*Journal of Theoretical*
*and Applied Mechanics*
*Sofia 1996, Year XXVI No 4*

# SOLID MECHANICS

## A NEW APPROACH TO DETERMINE THE YIELD LOCI OF HETEROGENEOUS MEDIA

A. Baltov M. Datcheva, R. Iankov

### I. Introduction

An opinion exists that the plastic anisotropy of polycrystalline bodies is induced by the nonhomogeneities on the micro level where the micro inclusions and defects have a place. A proper investigation of such effects needs to connect the level of the materials anisotropy and the inclusions orientation. For periodical nonhomogeneous structures there is a special mathematical technique for an asymptotic analysis called homogenization theory [1], [2], [3]. This theory is successfully applied for the elastic behaviour where the explicit and fixed relation between strains and stresses exists. Unfortunately for elasto-plastic materials the correct transition from micro to macro level by the homogenization procedures is too complicated [4].

Usually additional simplifications have to be made. There are several examples of such efforts to avoid the mathematical difficulties connected with the application of the homogenization procedure. The approximate method suggested in [4] supposes a constant plastic micro strain in each of the components (so-called a Generalized Standard Material) of the heterogeneous media. The other surmountable case is when the perfect plastic Hencky material is considered. Then (see f. e. [5]) the results are received in the frame of the deformation theory of plasticity and hence they are restricted to cases where this theory has a place. Obviously it is important to construct the homogenization procedure in frame of the flow rule plasticity. In the rigid-plastic case for the practical determination of the macroscopic yield locus a limit analysis

problem on the unit cell $Y$ is commonly required to be solved: the loading parameters are the 6 independent components of the macro stress tensor [4]. The macro yield locus $P^{hom}$ is defined through the micro one $P(\mathbf{y})$:

$$P^{hom}=\left\{\Sigma_{ij}:\exists\sigma_{ij}\in P(\mathbf{y}),\ \forall\mathbf{y}\in Y,\ div\sigma_{ij}=0,\ \sigma_{ij}n_j\Big|\partial Y-antip.,\ \left\langle\sigma_{ij}\right\rangle=\Sigma_{ij}\right\}.$$

Here <*> is the volume average of the variable inside. Most of the macro quantities are the volume average of the micro one. This assertion suffers, as it had been shown, counter exceptions: elastic coefficients, plastic strain, plastic work. Further the macro and micro fields are respectively denoted by capital and minor letters ($\Sigma_{ij}, E_{ij}$ — macro stress and strain tensors, $\sigma_{ij}$, $\varepsilon_{ij}$ — micro stress and strain tensors and etc.).

In [6] the yield locus has been constructed using the limit analysis technique and the concept of the "stress mode". The stress mode is a fixed direction in the $\Sigma$-stress space and the proportional loading process is realized till the point on the macro yield locus has been reached. The method is based on the supposition that the existence of the limit on the micro level leads to the existence of the limit on the macro level. These authors have investigated the heterogeneous structure with a basic cell which is a thin square plate with a rectangular cuboid as an inclusion.

When the homogenization procedure is applied the evaluation of the actual elastoplastic macro constitutive law requires the knowledge of an infinite number of internal variables namely the whole set of inelastic microstrains. An efficient solution of the limit analysis problem can be obtained when the macro-strain rate (in sense of perfect plasticity) can be kept constant. In this case the limit analysis is realized dashing $t$ (the relevant time parameter in the flow rule) — the only parameter of the problem — at an infinite.

This paper concerns the establishment of a method to predict the plastic anisotropy of a two-component on the micro level elasto-plastic material. The investigation is subjected to the following simplifying acceptances:

(1) The structure is a periodic one related to the chosen representative micro structural volume element $Y$ (r.v.e.) and hence the existence of a small parameter $\delta$ is supposed. The process of deformation is examined on two levels — macro and micro and hence the problem contains two scales: the "macroscopic" scale $\mathbf{x}$ (called "slow" coordinates) on which the macro characteristics of the material depend and the "microscopic" scale $\mathbf{y}=\frac{\mathbf{x}}{\delta}$ (called "fast" coordinates) on which the micro quantities depend;

(2) The displacements and strains are small and the assumptions of the linear geometric mechanics have a place. On the micro level material is supposed to be plastically incompressible;

(3) The mechanical properties of the matrix and the inclusion are different. On the micro level the materials of the components are elastoplastic and isotropic with

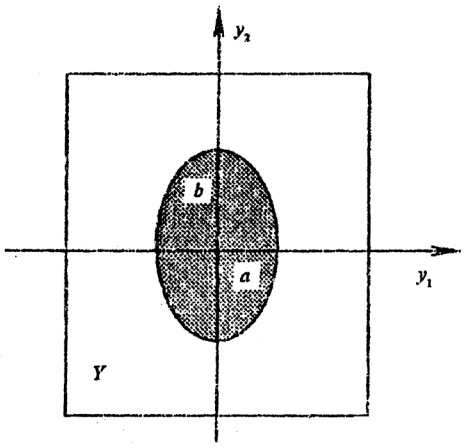



Fig. 1 Basic cell

different elastic characteristics and yield functions for the matrix and the inclusion.

The yield surface will be found through numerical experiments for the given micro structure and given material properties of the matrix and inclusion. In reality an effort to avoid the mathematical difficulty because of the absence of fixed relations between stresses and strains in inelastic case is made.

As an application the problem for a thin square plate subjected to a plane stress state is solved. The basic cell is composed of a matrix and centrally posed elliptical inclusion (see Fig. 1). The parameter $r=a/b$ ($a$, $b$ — semiaxes of the ellipse) is used to characterize the orientation of the structure.

The elastic and plastic anisotropy on the macro level will be shown strongly to depend on inclusions orientation. Inelastic macro characteristics giving a notion about the influence of the local structure on the heterogeneous quantities are searched for without passing the complete homogenization procedure.

## II. Relations between mechanical measures on macro and micro levels

On the micro level the basic equations for the representative volume element $Y$ are:

— The equilibrium equation:

$$\sigma_{ij,y_j} = 0, \quad (i,j = 1, 2, 3). \tag{1}$$

— The equation of geometry:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,y_j} + u_{j,y_i}\right), \tag{2}$$

where $u_i$ ($i$=1, 2, 3) is the displacement vector and $\varepsilon_{ij}$ is the strain tensor;

— The constitutive equations are:

$$\begin{aligned} \varepsilon_{ij} &= \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p}, \\ \sigma_{ij} &= C_{ijkl}^{0}\varepsilon_{kl}^{e}, \\ d\varepsilon_{ij}^{p} &= d\lambda \frac{\partial F}{\partial \sigma_{ij}}, \end{aligned} \tag{3}$$

where $C_{ijkl}^{0}(\mathbf{y})$ is the elastic modules tensor on the micro level, $\varepsilon_{ij}^{e}$ and $\varepsilon_{ij}^{p}$ are the elastic and plastic parts of the strain, the factor $d\lambda(\mathbf{y})$ is a plastic flow parameter and $F(\mathbf{y})$ is the yield function of the materials creating the volume element $Y$.

In the present investigation a method for homogenization is used which requires that the stress and strain fields are $Y$ — periodical functions [4]. That means:
1. For the displacement field the following decomposition is valid:

$$u_i = v_i + E_{ij}y_j + c_i, \quad (i = 1, 2, 3), \tag{4}$$

where $\mathbf{v}$ is a periodic vector function of $\mathbf{y}$; $\mathbf{c}$ is a constant vector characterizing the displacement field of the cell as a rigid body and may be missed without loosing the generality, $E_{ij} = \frac{1}{2}\left\langle u_{i,y_j} + u_{j,y_i}\right\rangle$.

2. For the points $M$ and $N$ situated on the opposite sides of the unit cell $Y$ the equalities have a place:

$$\begin{aligned} \sigma_{ij}(M) &= \sigma_{ij}(N), \\ \sigma_{ij}n_j(M) &= -\sigma_{ij}n_j(N), \end{aligned} \tag{5}$$

where $\mathbf{n}$ is the normal vector to the $Y$ boundary — $\partial Y$.
In the linear elastic case it is possible to represent the displacement field through the set of elementary desplacements $w_i^{kl}$:

$$u_i = E_{kl}w_i^{kl}. \tag{6}$$

The elementary displacement fields are induced by the unit macrostrains $I_{ij}^{kl} = \frac{1}{2}\left(\delta_i^k\delta_j^l + \delta_j^k\delta_i^l\right)$ ($k$, $l$=1, 2, 3 fixed for each of the six unit problems). The strain and the stress localization tensors are defined by:

$$B_{ij}^{kl} = \varepsilon_{ij}\left(\mathbf{w}^{kl}\right), \quad A_{ij}^{kl} = C_{ijpq}^{0}B_{pq}^{kl}. \tag{7}$$

The properties of the localization tensors are detailed discussed in f. e. [7]. The problem to find $B_{ij}^{kl}$ is formulated f. e. in [8].

The variation formulation of the problem to find the fields $\mathbf{w}^{kl}$ ($k$, $l$=1, 2, 3) and respectively the tensor $B_{ij}^{kl}$ is:

Find $\mathbf{w}^{kl}$ ($k$, $l$=1, 2, 3 fixed for each problem) such that:

$$
\begin{aligned}
& w_i^{kl} = v_i^{kl} + I_{ij}^{kl} y_j, \\
& \int_Y C_{ijpq}^0 \varepsilon_{pq}\left(\mathbf{v}^{kl}\right) \varepsilon_{ij}(\partial \mathbf{v}) d_{\mathbf{y}} = \int_S \left[C_{ijkl}^0\right] \partial v_i n_j \, dS,
\end{aligned} \tag{8}
$$

for every $\partial \mathbf{v}$ periodic function of $\mathbf{y}$.

It is supposed that the inclusion (media *I*) and the matrix (media *II*) have constant elastic characteristics and:

$$
\left[C_{ijkl}^0\right] = C_{ijkl}^0(\textit{media II}) - C_{ijkl}^0(\textit{media I}). \tag{9}
$$

With the help of the finite elements method, for the 2D strain state and the circular inclusion, the problem has been discussed in [8].

To find the average characteristics of the heterogeneous structure the procedure to express them through localization tensors is used [7].

Using the localization tensors for given macro strain and stres fields we can find the corresponding micro fields $\sigma_{ij}^{(el)}$ and $\varepsilon_{ij}^{(el)}$ such that:

$$
\begin{aligned}
& \sigma_{ij}^{(el)}(\mathbf{y}) = A_{ij}^{kl}(\mathbf{y}) \Sigma_{kl}, \\
& \varepsilon_{ij}^{(el)}(\mathbf{y}) = B_{ij}^{kl}(\mathbf{y}) E_{kl}, \\
& \sigma_{ij}^{(el)}(\mathbf{y}) = C_{ijkl}^0(\mathbf{y}) \varepsilon_{ij}^{(el)}(\mathbf{y}), \\
& \Sigma_{ij} = \left\langle \sigma_{ij}^{(el)} \right\rangle, \quad E_{ij} = \left\langle \varepsilon_{ij}^{(el)} \right\rangle.
\end{aligned} \tag{10}
$$

Than the true microstresses and strains may be represent as:

$$
\begin{aligned}
& \sigma_{ij} = \sigma_{ij}^{(el)} + r_{ij}, \\
& \varepsilon_{ij} = \varepsilon_{ij}^{(el)} + \varepsilon_{ij}^{*}.
\end{aligned} \tag{11}
$$

Therefore the residual stress $r_{ij}$ is connected with the appearance of a strain $\varepsilon_{ij}^{*}$:

$$
\varepsilon_{ij}^{*} = \left(C_{ijkl}^0\right)^{-1} r_{kl} + \varepsilon_{ij}^{p} = \varepsilon_{ij} - \left(C_{ijkl}^0\right)^{-1} \sigma_{ij}^{(el)}, \tag{12}
$$

which fulfills the conditions of the strain compatibility.

The conclusion is that in the elastoplastic case the macro equations depend on an infinite number of internal parameters. Really, if we admit the existence of the plastic potential for the homogenized media this potential will depend on the self equilibrium stress field (see also [9]).

### III. The yield surface for the homogenized structure

The aim of our investigation is to find the qualitative influence of the microstructure anisotropy on the plastification of the heterogeneous material. The method proposed to find the yield function uses only the connection between macro and micro plastic strains and the localization tensor. That is a possibility to avoid the difficulties of the limit analysis or the restrictions on the materials response. For the plastic part of the macro strain the following is held:

The periodic conditions, (2), (3) and (10) give that the plastic part of the macrostrain is connected by $\varepsilon_{ij}^{p}$ as follows:

$$E_{ij}^{p} = \left\langle A_{ijkl}\varepsilon_{kl}^{p} \right\rangle. \tag{13}$$

This result is a classic one for the homogenization theory in elasto-plastic case [10]. To characterize the macro plastic strain tensor we use its Euclidean norm:

$$E^{p} = \left\| E_{ij}^{p} \right\|_{2} = \left\| \left\langle A_{ijkl}\varepsilon_{kl}^{p} \right\rangle \right\|_{2}. \tag{14}$$

Therefore it is possible to find the point $\Sigma$ in $\{\Sigma_{ij}\}$ space corresponding to the given micro plastic state assessing the $E^{p}$. This point $\Sigma$ is assuming to belong to the initial macro yield locus. Thus it is possible to have a plastic flow on the macrolevel and uncompleted plastification on the microlevel. In [11], for the polycrystalline structures, the yield surface have been searched estimating the strain but the applied method requires the micro strain to be a constant ($\varepsilon_{ij}=E_{ij}$).

It will be shown that using the $E^{p}$-estimation it is easy to observe the influence of the size, geometry and the orientation of the micro inclusions on the plastic properties of the heterogeneous material.

### IV. An algorithm to define the yield loci

The following algorithm is proposed to find the plastic part of the macrostrain tensor:

Step 1. Find $\mathbf{w}^{kl}$ ($k$, $l$ fixed for each problem) such that:

$$\begin{aligned} & w_i^{kl} = v_i^{kl} + I_{ij}^{kl} y_j, \\ & \int_Y C_{ijpq}^{0} \varepsilon_{pq}\left(\mathbf{v}^{kl}\right) \varepsilon_{ij}(\partial \mathbf{v})\, d\mathbf{y} = \int_S \left[C_{ijkl}^{0}\right] \partial v_i n_j\, dS, \\ & \left[C_{ijkl}^{0}\right] = C_{ijkl}^{0}(\textit{media II}) - C_{ijkl}^{0}(\textit{media I}), \end{aligned} \tag{15}$$

for every $\partial \mathbf{v}$ periodic function of $\mathbf{y}$. Further we denote: $\left[C_{ijkl}^{0}\right] n_j = F_i^{kl}$. $F_i^{kl}$ have a meaning of forces on the inclusion contour [8].

Step 2. Using the solution of "Step 1" determine the localization tensors from (7).
Step 3. Solve the system (1), (2), (3) with boundary conditions:

$$\sigma_{ij} n_j \big|_{\partial Y} = \Sigma_{ij} n_j \tag{16}$$

Step 4. Determine the plastic part of microstrain tensor.
Step 5. Using the formula (14) calculate $E^p$.
Step 6. Realize a proportional loading until $E^p$ reaches the given limit value (f. e. 0.12%). The corresponding point $\Sigma_{ij}$ is assumed to belong to the initial macro yield locus.

Then approximating the results from "Step 3-6" for different directions in $\{\Sigma_{ij}\}$ space we receive the yield surface.

It is possible to apply this algorithm for arbitrary yield functions on the micro level. A macro plastic flow can be realized without completing the plastification on the microlevel. It will be shown that using the proposed algorithm it is easy to observe the influence of the size of the inclusion, geometry and orientation on the material behaviour.

### V. A numerical example

To demonstrate the applicability of the proposed in the previous section algorithm a thin plate of the heterogeneous material containing periodically distributed elliptical micro inclusions is considered. In the case of **2D** stress state and inclusion's form — ellipse the forces are:

$$F_i^{kl} = -\left[C^0_{klij}\right] \frac{by_j}{a\sqrt{b^2 + a^2 - y_1^2 - y_2^2}}, \tag{17}$$

where $i,j=1, 2$ and $k,l=1, 2$.

Because of the existing symmetry it is obviously that only three problems among standing in "Step 1" need to be solved: **P1**: $k=l=1$; **P2**: $k=l=2$ and **P3**: $k=1, l=2$. The finite element method is used to solve these three problems. The Fig. 2 shows the kinematics boundary conditions (fixed displacements on $\partial Y$) and the applied loads (uniformly distributed on the inclusion's contour forces $F_i^{kl}$).

A four node isoparametric finite element with four Gauss integration points is used. The finite element discretization is shown in Fig. 3. The material characteristics of the matrix and the inclusion (elasticity modules, Poisson's ratio, yield stress) are given in the Table 1.

Following the "Step 1, 2" in every finite element Gauss integration point the strain localization tensor $B_{ij}^{kl}$ and respectively the stress localization tensor $A_{ij}^{kl}$ are calculated.

The finite element mesh used in "Step 3" is the same as in the previous steps. In such a way the points where the localization matrix is calculated are the same where the plastification is checked up.

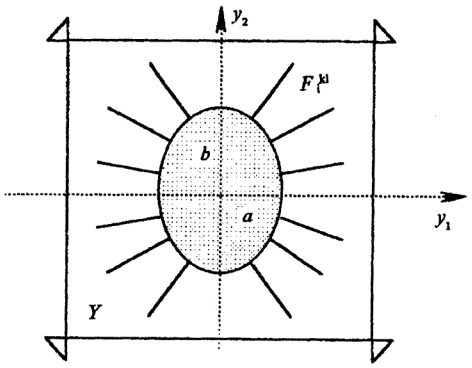



Fig. 2. Kinematic and static boundary conditions for "Step 1-2" problem

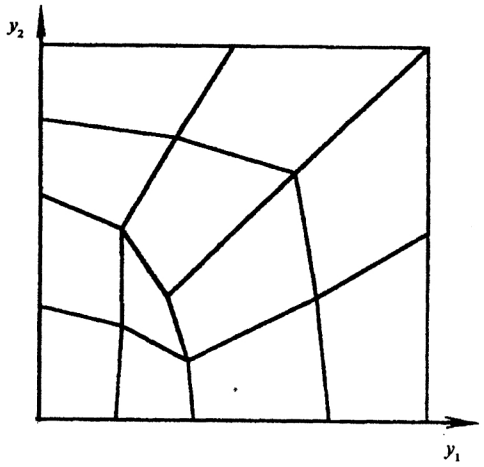



Fig. 3. The finite element discretization

**Table I.** Material constants for the matrix and inclusion.

| | elasticity module [MPa] | Poisson's ratio | yield stress [MPa] |
|---|---|---|---|
| inclusion | 22000 | 0.31 | 210 |
| matrix | 18000 | 0.33 | 200 |

The aim is to find such values of the load for which the plastic macro strain measure $E^p$ reaches 0.12%. A proportional loading is simulated and the elastoplastic problem (1)-(3) with Misses type flow rule is solved until the micro yield locus point is reached. If using (13) and (14) we receive that $E^p$ reaches 0.12% than we can say that the corresponding to the given loading state point in macrostress space $\Sigma_{ij}$ belongs to the macro yield locus. In the numerical procedure $E^p$ is estimated by:

$$E^p = \left\| \left\langle A_{ijkl}\varepsilon^p_{kl} \right\rangle \right\|_2 \leq \frac{1}{V^{fe}} \sum_{fe=1}^{NE} \left\| \left\langle A_{ijkl}\varepsilon^p_{kl} \right\rangle \right\|_2^{fe} dV^{fe} \tag{18}$$

where $\|*\|_2^{fe}$ is an Euclidean norm in the Gauss integration point of the corresponding quantity, $V^{fe}$ is the finite element area, $NE$ is the number of the used finite elements.

Fig. 4. presents the yield surface of the homogenized material for three different values of the inclusions form parameter $r$ ($r$=2.67; 0.67; 1).

The results demonstrate a significant influence of the microstructure on the plastic behaviour of the heterogeneous material. We can approximate the numerically

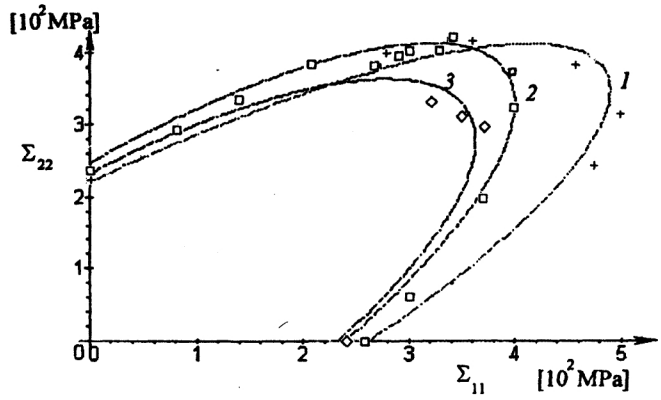



Fig. 4. Yield surfaces: 1(+) — $r$=2.67, 2(□) — $r$=0.67, 3(◇) — $r$=1

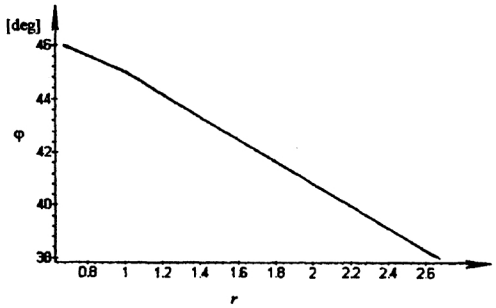



Fig. 5. The yield surface orientation: φ versus $r$

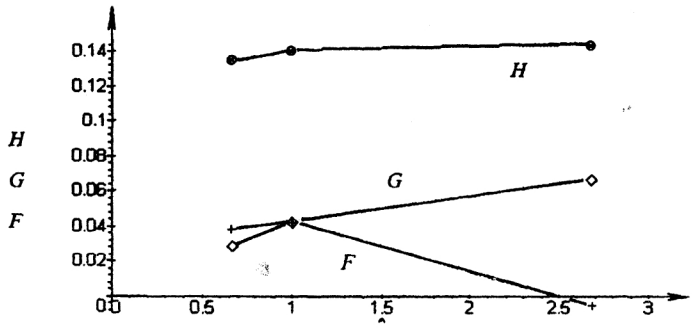



Fig. 6. $H$(○), $G$(◇) and $F$(+) versus $r$

received yield stress points by a central ellipse. When $r$ changes its value the yield locus changes not only its sizes but it is turned on the axis origin (Fig. 4). This result shows that the plastic anisotropy is in an accordance with the inclusion's orientation (Fig. 5). The theoretical results and conclusions drawn in the present work are in a qualitative agreement with the experimental results reported in [12].

The approximation by the central ellipse of the yield locus leads to the well known Hill's model for the orthotropic plasticity in sheet metals. Then in terms of standard parameters $F$, $G$, $H$ [13] the equation of the macro yield loci is:

$$(G+H)\Sigma_{11}^2 - 2H\Sigma_{11}\Sigma_{22} + (F+H)\Sigma_{22}^2 = 1. \tag{19}$$

Fig. 6 presents how the numerically received values for the $F$, $G$, $H$ depend on the ratio $r$. It is seen that predictions of the used method to construct the yield locus are in a good agreement with the Hill's model for the orthotropic plasticity.

## VI. Conclusions

The proposed simple method used to estimate the macro yield locus of the heterogeneous material leads to the reasonable results. This method can be successfully used to model the initial anisotropy due to the microstructure.

## Acknowledgement

Support of this research was provided by the National Foundation "Scientific Research" under the Grant No MM16.

## REFERENCES

[1] Bakhvalov, N. S., G. P. Panasenko. Homogenization in Periodic Media, Nauka, Moscow, 1984.

[2] Sanches-Palencia, E. Homogenization in Mechanics: A Survey of Solved and Open Problems. *Rend. Sem. Met.*, **44(1)** (1986), 1-45.

[3] Telega, J. J. Some Results of Homogenization in Plasticity: Plates and Fissured Solids, In: Inelastic Solids and Structures, (Ed. Kleiber, M., Konig J. A.), 1990.

[4] Suquet, P. M. Local and Global Aspects in the Mathematical Theory of Plasticity, In: Plasticity Today (Ed. Sawczuk, A., G. Bianchi), Elsevier Appl. Sci. Publ., London, 1985.

[5] Jansson, S. Homogenized Nonlinear Constitutive Properties and Local Stress Concentrations for Composites with Periodic Internal Structure. *Int. J. Solids Structures,* **29** (1962), 2181-2200.

[6] Marigo, J.-J., P. Mialon, J.-C. Michel, P. Suquet. Plasticité et homogénéisation: un exemple de prévision des charges limites d'une structure hétérogène périodique. *Int. J. Mécan. Theor. Appliquée,* **6(1)** (1987), 47-75.

[7] SUQUET, P. M. Une méthode duale en homogénéisation: Application aux milieux élastiques. *Journal de Méc. Théor. Appliquée,* **Numero special** (1982), 79-98.

[8] KRUCH, S. Comportement global des Materiaux Copmposites viscoelastiques, PhD thesis, Université Pierre et Marie Curie, 1988.

[9] STOLZ, C. Anélasticité et Stabilité, These de doctorat d'état, Unicersité Pierre et Marie Curie, 1987.

[10] MANDEL, J. Contribution théoretique à l'étude de l'écrouissage et des lois d'écoulement plastique, In: Proc. 11th Int. Congr. Appl. Mech., Springer Verlag, 1964, 502-509.

[11] VAN HOUTTE, P. Calculation of the Yield Locus of Textured Policrystals Using the Taylor and the Relaxed Taylor Theory. *Textures and Microstructures,* **7** (1987), 29-72.

[12] MALLICK, K., K. SHYAM SAMANTA, A. KUMAR. An Experimental Study of the Evolution of Yield Loci for Anisotropic Materials Subjected to Finite Shear Deformation. *ASME Journal of Engineering Materials and Technology,* **113** (1991), 192-198.

[13] HILL, R., J. W. HUTCHINSON. Differential Hardening in Sheet Metal under Biaxial Loading: A Theoretical Framework Composites. *ASME Journal of Applied Mechanics,* **59** (1992), 1-9.