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FRACTURE MACHANICS

COHESIVE QUADRATIC LOAD DISTRIBUTION ARRESTING CRACK OPENING – THE DUGDALE APPROACH

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1. Introduction

The Dugdale model solution for two collinear unequal cracks has been investigated in [1]. A stepwise approximation is used in [2] for obtaining the solution of two collinear straight cracks where the plastic zones developed are closed by variable load distribution over them. The original Dugdale model [3], which assumes constant stresses within the plastic zone ahead of crack tips, has been generalised for the case when a parabolic stress distribution closes the plastic zones ahead of the crack tips by Harrop [4].

In the present paper the problem of two collinear, symmetrically situated straight cracks in an infinite, homogeneous, elastic-perfectly plastic matrix is studied when the crack faces are opened by remotely applied uniform tension acting normal to the rims of cracks. The plastic zone thus formed ahead of cracks tips are then closed by distributing cohesive quadratically varing normal stress distribution over them. Thus the cracks are arrested.

2. The constitutive equations

As is well-known [5], the stress components P_{ij} $(i,j = x,y)$ and displacement components u_i $(i = x,y)$ may be expressed in-terms of two potentials $\phi(z)$ and $\Omega(z)$, of complex variable z, as

(1) $$P_{yy} - iP_{xy} = \phi(z) + \Omega(\bar{z}) + (z-\bar{z})\phi'(z)$$

(2) $$2\mu\left\{\frac{\partial u_x}{\partial x} + i\frac{\partial u_y}{\partial x}\right\} = \kappa\phi(z) - \Omega(\bar{z}) - (z-\bar{z})\phi'(z)$$

where μ is the shear modulus, $\kappa=3-4\sigma$ for plane strain case and $\kappa=(3-\sigma)/(1+\sigma)$ for plane stress case, σ being the Possion's ratio. The bar over the function denotes its complex conjugate operation and prime denotes the differentiation of the function with respect to z.

Let an elastic body occupying the entire xy-plane cut along n segments $L_k=a_kb_k$ ($k=1, 2, \ldots n$) on the real x-axis, the union of these segments $L_k=(k=1, 2, \ldots n)$ is denoted by L. If the stresses $P_{yy}^{\pm}-iP_{xy}^{\pm}$ are prescribed over the rims of crack $L_{\dot{\kappa}}$ then using equation (1) following Hilbert problems are obtained

(3) $$\phi^+(t) + \Omega^-(t) = P_{yy}^+ - iP_{xy}^-, \quad \text{on } L$$

$$\phi^-(t) + \Omega^+(t) = P_{yy}^- - iP_{xy}^-,$$

under the assumption

(4) $$\lim_{y\to 0}\{y\phi'(t+iy)\} = 0,$$

where superscripts + and — refer to the value of the function when approached from upper ($y>0$) and lower ($y<0$) rims of the cuts. And any point on cut L_k, is denoted by t.

The solution of problem in equation (3), in the absence of body forces and stress free infinite boundary may be written, as in [5].

(5) $$\phi(z) = \phi_0(z) + \frac{P_n(z)}{X(z)},$$

and

(6) $$\Omega(z) = \Omega_0(z) + \frac{P_n(z)}{X(z)},$$

where

(7) $$\phi_0(z) = \frac{1}{2\pi i X(z)}\int_L \frac{X(t)p(t)}{t-z}dt + \frac{1}{2\pi i}\int_L \frac{q(t)}{t-z}dt,$$

(8) $$\Omega_0(z) = \frac{1}{2\pi i X(z)}\int_L \frac{X(t)p(t)}{t-z}dt - \frac{1}{2\pi i}\int_L \frac{q(t)}{t-z}dt,$$

where

(9) $$X(z) = \prod_{k=1}^{n}(z-a_k)^{0.5}(z-b_k)^{0.5},$$

and $P_n(z)$ is the polynomial of degree not greater than n

$$P_n(z) = C_0 z^n + C_1 z^{n-1} + \ldots\ldots + C_n, \tag{10}$$

$i = \sqrt{-1}$ and $p(t)$, $q(t)$ are defined as follows

$$p(t) = \frac{1}{2}\left(P_{yy}^{+} + P_{yy}^{-}\right) - \frac{i}{2}\left(P_{xy}^{+} + P_{xy}^{-}\right), \tag{11}$$

$$q(t) = \frac{1}{2}\left(P_{yy}^{+} - P_{yy}^{-}\right) - \frac{i}{2}\left(P_{xy}^{+} - P_{xy}^{-}\right), \tag{12}$$

and C_i (i=1,2,...n) are the constants determined from conditions prescribed at infinite boundary and single-valueless of displacement on cut which could be expressed mathematically as

$$2(\kappa+1)\int_{L_k} \frac{P_n(t)}{X(t)}dt + \kappa\int_{L_k}\left[\phi_0^{+}(t) - \phi_0^{-}(t)\right]dt + \int_{L_k}\left[\Omega_0^{+}(t) - \Omega_0^{-}(t)\right]dt, \tag{13}$$

$$k = 1,2...n.$$

Stress intensity at the cut's tip $z = z_1$ is defined [6], as

$$K = K_1 - iK_2 = 2\sqrt{2\pi}\operatorname*{Lim}_{z\to z_1}\left\{\sqrt{z - z_1}\,\phi(z)\right\}. \tag{14}$$

3. Statement of the problem

An infinite, homogeneous, isotropic, elastic-perfectly plastic matrix, occupies xoy-plane, ref. Fig. 1, cut along two equal, symmetrically situated straight cracks L_1=[$-b,-a$] and L_2=[a,b] on x-axis. The matrix is subjected to remotely applied unidirectional uniform tension, σ_∞, in the direction normal to the rims of the cracks L_1 and L_2. Consequently the rims of the cracks open and the plastic zones Γ_1, Γ_2, Γ_3, Γ_4 are formed ahead of the crack tips $-b$, $-a$, a and b respectively. These plastic

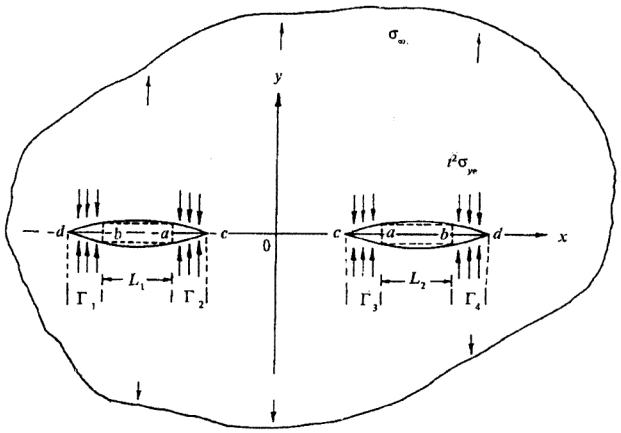


Fig. 1. Configuration

zones occupy the regions $\Gamma_1=[-d,-b]$; $\Gamma_2=[-a,-c]$; $\Gamma_3=[c,a]$ and $\Gamma_4=[b,\ d]$. Each of them is subjected to normal cohesive stress distribution $t^2\sigma_{yc}$, where σ_{yc} is the yield point stress and t is any point on any of the plastic zones.

4. Construction of the solution

The solution of the above problem is obtained by superposing the solutions of two component problems, which contribute towards the stress singularity of the problem stated in section 3. These two problems are appropriately derived from original problem stated in section 3, and are termed as (*i*) *Problem I* and (*ii*) *Problem II*. These are stated and solved below.

Problem I.

A stress free, infinite, homogeneous, isotropic, elastic matrix occupies xoy plane. It contains two straight cracks R_1 $\{=\Gamma_1 \cup L_1 \cup \Gamma_2\}$ and R_2 $\{=\Gamma_3 \cup L_2 \cup \Gamma_4\}$ formed by the union of actual cuts L_1 and L_2 and the relevent plastic zones. The cracks R_1 extends from $-d$ to $-c$ and the crack R_2 runs through c to d. The rims of crack R_1 and R_2 are subjected to unidirectional tension, σ_∞, acting along y-direction.

The solution of this classical problem may directly be written using [5]

$$\phi_1(z)=\frac{\sigma_\infty}{2}\left[\frac{1}{iX(z)}\left\{z^2-\frac{d^2E(k')}{K(k')}\right\}-\frac{1}{2}\right] \tag{15}$$

where $K(k')$, $E(k')$ are complete elliptic integral of first and second kind, respectively, with modulus k'. Subscript 1 denotes that the potential $\phi(z)$ refers to *Problem I*.

$$k'=\left(1-c^2/d^2\right)^{1/2}. \tag{A0}$$

$$X(z)=\left[\left(z^2-c^2\right)\left(d^2-z^2\right)\right]^{1/2}. \tag{16}$$

Problem II.

A homogeneous, isotropic, elastic-perfectly plastic, infinite matrix contains two equal, symmetrical sraight cracks $L_1\{=[-b,-a]\}$ and L_2 $\{=[a,b]\}$ lying on the x-axis. The plastic zone Γ_1 $[-d,-b]$ and Γ_2 $[-a,-c]$ exist ahead of the tips $-b$ and $-a$ of the crack L_1, respectively. Ahead of tips a and b of crack L_2 the plastic zones Γ_3 $[c,a]$ and Γ_4 $[b,d]$ exist. The boundary conditions of the problem are

(a) No stress are acting at infinite boundary of the martix.

(b) Each of the plastic zones Γ_i (i=1, 2, 3, 4) is acted upon by cohesive stress

$$P_{yy}^{\pm}=t^2\sigma_{ye},\ P_{xy}^{\pm}=0,$$

where σ_{yc} is the yield point stress and t is any point on any of the plastic zones.

(c) The rims of the cracks L_1 and L_2 are stress free.

(d) The displacements are single valued on and around the cracks.

Using the conditions (b) and (c), equations (1) and (4), the following Hilbert

problems are obtained

$$[\phi_2(t)+\Omega_2(t)]^+ + [\phi_2(t)+\Omega_2(t)]^- = t^2\sigma_{ye}, \tag{17}$$

on L

$$[\phi_2(t)-\Omega_2(t)]^+ - [\phi_2(t)-\Omega_2(t)]^- = 0, \tag{18}$$

where $L = \bigcup_{i=1}^{4} \Gamma_i$. Subscript 2 after potential ϕ and Ω denotes that these functions refer to *Problem II.*

The solution of above equations written using equations (5) and (6)

$$\phi_2(z) = \phi_0(z) + \frac{C_0 z^2 + C_1 z + C_2}{X(z)}, \tag{19}$$

$$\phi_2(z) = \Omega_2(z) \tag{20}$$

where

$$\phi_0(z) = \frac{\sigma_{ye}}{2\pi i X(z)} \int \frac{t^2 X(t)}{t-z} dt = \Omega_0(z), \tag{21}$$

$X(z)$ is defined by equation (16), constants C_0, C_1 and C_2 are yet to be determinec The condition (a) gives, (ref. [5])

$$C_0 = 0. \tag{22}$$

The values of constants C_1 and C_2 are obtained from the condition of singl valueless, by substituting $\phi_0(z)$ and $\Omega_0(z)$ from equation (21) into equation (13) and integrating one obtains

$$C_1 = 0 \tag{23}$$

$$C_2 = \frac{\sigma_{ye} z}{\pi i}\left[-\frac{a^2+2c^2+2d^2}{3a} X(a) + \frac{bX(b)}{3} + \frac{c^2+d^2-z^2}{a} X(a)\right]. \tag{24}$$

Thus the potential $\phi_2(z)$ is completely determined

$$\phi_2(z) = -\frac{\sigma_{ye} z}{\pi i X(z)}\left[\frac{d}{3}\left\{2\left(c^2+d^2\right)H_0 - c^2 G_0\right\} - H_1 d H_0 - \frac{H_2}{d} G_1 - \frac{H_2 z^2}{d\left(d^2-z^2\right)} P_0\right]. \tag{25}$$

where

$$H_0 = E(\nu, q) + E(\lambda, q), \tag{A1}$$

$$H_1 = c^2 + d^2 - z^2, \tag{A2}$$

$$H_2 = H_1 z^2 - c^2 d^2, \tag{A3}$$

$$H_3 = (1 - a^2/d^2)^{0.5}/k', \tag{A4}$$

$$H_4 = (1 - b^2/d^2)^{0.5}/k', \tag{A5}$$

(A6) $$H_5 = (1-z^2/d^2),$$

(A7) $$H_6 = -k'^2/H_5,$$

(A8) $$G_0 = F(\nu, q) + F(\lambda, q),$$

(A9) $$G_1 = F(H_3) - F(H_4) - F(1),$$

(A10) $$P_0 = \Pi(H_6, H_3) - \Pi(H_6, H_4) - \Pi(H_6, 1),$$

(A11) $$\nu = \mathrm{Sin}^{-1}\frac{d}{a}\sqrt{\frac{a^2-c^2}{d^2-c^2}},$$

(A12) $$\lambda = \mathrm{Sin}^{-1}\sqrt{\frac{d^2-b^2}{d^2-c^2}},$$

(A13) $$q=(1-c^2/d^2)^{1/2},$$

k' is same as in equation (AO). $F(\nu, q)$, $E(\lambda, q)$ are the incomplete elliptic integrals of first and second kind and $F(H_3)$ and $\Pi(H_6,H_3)$ are the complete elliptic integrals of first and third kind.

5. Plastic zone size and crack opening displacement

Size of the plastic zones at interior and exterior tip of the crack L_1, for problem stated in section 3, is obtained by superposing the corresponding stress intensity factor (*SIF*) for *Problem I* and *Problem II*, discussed in section 4.

The *SIF* for each *Problem I* and *II* at the interior tip ($x = c$) and the exterior tip ($x = d$) of the extended crack $R_2\{=\Gamma_3 \cup L_2 \cup \Gamma_4\}$ is obtained using equation (14) and appropriate $\phi_i(z)$ ($i = 1, 2$). For *Problem I*, $\phi_1(z)$ is taken from equation (15) while for *Problem II* $\phi_2(z)$ is substituted from equation (25).

Superposing the *SIFs* for *Problem I* and *II* calculated at $x = c$ the following nonlinear equation is obtained

(26) $$d^2\left[\frac{1}{3}\left\{H_0\left(2c^2/d^2-1\right)-c^2/d^2G_0\right\}\right]-\frac{\pi}{2}\left\{\frac{c^2}{d^2}-\frac{E(k')}{K(k')}\right\}=0.$$

At the exterior tip $x = d$, superposing the *SIFs* for *Problem I* and *II* following transcendental equation is obtained

(27) $$d^2\left[\frac{1}{3}\left\{H_0\left(2-c^2/d^2\right)-c^2/d^2G_0\right\}\right]-\frac{\pi}{2}\left\{1-\frac{E(k')}{K(k')}\right\}=0.$$

The non-linear simultaneous equations (26) and (27) are solved approximately for c and d for prescribed $\sigma_\infty/\sigma_{ye}$, b,a. The plastic zone length, P_i, at interior tip of the crack is calculated from $P_i = b - a$ and at exterior tip the plastic zone length, P_e, is obtained from $P_e = d - c$.

For the plastic zone lengths calculated above corresponding crack opening displacement (*COD*) at the crack tips $x = a$ and $x = b$ of crack L_2 are obtained from relation, ref. [4].

$$u_y = \frac{4}{E} Im\left\{\int \Phi(z)dz\right\}, \tag{28}$$

where E is the Young's modulus and Im [ ] denotes the imaginary part of the function in parentheses. The potential $\Phi(z)$ is obtained superposing non-singular terms of $\phi_1(z)$ and $\phi_2(z)$. Substituting values of $\phi_1(z)$ and $\phi_2(z)$ from equations (15) and (25) into equation (28) and integrating the crack face opening displacement u_y is obtained. *COD* at interior tip $x = a$ of the crack L_2 is

$$(u_y)_{x=a} = \frac{4\sigma_\infty}{3\pi E} G_1 c a^2 \left\{\left(a^2/c^2 - 1\right)\left(1 - a^2/d^2\right)\right\}^{1/2}, \tag{29}$$

and *COD* at exterior tip $x = b$

$$(u_y)_{x=b} = \frac{4\sigma_\infty}{3\pi E} G_1 c b^2 \left\{\left(b^2/c^2 - 1\right)\left(1 - b^2/d^2\right)\right\}^{1/2}. \tag{30}$$

6. Numerical example

The above analysis is applied to study qualitative behaviour of *COD* at the interior and exterior crack tips. The variation of load ratio, required to arrest the crack, for prescribed plastic zone length and increasing crack length is studied and plotted.

Fig. 2 depicts the normalized load ratio at the interior crack tip as size of the crack is increased. It is observed from the curves that for a prescribed plastic zone as

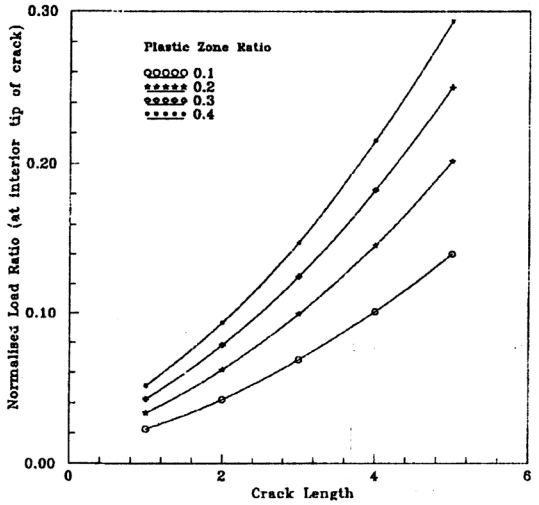


Fig. 2. Variation of normalized load ratio at interior tip of crack versus crack length

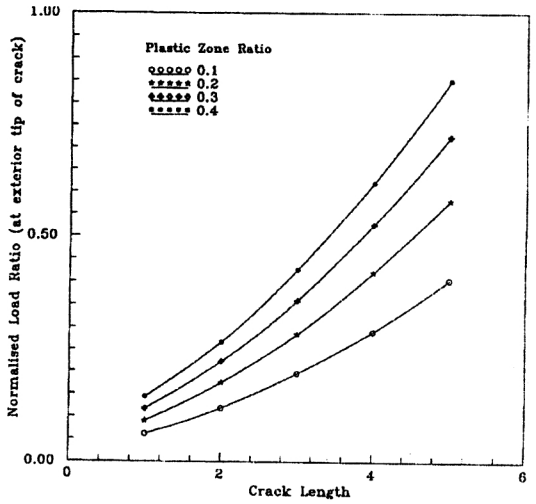


Fig. 3. Variation of normalized load ratio at exterior tip of crack versus crack length

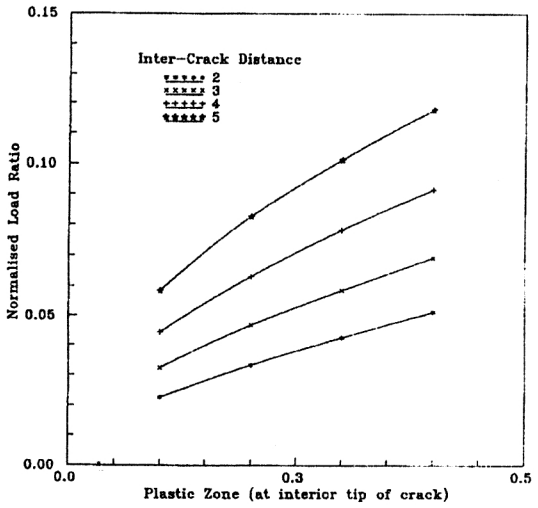


Fig. 4. Variation of normalized load ratio versus plastic zone at interior tip of the crack

the crack length is increased, more load is required to close it, as expected. Also it may be noted that as the plastic zone size is increased, more load is required for their closure and consequently for crack arrest.

The load ratio is plotted against increasing crack length, increments in plastic zone at the exterior tip in Fig. 3 the variation is observed but in this case bigger load

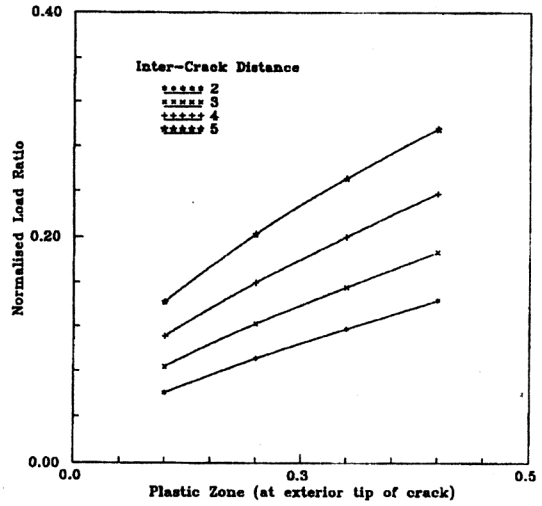


Fig. 5. Variation of normalized load ratio versus plastic zone at exterior tip of the crack

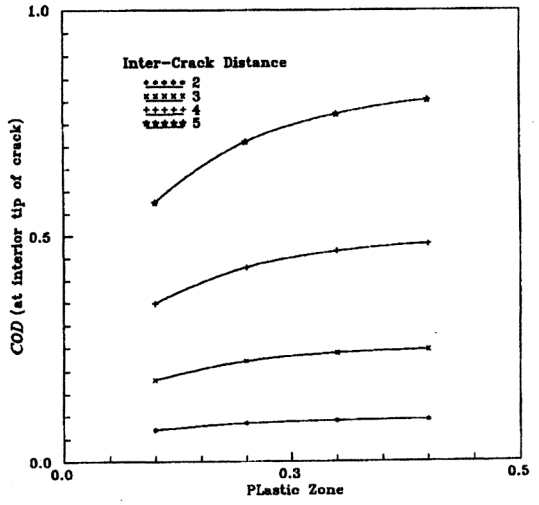


Fig. 6. Variation of *COD* at interior tip of the crack versus plastic zone

is required to close the plastic zone. This may denote the fact that two exterior plastic zones do not effect each other while the two interior plastic zones do affect each other.

Fig. 4 and 5 show the loads required to arrest the crack when plastic zone is increased for fixed crack length. In this case crack length is taken unity. It is seen from these figures that if the cracks are near to each other for a fixed plastic zone the less load is required for closure and while if they are moved apart, more load is

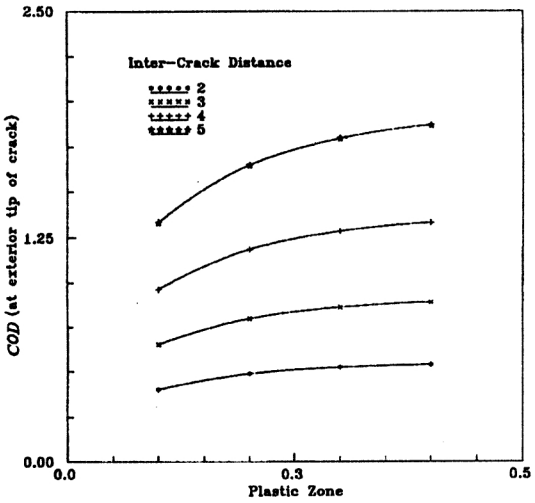


Fig. 7. Variation of *COD* at exterior tip of crack versus plastic zone

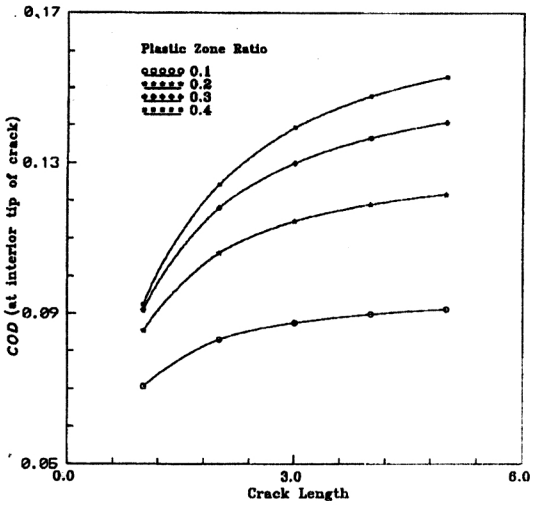


Fig. 8. Variation of *COD* at interior tip of the crack versus crack length

required for their closure. Also it may be noted in Fig. 5 that at the exterior up more load is required for crack arrest.

The *COD* at interior and exterior tips of the crack as crack length is increased, and the distance between them is kept fixed, is plotted in Fig. 6 and 7. It is seen as the crack length is increased the crack opens more at both the tips. But the path taken by *COD* at two tips have different behaviour. This is because of the fact that at the interior tip the two plastic zones tend to coalesces.

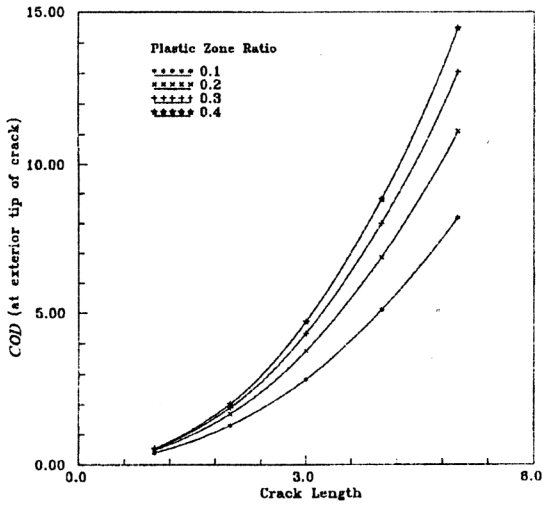


Fig. 9. Variation of *COD* at exterior tip of the crack versus crack length

Fig. 8 and 9 show the variation of *COD* at the interior and exterior tips of the crack, respectively, as the inter-crack distance between the two cracks is increased. It may be noted as the cracks are moved apart, the behaviour of *COD* at interior and exterior tips is similar.

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