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GENERAL MECHANICS

GENERALIZED FORCES AND ROBUSTNESS OF MECHANICAL SYSTEMS*

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1. Introduction

Let a mechanical system with stationary holonomic relations be represented in the form

$$Lq = Q \tag{1}$$

where the "control" force Q must ensure the property of zero equilibrium asymptotical stability; L is a Lagrange operator defined by the kinetic energy $T = p^*A(q)p$:

$$Lq = \frac{d}{dt}\frac{\partial}{\partial \dot{q}}T - \frac{\partial}{\partial q}T. \tag{2}$$

Here q and $p = \dot{q}$ are the n-dimensional vector Lagrange variables; the bounded (by the whole of its arguments) positive definite matrix $A(q)$ is assumed to be differentiable.

Many design procedures define the "control" force by the formula

$$Q = (L + L_1)q \tag{3}$$

where L_1 is some operator defining the system

$$L_1 q = 0 \tag{4}$$

having the asymptotical stable equilibrium $q = p = 0$.

It is clear that there are various operators defining such systems, e.g.,

$$L_1 q = \dot{q} + q. \tag{5}$$

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The system (4) and (5) is equivalent to the system (1), (3) and (5) if initial data q and p are connected by the equality $p + q = 0$. This fact is in conflict with that in general, initial data are arbitrary for the original system.

Thus, the first problem is to motivate choosing some operator L_1 or (just the same) the structure of Q in order to avoid similar and other conflicts. The second one is that it is impossible to realize exactly force (3). That is why instead of (3), we have

$$Q = (L_1 + L_2)q \tag{6}$$

where L_2 is the operator close to L in some meaning.

The "control" force (6) generates the next closed-loop system

$$L_1 q + (L - L_2)q = 0 \tag{7}$$

where the term $(L - L_2)q$ can be interpreted as the perturbation of system (4).

Thus the second problem is the robust one [1, 2].

Formally the robust problem solution has a positive solution under the following condition

$$\left\| L_1^{-1}(L - L_2) \right\| < 1 \tag{8}$$

where $\|.\|$ is some operator norm.

In general, this condition is not constructive, it being very important to give other conditions when inequality (8) holds.

It is an important case of the second problem where the synthesized forces are often put in practice with the help of digital techniques. Already for the rather simple case of linear time-constant control systems there is the set of isolated values of quantization intervals such that they can loss their controllability with respect to discrete controls [3]. That is why a priori, without corresponding proofs (corresponding conditions) it is impossible to think that the term $(L - L_2)q$ is some bounded vector-function of time, and in this way to estimate the perturbation influence.

In the case of linear time-invariant system of ordinary differential equations where the perturbation does not affect on the highest derivatives and has character of quantization by time and level or one of time-delay, the corresponding robust theory is given in [4] (see also [5]). In such systems, perturbations containing the highest derivatives can produce the catastrophic loss of stability, e.g., when the highest derivatives have time-delay [6].

2. Generalized force structure and asymptotical stability

It is clear that the both problems are closely connected. Moreover if the first one is solved in the "natural" way, there is the hope that the second one may be solved without any difficulties.

Force structure influence upon the motion stability has started to be studied, in essence, in works of Lagrange who proved the theorem about stability of conservative

system equilibrium in 1788. Rayleigh introduced the dissipation function in 1873, and after that the dissipative force influence upon the motion stability and oscillations of potential systems has studied in detail. The next step in analysis of force structure influence upon the motion stability was made by Thomson and Tait in 1879. They gave the general definition of gyroscopic forces and proved four theorems about motion stability. In the end of 20's of this century, the non-conservative position forces have been taken into account.

Many results in this field of interest are produced by the Russian school. Here A. M. Lyapunov, S. A. Chaplygin, N. G. Chetaev, I. G. Malkin, V. V. Rumyantsev, V. M. Matrosov and etc. can be recalled.

Main results can be characterized by the fact that generalized forces are arbitrary in the framework of their fixed structure and ensure the stability property regardless to inertial characteristics of mechanical systems. In the other words, these forces are adaptive (i.e. they ensure the stability property for the class of systems). Under the condition that the force structure is fixed and the force perturbations are small there is the robustness property, too.

In models studied in the analytical mechanics, forces can depend on position and velocity variables and do not depend on acceleration ones. That is why let us restrict with the following structure of forces

$$Q = -(B + C)q - (D + G)p. \tag{9}$$

Here the next components are used:

position non-conservative force $-Bq$ *where* B *is some bounded (by the whole of its arguments) skew-symmetric matrix function of time and Lagrange variables;*

conservative-potential forces $-Cq$ *where* C *is some differentiable and bounded (by the whole of its arguments) symmetric matrix function of Lagrange variables;*

velocity non-conservative forces $-D\dot{q}$ *where* D *is some bounded (by the whole of its arguments) symmetric matrix function of time and Lagrange variables;*

gyroscopic forces $-G\dot{q}$ *where* G *is some bounded (by the whole of its arguments) skew-symmetric matrix function of time and Lagrange variables.*

It is clear that the system (1) and (9) has the zero equilibrium $q = p = 0$. Let us state a result about its asymptotical stability.

Lemma 1. *The zero equilibrium* $q = p = 0$ *of the system (1) and (9) acted by any bounded (by the whole of its arguments) position non-conservative* ($B \not\equiv 0$) *and gyroscopic forces* ($C \not\equiv 0$) *can be made asymptotical stable, if dissipative forces with any (rather small) positive definite matrix* D *and conservative-potential forces with some bounded (by the whole of its arguments) positive definite matrix* C *will be added.*

Moreover if position non-conservative forces are absent ($B \equiv 0$), *then conservative-potential forces can be chosen with any arbitrary small positive definite matrix* C.

Proof. According to [7], take a Lyapunov function in the form

$$V = \begin{pmatrix} q \\ p \end{pmatrix}^* \begin{bmatrix} \Pi & \alpha\Pi A \\ \alpha A\Pi & A \end{bmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \tag{10}$$

where α is some positive constant; Π is a differentiable and bounded positive definite matrix depending on the position variable q.

It is easy to see that this form is positive definite if $\alpha^2\Pi$ is rather small.

Write the system (1) and (9) in the following form

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \mathfrak{A}\begin{pmatrix} q \\ p \end{pmatrix} \tag{11}$$

where

$$\mathfrak{A} = \begin{bmatrix} O & E_n \\ -A^{-1}(B+C) & -A^{-1}(D+G) \end{bmatrix}. \tag{12}$$

It is easy to see that the derivative of the Lyapunov function includes the square form with the following matrix

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{13}$$

where $W_{11} = -\alpha\,\Pi\,(B+C) - \alpha(B-C)\Pi$, $W_{12} = -\Pi + \alpha\,\Pi(G+D)+(B+C)^*$, $W_{22} = -2D+\alpha(A\Pi+\Pi A)$.

Let the matrix C be positive definite and $\Pi = C$. Then the matrix $-W_{22}$ is positive definite if the matrix D is the same type and the matrix $A(\alpha C)+(\alpha C)A$ is chosen to be arbitrary small with the help of a rather small matrix $C_1 = \alpha C$.

It is easy to represent the first block entry of matrix (13) in the following form

$$W_{11} = -\alpha^{-1}C_1\left[2E_n + \alpha\left(BC_1^{-1} - C_1^{-1}B\right)\right]C_1 \tag{14}$$

then $-W_{11}$ is positive definite for any bounded matrix positive definite C_1, if α is rather small. Let it be the case, then the matrix $-W$ is positive definite if $-W_{22} + W_{21}W_{11}^{-1}W_{12}$ is of the same type.

The last condition can be represented in the inequality form

$$2D \geq d_0E_n + AC_1 + C_1A + \alpha\left[(D-G) + BC_1^{-1}\right]$$

$$\left[2E_n + \alpha\left(BC_1^{-1} - C_1^{-1}B\right)\right]^{-1}\left[(D+G) - C_1^{-1}B\right] \tag{15}$$

where d_0 is any small positive constant.

It follows from the above that inequality (15) holds for some small matrix D if a rather small constant α is chosen, the matrix C being equal to $\alpha^{-1}C_1$. If $B \equiv 0$ then the

positive definite matrix C can be taken to be arbitrary small (this fact is known from the third theorem of Thomson-Tait-Chetaev [9]).

Thus, asymptotical stability is the result of jointly acting forces having different nature. We may assume that any forces are in a mechanical system. In particular, we cannot exclude that there are small position non-conservative forces in any system. By this reason it is important to know how small perturbations of the matrices B, C, D and G influence on the asymptotical stability property.

It follows from the above lemma that the next statement is true.

Theorem 1. *Let 1. the matrices B, C, D and G satisfy to inequality (15); 2. the matrices C and D be positive definite and be not small. Then the asymptotical stability property of the zero equilibrium is robust with respect to small perturbations of the matrices B, C, D and G (by any norm in a time-constant matrix space).*

Indeed, under the conditions of the stated theorem, we may use the Lyapunov function constructed above in order to restrict perturbations of the matrices B, C, D and G.

Remark 1. *The above results can be applied to the case of systems that are not mechanical, but have description of the kind (1) and (9). The "control force" (9) can be free for choosing or conversely some of its components can be fixed. The performed analysis shows the influence of a generalized force structure on the zero equilibrium stability and permits us to understand what components must be introduced or compensated.*

Remark 2. *The case of time-varying holonomic relations can be considered in the same way, but then we must take into account the partial derivative by time from the matrix A. It results that dissipative forces must overlap the influence of this derivative. The same is true if we study non-zero equilibrium $q = q(t)$; $p = p(t)$. For details, see [7].*

Remark 3. *The obtained result has an adaptive character as the zero equilibrium of a mechanical system is asymptotically stable regardless from its inertial characteristics. It is not the case in general [7] if a mechanical system has time-varying holonomic relations or it is stabilized to a dynamic equilibrium $q_e = q(t)$; $p_e = p(t)$. In the last case, in order to stabilize to this equilibrium, it is well known to be used the following control force*

$$Q = -(B+C)(q-q_e)-(D+G)(p-p_e)+A\dot{p}_e \tag{16}$$

where it is obligatory that the matrix-function coefficient A must be given while the other coefficients can be chosen.

Initial data of the system (1) and (16) are always at a short distance from the desired configuration (q_e, p_e). It means that the first two terms of (16) are small. Thus the control force (16) can be put in practice if the term $A\dot{p}_e$ is small, too. The last condition holds obviously in the case where the desired configuration is "slow", i.e., $\dot{p}_e$ is small. Conversely if the desired configuration is "fast", i. e., $\dot{p}_e$ is great, then actuators of a system must be specially constructed in order to take great control

signals. In other words the class of "fast" desired configurations makes high demands to the corresponding actuators.

3. Example

Let us consider a rotor in aerodynamic medium under the condition that its centre coincides with the centre of its casing. Then its forced motion is described by the next equations [9]

$$\ddot{x}+b\dot{x}+k^2x=X;\quad \ddot{y}+b\dot{y}+k^2y=Y \tag{17}$$

where b and k are some positive constants; X and Y are non-linear forces.

Due to the third theorem of Thomson-Tait-Chetaev the zero equilibrium of system (17) is asymptotically stable.

If these centres are displaced then non-conservative position forces appears, there being the following equations [9]

$$\ddot{x}+b\dot{x}+k^2x-py=X;\quad \ddot{y}+b\dot{y}+k^2y+py=Y \tag{18}$$

where p is some positive constant.

Let us identify these equations with the system (1) and (9):

$$A=E_2;\ B=p\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix};\ C=k^2E_2;\ D=bE_2;\ G=0. \tag{19}$$

Then it follows from inequality (15) that the zero equilibrium is asymptotically stable if

$$b>\alpha k^2+\frac{\alpha}{4}\left(b^2+\frac{p^2}{\alpha^2k^4}\right). \tag{20}$$

Hence we have inequality

$$\left(4k^2+b^2\right)\alpha^2-4b\alpha+\frac{p^2}{k^4}<0. \tag{21}$$

It holds obviously for some positive α if

$$b^2\left(1-\frac{p^2}{4k^4}\right)>\left(\frac{p}{k}\right)^2;\quad p>2k^2. \tag{22}$$

This estimation is not very precise, as it gives only a sufficient condition of asymptotical stability of the zero equilibrium. It can be made more precise due to the fact that here Hurwitz criterion can be applied to the system under consideration. In this case we have the next necessary and sufficient condition of asymptotical stability of the zero equilibrium [9]:

$$b > \frac{p}{k}. \tag{23}$$

In the both case the property of asymptotical stability is robust in the respect to small perturbations of the acting forces.

4. Time-quantization

The well-known Kotelnikov's theorem (1933) gives conditions that no information loss when continuous functions are replaced with their discrete values. This fact is often implied to systems with discrete controls. In general, this approach to use discrete feedbacks is absolute groundless as systems themselves do not imply the property to convert discrete data in the continuous form. Proving system robustness with respect to time-discreteness in feedbacks can be correctly given within the framework of Lyapunov method for the class of quasi-linear systems of the kind (1) and (9). The same approach is true also for proving system robustness with respect to time-delay in feedbacks. It has been worked out by V. Zubov [4] for linear time-constant closed-loop systems (see also [8]).

Let us consider the system

$$\dot{x}=f(t,\ x)x+g(t,\ x)u \tag{24}$$

with the continuous feedback

$$u=r\ (t,\ x)\ x \tag{25}$$

where the finite-dimensional vector-valued functions x and u are the system output and input, respectively; the functions $f(t,\ x)$, $g(t,\ x)$ and $r(t,\ x)$ satisfy the uniqueness and existence theorems. In particular, we assume that they are continuous and bounded on the real half-axis $[0, +\infty)$. Using some time-discreteness let us introduce the discrete feedback

$$u_k=r\ (t_k,\ x_k)x_k; \qquad t\in[t_k,\ t_{k+1}). \tag{26}$$

The structure of the system (24)-(25) is the key moment that ensures the following assertion for every interval $[t_k,\ t_{k+1})$.

Lemma 2. *The function* $z(t) = \|x - x_k\|$ *has the estimation*

$$z(t) \le (F + GR)(t - t_k)\exp\left[F(t - t_k)\right]\|x_k\| \tag{27}$$

where for all $t \in [0, \infty)$ *and arbitrary continuous vector-functions* x

$$F = \sup_{t,x}\|f(t,x)\|, \quad G = \sup_{t,x}\|g(t,x)\|, \quad R = \sup_{t,x}\|r(t,x)\|. \tag{28}$$

Proof. Let us write the system (24) and **(26)** in the integral equation form

$$x - x_k = \int_{t_k}^{t} f(\tau,x)(x-x_k)d\tau + \int_{t_k}^{t}\left[f(\tau,x)+g(\tau,x)r(t_k,x_k)\right]d\tau\, x_k. \tag{29}$$

It follows from (28) and (29) that $z(t) \le \int_{t_k}^{t} Fz(\tau)\,d\tau + \int_{t_k}^{t}(F+GR)\,d\tau\|x_k\|$. Using Gronwall lemma we see that inequality (27) is true.

Moreover it follows from the triangle inequality that there is the next estimation

$$\|x\| \le z(t) + \|x_k\| \le \left\{1+(F+GR)(t-t_k)\exp\left[F(t-t_k)\right]\right\}\|x_k\|. \tag{30}$$

Let the system (24) and (25) be asymptotic stable. Does system (24) with the discrete feedback (26) inherits this property? Unfortunately the above estimations themselves do not guarantee the positive answer as the set $\{x_k\}$ is not defined by the continuous closed-loop system (24)-(25). That is why there it is necessary to use some additional conditions. In particular, we may state the next assertion.

Theorem 2. *Assume that 1. there are the functions*

$$v(t,x) = x^{*}V(t,x)x, \quad w(t,x) = x^{*}W(t,x)x$$

such that on the solutions of the system (24) and (25) there holds the next relation

$$\frac{d}{dt}v(t,x) = -w(t,x) \tag{31}$$

with the inequalities

$$\alpha_1 < V(t,x) < \alpha_2, \quad \beta_1 < W(t,x) < \beta_2, \quad \left\|\mathrm{grad}_x v(t,x)\right\| < \gamma\|x\| \tag{32}$$

where α_1, α_2, β_1, β_2 and γ are positive constants.
2. for all $t \in [0, \infty)$ and arbitrary differentiable vector-functions x there exists the positive constant

$$L = \sup\left\|\frac{d}{dt}r(t,x)\right\|.$$

Then the time-discreteness set such that system (24) with the discrete feedback (26) inherits the asymptotic stability of the system (24) and (25) is not empty. Moreover there exists some sufficiently small positive constant h_1 such that this set involves a time-discreteness $\{t_k\}$ with increments supremum

$$h = \sup_k (t_{k+1} - t_k)$$

satisfies the inequalities

$$0 < h < h_*, \quad 0 < p(h) < 1 - q(h_*)h$$

where the following positive functions

$$q(h_*)=\frac{1}{h_*}\left[1-\exp\left(\frac{-\beta_1}{\alpha_2}h_*\right)\right]\left[1-\frac{\alpha_2}{\beta_1}a(h_*)\right]$$

$$p(h)=\left[1-\frac{\alpha_2}{\beta_1}a(h)\right]\exp\left(\frac{-\beta_1}{\alpha_2}h\right)+\frac{\alpha_2}{\beta_1}a(h)$$

$$a(h)=\frac{\gamma Gh}{\alpha_1}\left[1+hR(F+GR)\exp(Fh)\right]\left[L+R(F+GR)\exp(Fh)\right].$$

Proof. For $t\in[t_k,\ t_{k+1})$ let us represent the system (24) and (26) in the form

$$\dot{x}=[f(t,\ x)+g(t,\ x)r(t,\ x)]\ x+g(t,\ x)\tag{33}$$
$$\{[r(t_k,\ x_k)-r(t,\ x)]\ x_k+r(t,\ x)(x_k-x)\}.$$

Then we have for system (33) from relation (31) that

$$\dot{v}(t,\ x)=-w(t,\ x)+\mathrm{grad}_x v(t,\ x)g(t,\ x)\{[r(t_k,\ x_k)-r(t,\ x)]\ x_k+r(t,\ x)(x_k-x)\}.$$

Inequalities (32), Lemma 1 and the conditions of the theorem under proving imply that

$$\dot{v}(t,x)\le-\frac{\beta_1}{\alpha_2}v(t,x)+\frac{\gamma G(t-t_k)}{\alpha_1}\left\{1+R(F+GR)(t-t_k)\exp\left[F(t-t_k)\right]\right\}$$
$$\left\{L+R(F+GR)\exp\left[F(t-t_k)\right]\right\}v_k$$

where $v_k=v(t_k,\ x_k)$.

Therefore

$$\dot{v}(t,x)\le-\frac{\beta_1}{\alpha_2}v(t,x)+a(h)v_k.$$

Solving this integral inequality we have

$$v_{k+1}\le p(h)v_k.\tag{34}$$

It follows from the inequalities (32) and (34) that for any k

$$\|x_k\|^2\le\frac{v_k}{\alpha_1}\le\frac{p^k(h)}{\alpha_1}v_0.$$

Hence due to inequality (30)

$$\|x\|^2\le\left[1+R(F+GR)h\exp(Fh)\right]^2\frac{p^k(h)}{\alpha_1}v_0\le c\|x_0\|^2\exp(-\alpha t)$$

where c and α are some positive constants not depending on k. Thus x tends to zero for $t\to\infty$.

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