БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ
ТЕОРЕТИЧНА И ПРИЛОЖНА МЕХАНИКА
София · 1970, год. I, № 1

BULGARIAN ACADEMY OF SCIENCES
THEORETICAL AND APPLIED MECHANICS
Sofia · 1970, year. I, Np. 1

On the periodic solutions of an autonomous system of differential equations of second order in a critical case

G. Bradistilov, G. Boyadjiev

1. Introduction

We consider the quasilinear autonomous system of differential equations

$$\ddot{x}+\omega^2 x=\varepsilon f(x,\dot{x},\varepsilon), \tag{1.1}$$

where $x=(x_1,\ldots,x_n)$ and $f=(f_1,\ldots,f_n)$ are n-dimensional vectors and $\omega=\mathrm{diag}\,(\omega_1,\ldots,\omega_n)$. The vector f is an analytical function of $x(t)$, $\dot{x}(t)$ and of the small positive parameter ε in a domain G of the space $(x,\dot{x},\varepsilon)$ containing $(x^0,\dot{x}^0.0)$. Here $x^0(t)$ is a solution of the linear system

$$\ddot{x}+\omega^2 x=0, \tag{1.2}$$

which characteristic equation has purely imaginary roots $\pm i\omega$.

It is assumed that f is an odd function with respect to x

$$f(x,\dot{x},\varepsilon)=-f(-x,\dot{x}\,\varepsilon). \tag{1.3}$$

If $x(t)$ is a solution of the equation (1.1), then $-x-(2T-t)$ is a solution of (1.1), too. At $x(0)=0$ the solution $x(t)$ of (1) is periodical with period $2T$, if the equation

$$x(T)=0 \tag{1.4}$$

is fulfilled.

The critical case is considered, when $p-1\,(p\leq n)$ of the frequencies of the linear system (1.2) are multiples of one frequency, for example

$$\omega_\mu=k_\mu\omega_1\,(\mu=2,\ldots,p;\ k_\mu). \tag{1.5}$$

The condition (1.3) for f is used by Hale [1] to find periodical solutions of quasilinear autonomous systems. The method in [1] is based on a convergent process of successive approximations developed by L. Cesari — J. Hale —R. Gambill.

The condition (1.4) of periodicity was applied for the first time to the study of the periodical vibrations of successively connected pendulums by Bradistilov [2] and afterwards in a critical case Bradistilov and Boyadjiev [3] and Manolov [4]. Later on, condition (1.4) underlay papers on systems of differential equations of more general type in critical cases [5, 6].

In this paper we prove the existence of periodical solutions of (1.1) with period $2T=\frac{2\pi}{\omega_1}+o(\varepsilon)$ which correspond to the frequency ω_1. Condition (1.3), which at $x(0)=0$ reduces to the condition (1.4) of periodicity, was used.

2. Existence of periodical solutions

We consider the following particular integral of the linear system (1.2)

$$(2.1)\qquad \begin{cases} x_\nu^0(t)=N_\nu \sin\omega_\nu t, & \dot{x}_\nu^0(t)=\omega_\nu N_\nu \cos\omega_\nu t \quad (\nu=1,\ldots,p), \\ x_s^0(t)=0 & \dot{x}_s^0(t)=0 \qquad (s=p+1,\ldots,n) \end{cases}$$

with period $\frac{2\pi}{\omega_1}$ and initial values

$$(2.2)\qquad \begin{cases} x_\nu^0(0)=0, & \dot{x}_\nu^0(0)=\omega_\nu N_\nu \quad (\nu=1,\ldots,p), \\ x_s^0(0)=0, & \dot{x}_s^0(0)=0 \qquad (s=p+1,\ldots,n). \end{cases}$$

Here $N_1,\ldots,N_p$ are constants, which we are going to determine so that the basic system (1.1) will have a periodical solution, which for $\varepsilon=0$ reduces to (2.1).

Because of $x(0)=0$ we look for an integral $x(t)=x[t,\beta(\varepsilon),\varepsilon]$ of (1.1) with modified initial conditions instead of (2.2) as follows

$$(2.3)\qquad \begin{cases} x_\nu(0)=0, & \dot{x}_\nu(0)=\omega_\nu N_\nu+\beta_\nu(\varepsilon) \quad (\nu=1,\ldots,p), \\ x_s(0)=0, & \dot{x}_s(0)=\beta_s(\varepsilon) \qquad (s=p+1,\ldots,n). \end{cases}$$

Additionally we shall find $\beta(\varepsilon)=(\beta_1(\varepsilon),\ldots,\beta_n(\varepsilon))$, so that $\beta(0)=0$. As the system (1.1) is autonomous, we suppose $\beta_1=0$.

The integral $x(t)$ of (1.1) for sufficiently small values of ε and $\beta(\varepsilon)$ is represented according to Poincare's method by a series in the form of

$$(2.4)\qquad x_\nu(t)=x_\nu^0(t)+\frac{\beta_\nu}{\omega_\nu}\sin\omega_\nu t+\varepsilon\left\{F_\nu(t)+\sum_{\mu=2}^{n}G_{\nu\mu}(t)\beta_\mu+\cdots\right\}$$

$$(\nu=1,\ldots,n),$$

where for the functions $F_\nu(t)$ and $G_{\nu\mu}(t)$ we find

$$(2.5)\qquad F_\nu(t)=\frac{1}{\omega_\nu}\int_0^t f_\nu[x^0(u),\dot{x}^0(u),0]\sin\omega_\nu(t-u)du,$$

$$(2.6)\qquad G_{\nu i}(t)=\frac{1}{\omega_i}\frac{\partial F_\nu(t)}{\partial N_i} \qquad (\nu=1,\ldots,n;\ i=2,\ldots,p).$$

In order that the system (1.1) may have a periodical solution with period $2T=\frac{2\pi+2\delta(\varepsilon)}{\omega_1}$, it is sufficient that condition (1.4) be fulfilled, i. e. in this case $x\left(\frac{\pi+\delta}{\omega_1}\right)=0$. For $\varepsilon=0$ this period must reduce to the period $\frac{2\pi}{\omega_1}$ of solution (2.1), i. e. $\delta(0)=0$. Using (2.4), the condition of periodicity, after developing into a power series of δ, assumes the form

$$(2.7)\quad \begin{cases} (-1)^{k_i}k_i\delta\left(N_i+\frac{\beta_i}{\omega_i}\right)+\varepsilon\left\{F_i\left(\frac{\pi}{\omega_1}\right)+\sum_{\mu=2}^{n}G_{i\mu}\left(\frac{\pi}{\omega_1}\right)\beta_\mu+\cdots\right\}+\cdots=0, \\ \frac{\beta_s}{\omega_s}\sin\Omega_s+\frac{\delta\beta_s}{\omega_s}\cos\Omega_s+\varepsilon\left\{F_s\left(\frac{\pi}{\omega_1}\right)+\sum_{\mu=2}^{n}G_{s\mu}\left(\frac{\pi}{\omega_1}\right)\beta_s+\cdots\right\}+\cdots=0, \end{cases}$$

$$k_1=1,\quad \Omega_s=\frac{\omega_s\pi}{\omega_1}\quad (i=1,\ldots,p;\ s=p+1,\ldots,n).$$

From the first and last $p+1$ equations of (2.7) we can find $\delta, \beta_s (s=p+1,\ldots,n)$ as analytical functions of $\varepsilon, \beta_i (i=2,\ldots,p)$, which at $\varepsilon=\beta_i=0$ became zero. We substitute δ and $\beta_i (i=2,\ldots,p)$ into the remaining $p-1$ equations of (2.7) and divide by ε. In order to have a solution of β_i, for which $\beta_i(0)=0$, it is necessary that the conditions

$$(2.8)\quad \psi_i(N_1,\ldots,N_p)\equiv(-1)^{k_i}k_i\frac{N_i}{N_1}F_1\left(\frac{\pi}{\omega_1}\right)+F_i\left(\frac{\pi}{\omega_1}\right)=0\quad (i=2,\ldots,p)$$

be fulfilled. Using (2.5) we obtain

$$(2.9)\quad \int_0^{\pi}\left\{k_i^2N_if_1\left|x^0\left(\frac{z}{\omega_1}\right),\dot{x}_0\left(\frac{z}{\omega_1}\right),0\right|\sin z+N_1f_i\left|x^0\left(\frac{z}{\omega_1}\right),\dot{x}^0\left(\frac{z}{\omega_1}\right),0\right|\sin k_iz\right\}dz=0$$

$$(i=2,\ldots,p).$$

A sufficient condition for finding $\beta_i (i=2,\ldots,p)$ from (2.7) is that the following functional determinant differs from zero

$$(2.10)\quad \frac{D(\psi_2,\ldots,\psi_p)}{D(N_2,\ldots,N_p)}\neq 0.$$

The conditions (2.9) form a system of $p-1$ equations (sometimes called amplitude equations) with unknowns $N_1, N_2,\ldots, N_p$. Usually for autonomous systems of differential equations there are as many amplitude equations as is the number of parameters in the solution of the linear system (in our case p). But here we have the interesting case in which $p-1$ amplitude equations are obtained. The reason for this is that the function $f(x,\dot{x},\varepsilon)$ is an odd one after (1.3). From these $p-1$ equations we obtain $N_2,\ldots,N_p$ as functions of $N_1\neq 0$. Thus we find the following.

Theorem. If $N_2^{(m)},\ldots,N_p^{(m)}$ are m real solutions of the amplitude system (2.9) for which the inequality (2.10) is valid, then the system of differential equations (1.1) in the critical case (1.5) has m families of periodical solutions with period $\frac{2\pi+2\delta(\varepsilon)}{\omega_1}$, which depend on the parameter N_1. These periodical solutions are represented by the series (2.4) and for $\varepsilon=0$ reduce to the solutions (2.1) of the linear system (1.2) with period $\frac{2\pi}{\omega_1}$.

3. Construction of the periodical solutions

The periodic solutions of the differential equations (1.1) and its period are analytic functions of ε for the functions f are analytic with respect to $x, \dot{x}$ and ε. Then the period is represented in the form

(3.1) $$\frac{2\pi+2\delta}{\omega_1}=\frac{2\pi}{\omega_1}(1+a_1\varepsilon+a_2\varepsilon^2+\cdots),\quad \delta=\pi(a_1\varepsilon-a_2\varepsilon^2+\cdots),$$

where the coefficients a_1, a_2, ... have to be found.

In the system (1.1) we change the independent variable t with τ by the dependency

(3.2) $$t=\tau(1+a_1\varepsilon+a_2\varepsilon^2+\cdots),$$

after that we obtain

(3.3) $$\ddot{X}+\omega^2X(1+a_1\varepsilon+a_2\varepsilon^2+\cdots)^2=\varepsilon(1+a_1\varepsilon+a_2\varepsilon^2+\cdots)^2f(X,\dot{X}\frac{1}{1+a_1\varepsilon+a_2\varepsilon^2+\cdots},\varepsilon),$$

where $X(\tau)=x(t)=x[\tau(1+a_1\varepsilon+a_2\varepsilon^2+\cdots)]$, $\dot{X}(\tau)=\dot{x}[\tau(1+a_1\varepsilon+a_2\varepsilon^2+\cdots)]$.

The system (3.3) has periodic solution with constant period $\frac{2\pi}{\omega_1}$ which we are looking for in the form of

(3.4) $$X(\tau)=X^{(0)}(\tau)+\varepsilon X^{(1)}(\tau)+\varepsilon^2X^{(2)}(\tau)-\cdots$$

The system (3.3) possesses the same property of periodicity as the system (1.1). The functions f in its right side fulfil the condition of oddicity (1.3). Hence if $X(0)=0$, for the solution (3.4) to be periodical with period $\frac{2\pi}{\omega_1}$ it is sufficient to realize the equality

(3.5) $$X\left(\frac{\pi}{\omega_1}\right)=0 \quad \text{or} \quad X^{(\mu)}\left(\frac{\pi}{\omega_1}\right)=0 \quad (\mu=0,\ 1,\ 2,\ldots).$$

We substitute (3.4) in the equations (3.3) and compare the coefficients before the same degrees of ε, ε^2 and so on. So we find

(3.6) $$\ddot{X}^{(0)}+\omega^2X^{(0)}=0,$$

(3.7) $$\ddot{X}^{(1)}+\omega^2X^{(1)}=f(X^{(0)},\dot{X}^{(0)},\ \varepsilon)-2\omega a_1X^0\ ,$$

(3.8) $$\ddot{X}_\nu^{(2)}+\omega_\nu^2X_\nu^{(2)}=\sum_{\mu=1}^{n}\left(\frac{\partial f_\nu}{\partial X_\mu^{(0)}}X_\mu^{(1)}+\frac{\partial f_\nu}{\partial \dot{X}_\mu^{(0)}}\dot{X}_\mu^{(1)}\right)-\omega_\nu^2(2a_2+a_1^2)X_\nu^{(0)}-2\omega_\nu^2a_1X_\nu^{(1)}$$

$$(\nu=1,\ldots,\ n).$$

From the equation (3.6) by (3.5) readily we obtain

$$X_i^{(0)}(\tau)=N_i\sin\omega_i\tau,\quad X_s^{(0)}(\tau)=0\quad (i=1,\ldots,\ p;\ s=p+1,\ldots,\ n).$$

The general integral (3.7) is

(3.9) $$X_\nu^{(1)}(\tau)=D_\nu\sin\omega_\nu\tau+\xi_\nu^{(1)}(\tau),$$

where

(3.10) $$\xi_\nu^{(1)}(\tau)=\frac{1}{\omega_\nu}\int_0^\tau\left\{f_\nu(X^{(0)},\dot{X}^{(0)},0)-2\omega a_1X_\nu^0\right\}\sin\omega_\nu(\tau-u)du,$$

and $D_\nu(\nu=1,\ldots,\ n)$ are constants.

According to the condition of the periodicity (3.5) of $X_\nu^{(1)}(\tau)$ we have

$$X_\nu^{(1)}\left(\frac{\pi}{\omega_1}\right)=0=\xi_\nu^{(1)}\left(\frac{\pi}{\omega_1}\right)+D_\nu \sin\frac{\omega_\nu\pi}{\omega_1}$$

or

$$(3.11)\quad \xi_i^{(1)}\left(\frac{\pi}{\omega_1}\right)=0,\quad D_s\sin\frac{\pi\omega_s}{\omega_1}+\xi_s^{(1)}\left(\frac{\pi}{\omega_1}\right)=0\quad (i=1,\ldots,\ p;\ s=p+1,\ldots,\ n).$$

In this system from n equations take part the quantities a_1, $N_1,\ldots,$ N_p, $D_{p+1},\ldots,$ D_n. The first p equations of the system (3.11) contain $a_1, N_1,\ldots,$ N_p and after coresponding elaboration, the first equation gives us $a_1=\frac{F_1\left(\frac{\pi}{\omega_1}\right)}{\pi N_1}$. Replacing this value in the following $p-1$ equations we obtain again the system (2.9) for finding the unknown quantities $N_1, N_2,\ldots, N_p$ from where we find $N_2,\ldots, N_p$ as functions of N_1. From the last $n-p$ equations of the system (3.11) we find D_s as functions of the parameter N_1. Therefore from the solution (3.9) remain indefinit the constants $D_1, D_2,\ldots, D_p$, yet. One from this constants in the case D_1 we can find, for the system (1.1) is autonomous and by the prove for existence of the periodic solutions one from the quantity β_ν (modification of the initial conditons of the linear system) in our case β_1 was taken zero. Then from the initial conditions (2.3) by $\nu=1$ we obtain

$$(3.12)\quad \dot X_1(0)=\omega_1N_1(1+a_1\varepsilon+a_2\varepsilon^2+\cdots)=\dot X_1^{(0)}(0)+\varepsilon\dot X_1^{(1)}(0)+\varepsilon^2\dot X_1^{(2)}(0)+\cdots$$

From the equality (3.9) we have $\dot X_1^{(1)}(0)=\omega_1D_1$ and after replacing this term in (3.12) and comparing the coefficients before ε we derive $D_1=N_1a_1$. The other constants $D_2,\ldots, D_p$ shall be found from the following approximation

The general integral of the equation (3.8) by $X_\nu^{(2)}(0)=0$ is

$$(3.13)\quad X_\nu^{(2)}(\tau)=E_\nu\sin\omega_\nu\tau+\xi_\nu^{(2)}(\tau).$$

Here

$$(3.14)\quad \xi_\nu^{(2)}(\tau)=\frac{1}{\omega_\nu}\int_0^\tau\left\{\sum_{\mu=1}^{n}\left(\frac{\partial f_\nu}{\partial X_\mu^{(0)}}X_\mu^{(1)}+\frac{\partial f_\nu}{\partial\dot X_\mu^{(0)}}\dot X_\mu^{(1)}\right)-\omega_\nu^2(2a_2+a_1^2)X_\nu^{(0)}\right.$$

$$\left.-2\omega_\nu^2a_1X_\nu^{(1)}\right\}\sin\omega_\nu(\tau-u)du.$$

According to the condition for periodicity of $X_\nu^{(2)}(\tau)$ we have

$$(3.15)\quad \xi_i^{(2)}\left(\frac{\pi}{\omega_1}\right)=0,\quad E_s\sin\frac{\pi\omega_s}{\omega_1}+\xi_s^{(2)}\left(\frac{\pi}{\omega_1}\right)=0\quad (i=1,\ldots,\ p;\ s=p+1,\ldots,\ n).$$

In this system take part the quantities $a_2, D_2,\ldots, D_p, E_{p+1},\ldots, E_n$, which may be found as functions of the parameter N_1. In the equality (3.13) remain indefinit the quantities $E_2,\ldots, E_p$ because from (3.12) and (3.13) by having $\nu=1$ we derive $E_1=a_2N_1$. They can be found analogically from the following approximation.

If we continue this process we can recon any function $X^{(\omega)}(\tau)$ from (3.4), i. e. to find the approximate periodic solution of the system (33) and to compute the coefficients $a_1, a_2,\ldots,$ which take part in the period $\frac{2\pi+2\delta}{\omega_1}$.

By changing (3.2) we can transform the periodic solution (3.4) in the form of

$$x(t)=x^0(t)+\varepsilon x^{(1)}(t)+\varepsilon^2 x^{2)}(t)+\cdots, \tag{3.16}$$

but here the functions $x^{(\nu)}(t)$ are not periodical with constant period $\frac{2\pi}{\omega_1}$.

4. Example for double physical pendulum

Let us consider the system of differential equations which describe the motion of the double physical pendulum placed in vertical plane [7]

$$\begin{cases} (A_1+I_1)\ddot{\psi}_1+b_1B_2[\cos\lambda(\psi_1-\psi_2)\cdot\psi_2''+\lambda\sin\lambda(\psi_1-\psi_2)\cdot\dot{\psi}^2] \\ \qquad =-gB_1\psi_1+\lambda gB_1\sum_{n=2}^{\infty}(-1)^n\frac{\lambda^{2n-3}\psi_1^{2n-1}}{(2n-1)!}, \\ B_2b_1[\cos\lambda(\psi_2-\psi_1)\cdot\ddot{\psi}_1+\lambda\sin\lambda(\psi_2-\psi_1)\cdot\dot{\psi}_1^2]+(A_2+I_2)\ddot{\psi}_2 \\ \qquad =-gB_2\psi_2+\lambda gB_2\sum_{n=2}^{\infty}(-1)^n\frac{\lambda^{2n-3}\psi_1^{2n-1}}{(2n-1)!}, \end{cases} \tag{4.1}$$

where λ is a small parameter, $\lambda\psi_s$ is the angel which the ν-th pendulum ($\nu=1,2$) closed with the vertical axis, I_ν — the inertial moment of the ν-th pendulum with respect to the center of the gravity and $A_1=a_1^2m_1+b_1^2m_2$, $A_2=a_2^2m_2$, $B_1=a_1m_1+b_1m_2$, $B_2=a_2m_2$.

Here a_ν is the distance between the axis of the revolution and the center of gravity of the ν-th pendulum, m_ν its mass and b_1 — the distance between the axes of the revolutions of the two pendulums.

In order to transform the system (4.1) to the type (1.1) we replace

$$\psi_\nu=L_{\nu1}x_1+L_{\nu2}x_2, \quad (\nu=1,2), \tag{4.2}$$

where $L_{\nu\mu}$ are the solutions of the system

$$\begin{cases} [(A_1+I_1)\omega_\mu^2-gB_1]L_{1\mu}+b_1B_2\omega_\mu^2L_{2\mu}=0, \\ b_1B_2\omega_\mu^2L_{1\mu}+[(A_2+I_2)\omega_\mu^2-gB_2]L_{2\mu}=0, \end{cases} (\mu=1,2), \qquad \begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix}\neq 0$$

and $\omega_\nu(\nu=1,2)$ are the roots of the characteristic equation

$$\begin{vmatrix} (A_1+I_1)\omega^2-gB_1 & b_1B_2\omega_2 \\ b_1B_2\omega^2 & (A_2+I_2)\omega^2-gB_2 \end{vmatrix}=0.$$

After developing $\cos\lambda(\psi_1-\psi_2)$ and $\sin\lambda(\psi_1-\psi_2)$ by the exponents of λ and replacing (4.2) in (4.1) we obtain the following system

$$
(4.3)\quad \begin{cases} \ddot{x}_1+\omega_1^2 x_1=\dfrac{\varepsilon\omega_1^2}{gB_1}\Big\{b_1B_2\Big[\dfrac{1}{2}((L_{11}-L_{21})x_1+(L_{12}-L_{22})x_2)^2(L_{21}\ddot{x}_1+L_{22}\ddot{x}_2) \\ \qquad -((L_{11}-L_{21})x_1+(L_{12}-L_{22})x_2)(L_{21}\dot{x}_1+L_{22}\dot{x}_2)^2\Big] \\ \qquad +\dfrac{gB_1}{6}(L_{11}x_1+L_{12}x_2)^3\Big\}+\ldots \\ \ddot{x}_2+\omega_2^2 x_2=\dfrac{\varepsilon\omega_2^2}{gB_2}\Big\{b_1B_2\Big[\dfrac{1}{2}(L_{11}-L_{21})x_1+(L_{12}-L_{22})x_2)^2(L_{11}\ddot{x}_1+L_{12}\ddot{x}_2) \\ \qquad +((L_{11}-L_{21})x_1+(L_{12}-L_{22})x_2)(L_{11}\dot{x}_1+L_{12}\dot{x}_2)^2\Big] \\ \qquad +\dfrac{gB_2}{6}(L_{21}x_1+L_{22}x_2)^3\Big\}+\ldots,\ \varepsilon=\lambda^2, \end{cases}
$$

which is a particular case from (1.1) and for which the condition (1.3) holds. We suppose that the equality (1.5) is fulfilled, i. e. $\omega_2=k\omega_1$ ($k>1$ and integral number).

The amplitude system (2.9) in this case in one equation is reduced. After solving the corresponding integrals it is seen that this equation is from third exponent with respect to N_2

$$
(4.4)\qquad \psi_2(N_1,N_2)\equiv aN_1N_2^3+bN_1^2N_2^2+cN_1^3N_2+dN_1^4=0
$$

with coefficients complex terms in which the parameters definiting the double pendulums take place. In dependency from the values of k we have the following cases:

1. By $k\neq 3$ the coefficients b and d of the equation (4.4) are zero. Hence it has one root zero for which we verify that $\frac{\partial\psi_2}{\partial N_2}\neq 0$, i. e. the condition (2.10) is fulfilled. Then the double physical pendulum has periodical solutions which correspond to the frequency ω_1 and approximately are with

$$
x_1(t)\approx N_1\sin\omega_1 t,\quad x_2\approx 0
$$

represented. If the left two roots $N_2^{(2,3)}$ are real and for their $\frac{\partial\psi_2}{\partial N_2}\neq 0$, for the frequency ω_1 exist still two families periodical solutions which approximately are with

$$
(4.5)\qquad x_1(t)\approx N_1\sin\omega_1 t,\quad x_2(t)\approx N_2\sin k\omega_1 t
$$

represented.

2. By $k=3$ the free term d of the equation (4.4) in general is unequal to zero and then it has one or three real roots. If for their $\frac{\partial\psi_2}{\partial N_2}\neq 0$ thus the double pendulum possesses periodical solutions from the form of (4.5).

References

1. Hale J. K., Sufficient Conditions for the Existence of Periodic Solutions of Systems of Weakly Nonlinear First and Second Order Differential Equations, J. Math. Mech., 2, 1958.
2. Bradistilov G., Über periodische und asymptotische Lösungen beim n-fachen Pendel in der Ebene, Math. Annalen, Bd. 116, H. 2, 1938.
2. Bradistilov G. und G. Boyadjiev, Existenz periodischer Bewegungen eines n-fachen Pendels im Falle, daß einige Wurzeln seiner charakteristischen Gleichung ein Vielfaches einer anderen sind, ZAMM, Bd. 39, H. 7/8, 1959.
4. Манолов С., О существовании малых периодических движений вокруг положения относительного устойчивого равновесия одной механической системы, ПММ, т. 19, 1955.
5. Брадистилов Г. и Г. Бояджиев, О периодических движениях одной консервативной системы с n степеней свободы в окружности равновесного положения, Zagad. dr. nieliiowych, 6, 1964.
6. Манолов С., Периодични решения при една класа от автономри нелинейни системи и и някои приложения, Год. на ВТУЗ, Математика, т. II, кн. 2, 1965.