 = 3$and$I_3 = 1$. In other words, it should meet there is no strain energy function at initial spontaneous configuration. For the incompressible material, when $n = 1$, the expression (4) can't be simplified to Neo-Hookean material, nor to Mooney-Rivlin material. A modified strain energy function for the incompressible rubber like materials from Gao has been proposed in [7] as follow:

$$W = A\left[(I_1^n - 3^n) + \alpha\,(I_2^n - 3^n)\right], \tag{6}$$

in which, α $(0 \le \alpha \le 1)$ is the material parameter reflecting the influence of the invariant of I_2 on the stress distribution.

From the new constitutive model (6), we can see that when $n = 1$and $\alpha = 0$, it transforms to Neo-Hookean model; when $n = 1$, it transforms to Mooney-Rivlin model. For incompressible rubber materials, we can get the expression of Cuachy stress tensor, based on strain energy function (6), as follow:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\frac{\partial W}{\partial I_1}\boldsymbol{B} - 2\frac{\partial W}{\partial I_2}\boldsymbol{B}^{-1}, \tag{7}$$

$\boldsymbol{I}$ is unit tensor. p is the undetermined scalar function, that justifies the incompressible internal constraint conditions.

For incompressible material, we also can get the following expressions:

$$\beta_1 = 2\frac{\partial W}{\partial I_1} = 2AnI_1^{n-1}, \quad \beta_{-1} = -2\frac{\partial W}{\partial I_2} = -2An\alpha I_2^{n-1}. \tag{8}$$

In the following, we discuss the rationality of the new constitutive model (6). First, Equation (6) meets the strain energy function $W = 0$, when $I_1 =$

$I_2 = 3$ and $I_3 = 1$, which indicates that the reference configuration is the spontaneous one. Second, Equation (7) meets the expression $tr(\boldsymbol{N} : \boldsymbol{D}) = p[tr(\boldsymbol{D})] = 0$, in which $\boldsymbol{N} = p\boldsymbol{I}$ is the undetermined stress; $\boldsymbol{D}$ stands for the strain rate tensor; for the incompressible material, we get $tr(D) = 0$. Third, β_1 and $\beta_2$meet the expression $\partial\beta_1/\partial I_2 + \partial\beta_2/\partial I_1 = 0$, that is to say, the material is hyper elastic. Besides, in 1975 Batra [13] claimed, that the constitutive model needs to meet the empirical inequalities $\beta_1 > 0; \beta_2 \leq 0$ in the uniaxial tensile test on isotropic materials. The empirical inequalities are justified by the testing data of Rivlin, Saunders and Treloar [14]. Obviously, it can be seen from Equation (8), when $A > 0$, $n > 0$, $\alpha \geq 0$, we can get $\beta_1 \geq 0$ and $\beta_2 \leq 0$. Finally, the new constitutive model proves to meet the Baker-Ericksen inequality with the biaxial tension under the plane stress, which was put forward by Truesdell and Noll [15] in 1962. For the isotropic material, the greater the principal stress is, the greater the principal stretch. Under the condition of $A > 0$, $n > 0$, $\alpha \geq 0$, the new constitutive model meets the expression $(\sigma_i - \sigma_j)(\lambda_i - \lambda_j) > 0$ $(\lambda_i \neq \lambda_j)$. Therefore, it is reasonable, that we use Equation (6) as the strain energy function, describing the incompressible rubber like materials, then construct the constitutive equations of the materials.

2. Theoretical analysis

In the practical engineering, the rigid inclusion size is much smaller than the size of the rubber membrane and it is sustained tension. Considering the convenience of analysis, a circular rubber membrane embedded a rigid inclusion in the center has been researched and uniform tensile stress is put on the rubber membrane boundary. Consider that the material is incompressible rubber membrane with a circular rigid inclusion, the radius of which is$a$. The uniform tensile stress imposes on infinity of rubber membrane, as illustrated in Fig. 1.

2.1. Geometrical analysis

If its coordinate before deformation is and coordinate after deformation is (r, θ, z), then the deformation pattern of the membrane can be expressed as:

$$r = r(R), \quad \theta = \Theta, \quad h = h(R) = H \cdot \lambda_3(R), \tag{9}$$

H stands for the thickness before deformation and h is the current thickness after deformation. The deformation gradient tensor $\boldsymbol{F}$, left Cuachy-Green de-

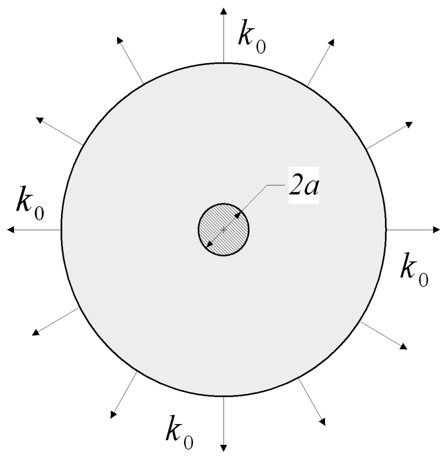

Fig. 1. The diagram of membrane structure

formation tensor $\boldsymbol{B}$ and $\boldsymbol{B}^{-1}$ are as follows, respectively:

(10) $$\boldsymbol{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix},$$

(11) $$\boldsymbol{B}^{-1} = \begin{bmatrix} \lambda_1^{-2} & 0 & 0 \\ 0 & \lambda_2^{-2} & 0 \\ 0 & 0 & \lambda_3^{-2} \end{bmatrix},$$

in which, λ_i $(i = 1, 2, 3)$ stands for the principal stretch:

(12) $$\lambda_1 = \frac{dr}{dR} = r', \lambda_2 = \frac{r}{R}, \lambda_3 = \frac{h}{H} = \lambda_3(R) = \frac{R}{rr'}.$$

2.2. Equilibrium equation

From the Equation (6), we can get, that the strain energy function $W = W(I_1, I_2)$ is the function of the first and second invariants of the left Cauchy-Green deformation tensor $\boldsymbol{B}$. If $W_1 = \partial W/\partial I_1$, $W_2 = \partial W/\partial I_2$, then we can get the following expressions from Equation (7):

(13) $$\begin{aligned} \sigma_1 &= 2\lambda_1^2 W_1 - 2\lambda_1^{-2} W_2 - p \\ \sigma_2 &= 2\lambda_2^2 W_1 - 2\lambda_2^{-2} W_2 - p \;, \\ \sigma_3 &= 2\lambda_3^2 W_1 - 2\lambda_3^{-2} W_2 - p \end{aligned}$$

in which, $\sigma_1 = \sigma_r$; $\sigma_2 = \sigma_\theta$; $\sigma_3 = \sigma_z$ are the radical Cauchy stresses, the circumferential Cauchy stress and the axial Cauchy stress of the membrane surface, respectively. The stress is free (referring to the plane stress) in the vertical direction of the membrane surface. With (13), we can get $p = 2\lambda_3^2 W_1 - 2\lambda_3^{-2} W_2$. Then, the Cauchy stress can be expressed as:

$$\begin{aligned} \sigma_1 &= 2(\lambda_1^2 - \lambda_3^2)(W_1 + \lambda_2^2 W_2), \\ \sigma_2 &= 2(\lambda_2^2 - \lambda_3^2)(W_1 + \lambda_1^2 W_2). \end{aligned} \tag{14}$$

If $T_i = H\lambda_3\sigma_i$, $i = 1, 2$, with the balance equation of the plane axial symmetry in the current configuration, the balance equation can be achieved as:

$$\frac{dT_1}{\lambda_1 dR} + \frac{T_1 - T_2}{r} = 0. \tag{15}$$

From the second expression of Equation (12), we can get the following expression:

$$\frac{d\lambda_2}{dR} = \frac{Rr' - r}{R^2} = \frac{\lambda_1 - \lambda_2}{R}. \tag{16}$$

2.3 Derivation of governing equation

Substitute the strain energy function (14) into Equation (15), then the governing equation of solving the problem can be obtained with Equation (16), as:

$$\frac{d\lambda_1}{d\lambda_2} = -\frac{\lambda_1 F_1(\lambda_1, \lambda_2)}{\lambda_2 F_2(\lambda_1, \lambda_2)}, \tag{17}$$

$F_1(\lambda_1, \lambda_2)$and $F_2(\lambda_1, \lambda_2)$ can be expressed just as follows:

$$\begin{aligned} F_1(\lambda_1, \lambda_2) = {} & \lambda_1^2\lambda_2^2 \left[\left(\lambda_1^3\lambda_2^3 + 3\right) W_1 + \alpha \left(\lambda_1^4\lambda_2^4 + \lambda_1 + \lambda_2 + \lambda_1\lambda_2\right) W_2 \right] \\ & + 2\left(\lambda_1^2\lambda_2^4 - 1\right)\left(\lambda_1^4\lambda_2^2 - 1\right)\left(W_{11} + \alpha\lambda_1^2\lambda_2^2 W_{22}\right), \end{aligned} \tag{18}$$

$$\begin{aligned} F_2(\lambda_1, \lambda_2) = {} & \lambda_1^2\lambda_2^2 \left(\lambda_1^4\lambda_2^2 + 3\right)\left(W_1 + \alpha\lambda_2^2 W_2\right) \\ & + 2\left(\lambda_1^4\lambda_2^2 - 1\right)^2 \left(W_{11} + \alpha\lambda_2^4 W_{22}\right), \end{aligned} \tag{19}$$

in which, $W_1 = \dfrac{\partial W}{\partial I_1}$, $W_2 = \dfrac{\partial W}{\partial I_2}$, $W_{11} = \dfrac{1}{\alpha} \cdot \dfrac{\partial^2 W}{\partial I_1^2}$, $W_{22} = \dfrac{1}{\alpha} \cdot \dfrac{\partial^2 W}{\partial I_2^2}$.

When$\alpha = 0$, that is $W = W(I_1)$, then $W_2 = W_{22} = 0$.

$$\frac{d\lambda_1}{d\lambda_2} = -\frac{\lambda_1}{\lambda_2} \cdot \frac{\lambda_1^2\lambda_2^2\left(\lambda_1^3\lambda_2^3 + 3\right)W_1 + 2\left(\lambda_1^2\lambda_2^4 - 1\right)\left(\lambda_1^4\lambda_2^2 - 1\right)W_{11}}{\lambda_1^2\lambda_2^2\left(\lambda_1^4\lambda_2^2 + 3\right)W_1 + 2\left(\lambda_1^4\lambda_2^2 - 1\right)^2 W_{11}}. \tag{20}$$

If $n = 1$, the constitutive relation transforms to Neo-Hookean material is on. Then, the governing equation is:

$$\frac{d\lambda_1}{d\lambda_2} = -\frac{\lambda_1}{\lambda_2} \cdot \frac{\lambda_1^3\lambda_2^3 + 3}{\lambda_1^4\lambda_2^2 + 3}. \tag{21}$$

The equation is in consistent with (14) of the document [9]. When$n = 1$, we have these two expressions: $W_1 = W_2 = 1$ and $W_{11} = W_{22} = 0$. The constitutive relation transforms to Mooney model and the governing equation is:

$$\frac{d\lambda_1}{d\lambda_2} = -\frac{\lambda_1}{\lambda_2} \cdot \frac{\lambda_1^2\lambda_2^2\left(\lambda_1^3\lambda_2^3 + 3\right) + \alpha\left(\lambda_1^4\lambda_2^4 + \lambda_1 + \lambda_2 + \lambda_1\lambda_2\right)}{\lambda_1^2\lambda_2^2\left(\lambda_1^4\lambda_2^2 + 3\right)\left(1 + \alpha\lambda_2^2\right)}. \tag{22}$$

The equation is in consistent with (15) of the document [9].

3. Calculation and discussion

3.1 Boundary conditions

From the Lame results of the small deformation, we can get the boundary condition in the infinite distance, that is:

$$\lambda_1 = \lambda_2 = \lambda_\propto. \tag{23}$$

If the radius of defect is a, we can get the boundary condition$\lambda_2 = 1$ when $R = r = a$, because of the rigid inclusion. In order to simplify the process, dimensionless is adopted with R/α and r/α. In the $\lambda_1 - \lambda_2$ plane, for the selected $\lambda_\propto$, we use Runge-Kutta method to solve the differential equation (17) and find out the $\lambda_\propto$, that can meet the boundary condition. From the solved (λ_1, λ_2), we can figure out the corresponding R/a before deformation from Equation (16). From $\lambda_2 = R/r = (R/a)/(r/a)$, we can work out (r/a). And from the solved (λ_1, λ_2), it is easy to figure out the stress distribution from Equation (14).

3.2 Results discussion

If the material parameters $n = 1$, $\alpha = 0.1$, for the different $\lambda_\propto$, the computed results, that meet the boundary are illustrated by Fig. 2. We can see, that the greater the $\lambda_\propto$, the larger the radial deformationλ_2 and it becomes stable value gradually far from the inclusion. When $n = 1$, $\alpha = 0.1$, the result is in consistent with the results of document [6], under the circumstance of rigid inclusion.

In order to study the effect on membrane deformation by the constitutive parameters α and n, we consider two circumstances, that is, when α

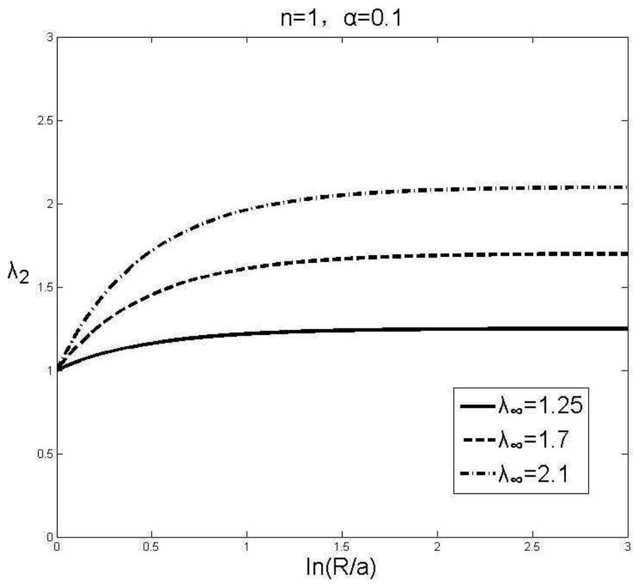

Fig. 2. The deformation distribution curves ofλ_2

Table. 1 Initial value of the boundary condition with the parameter n

n	α	$\lambda_\propto$	λ_1
1	0.1	5.197731445	12.43562
3	0.1	1.376368379	1.118210
5	0.1	1.038363275	1.076447

Table 2. Initial value of the boundary condition with the parameterα

n	α	$\lambda_\propto$	λ_1
3	0.1	1.376368379	1.718210
3	0.3	1.319231019	1.628187
3	0.5	1.283122808	1.564271

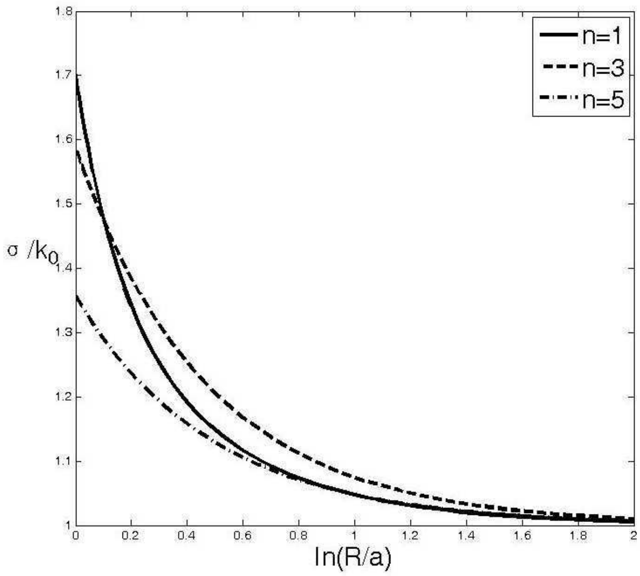

Fig. 3. The radial Cauchy stress distribution with the parameter n

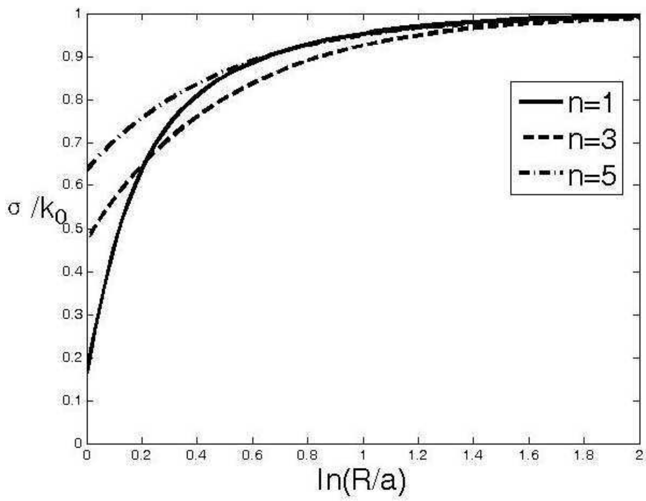

Fig. 4. The circumferential Cauchy stress distribution with the parameter n

is fixed, Cauchy stress distribution with the change of$n$has been researched. We simultaneously have a research on the Cauchy stress distribution with the change of α, when n is fixed. Tables 1 and 2 illustrate the initial value of the boundary condition and the internal boundary deformation $\lambda_1(R = r = a)$.

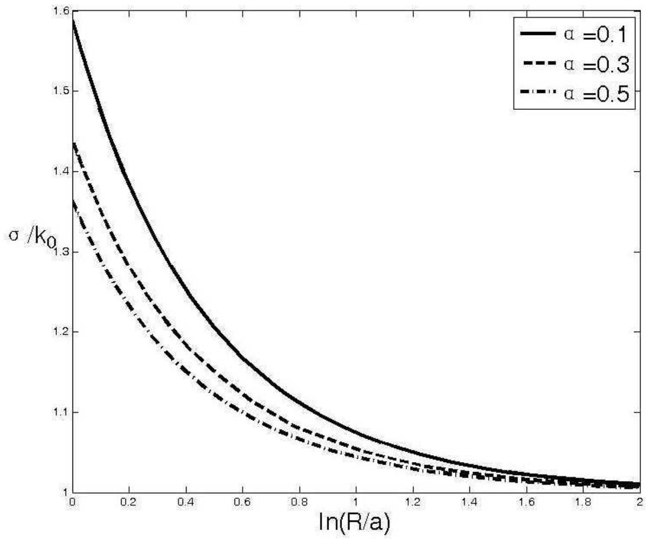

Fig. 5. The racial Cauchy stress distribution with the parameter α

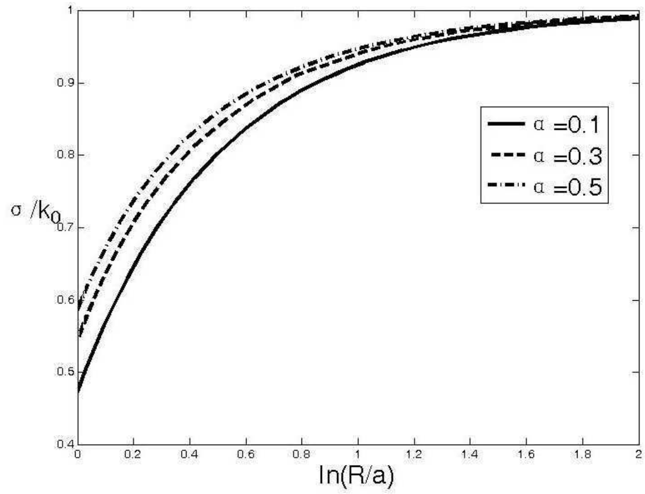

Fig. 6. The circumferential Cauchy stress distribution with the parameter α

Figures 3 and 4 show the radial and circumferential Cauchy stress distribution, respectively with the change of parameter n, when α is fixed. While, Figs 5 and 6 show the radial and the circumferential Cauchy stress distribution, respectively and the change of the membrane thickness with the change of parameter α, when n is fixed. Figures 7 and 8 show the change of membrane's thickness

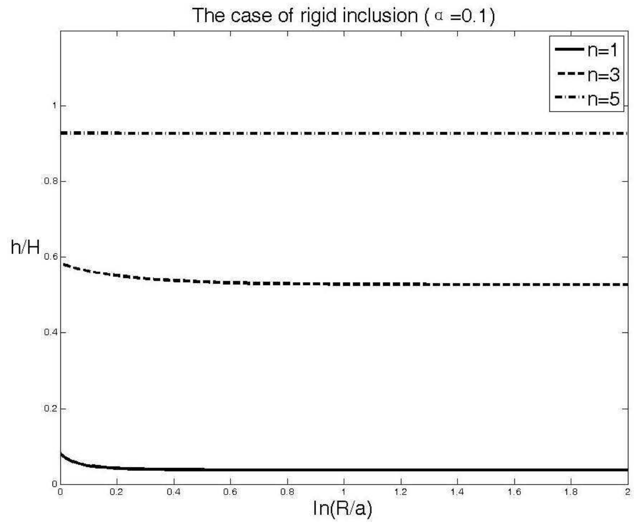

Fig. 7. Thickness change of membrane with the parameter n

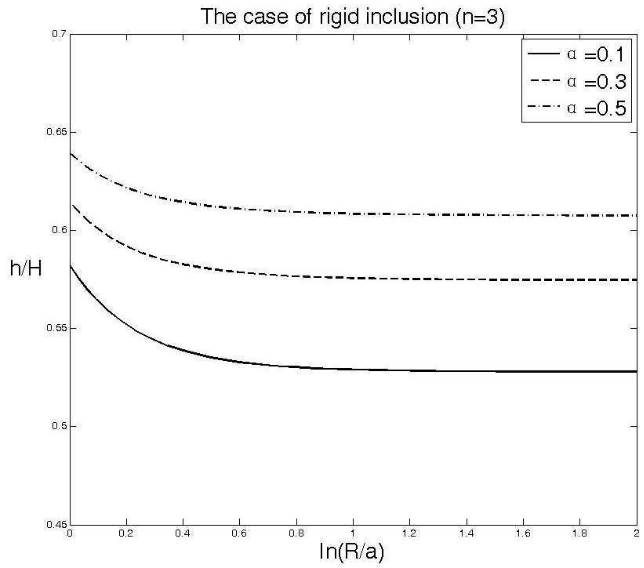

Fig. 8. Thickness change of membrane with the parameter α

with the change of parameters n and α, respectively.

From Figs 3 to 6, we can see when the stress is stretched inside the rubber membrane with rigid inclusion, the radial stress is greater than the circumferential one. As the distance from inclusion increases, the radial stress becomes smaller and the circumferential one increases. In the boundary of inclusion, the stress difference reaches its maximum. Finally, the stress difference becomes zero in the infinite distance and it is the same as external load, that is$\sigma_1 = \sigma_2 = k_0$.

From (12), we can get the expression $h = \lambda_3 H = H/\lambda_1\lambda_2$. From Figs 7 and 8, we can see the membrane's thickness becomes thinner and keeps stable gradually with the increase of the distance from the inclusion under the external force. At the joint between membrane and inclusion, the thickness of membrane becomes thin sharply, which indicates that the intensive deformation happens at the joint and that if there is no secure joint between the rubber membrane and inclusion, the membrane will be broken near the inclusion. Further, study will be carried out about the intensive deformation of the membrane's thickness at the joint between membrane and inclusion.

4. Conclusion

The finite deformation about the incompressible membrane under the uniform pressure has been researched. The results indicate, that the material constitutive parameters n and α have important influence on the mechanical property of rubber like materials, that is, the material's ability to resist deformation is strengthened with the increase of n and α. The radial stress is bigger than circumferential stress. As the distance from inclusion increases, the radial stress becomes smaller and the circumferential stress increases. Finally, the stress difference becomes zero in the infinite distance and it is the same as external load, that is $\sigma_1 = \sigma_2 = k_0$. At the joint between membrane and inclusion, the intensive deformation happens at the joint and if it is not firm agglutination at the joint, the membrane will be broken in the inclusion. Further, study will be carried out about the intensive deformation of the membrane's thickness at the joint between membrane and inclusion.

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