

Boundary layer growth on a yawed circular cylinder

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The effects due to sweepback of wings in modern aircraft have provided a practical need for study of the boundary layer flow over yawed cylinders. The flow over a yawed infinite cylinder may be determined by the superposition of two flows, one lying in planes normal to the axis, and the other parallel to the generators of the cylinder. Prandtl [1] and Struminsky [2] have studied the stationary case of such flows in the papers on three-dimensional boundary layers. Wundt [3] has discussed the early stages of the boundary layer growth on yawed cylinders when the potential velocity varies as the n -th power of time. All the above investigations were based on the boundary layer approximation and on the assumption that the irrotational flow field outside the boundary layer is that produced by the corresponding body in an inviscid fluid.

Thus, the slope of the inviscid outer flow due to the boundary layer was neglected.

It is worthwhile considering the higher-order solution for the growth of the boundary layer on an infinite yawed circular cylinder (Fig. 1) in spirit of higher-order boundary layer theory as it is presented in Van Dyke's book [4]. We assume [5, 6] large Reynolds number R , and short time of investigation, so that

$$\varepsilon = \frac{1}{\nu' R} = \frac{1}{S} \ll 1,$$

where ν' is a constant of order unity and S is the Strouhal number.

The aim of this paper is to extend the two-dimensional solution [5] by calculating the spanwise component of velocity in the layer. We can normalize the velocities by a characteristic velocity U_∞ , the length by the cylinder radius a , and the time by $\varepsilon \frac{a}{U_\infty}$. We suppose that the velocity component of the external flow in direction to the generators of the cylinder is a constant — W^0 . Then the Navier — Stokes equations became:

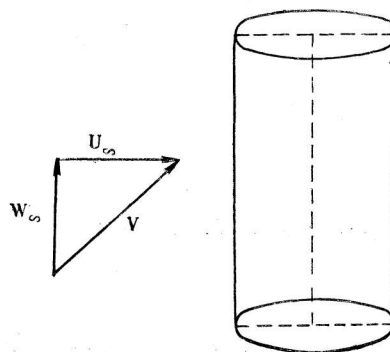


Fig. 1

$$(1) \quad \frac{\partial u}{\partial t} + \varepsilon \left(\frac{u}{r} \frac{\partial u}{\partial \theta} + v \frac{\partial u}{\partial r} + \frac{uv}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \nu' \varepsilon^2 \left(\nabla^2 u + \frac{2}{r^2} \frac{\partial v}{\partial \theta} - \frac{u}{r^2} \right),$$

$$(2) \quad \frac{\partial v}{\partial t} + \varepsilon \left(\frac{u}{r} \frac{\partial v}{\partial \theta} + v \frac{\partial v}{\partial r} - \frac{u^2}{r} \right) = -\frac{\partial p}{\partial r} + \nu' \varepsilon^2 \left(\nabla^2 v - \frac{v}{r^2} - \frac{2 \partial u}{r^2 \partial \theta} \right),$$

$$(3) \quad \frac{\partial w}{\partial t} + \varepsilon \left(\frac{u}{r} \frac{\partial w}{\partial \theta} + v \frac{\partial w}{\partial r} \right) = \nu' \varepsilon^2 \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right),$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$.

Here the pressure is normalized with respect to $\varepsilon^{-1}(\rho U_\infty^2)$. We use a cylindrical polar coordinate system (r, θ, z) which travels with the cylinder in (r, θ) -plane, so that the external rotational flow is of the form

$$(4) \quad U^0(r, \theta) = U_\infty \left(1 + \frac{1}{r^2} \right) \sin \theta.$$

The initial and boundary conditions to be satisfied are $u=v=w=0$ at $r=1$ and finite $t \geq 0$ and u, w matches the outer flow variables U^0, W^0 in an intermediate overlap region. We may satisfy the mass-conservation equation

$$(5) \quad \frac{\partial u}{\partial \theta} + \frac{\partial}{\partial r}(rv) = 0$$

by writing

$$(6) \quad u = \frac{\partial \psi}{\partial r}; \quad v = -\frac{1}{r} \frac{\partial \psi}{\partial \theta}.$$

The comparative simplicity of the mathematical treatment of the three dimensional laminar flow over yawed cylinders arises from the so called "independence principle".

An examination of the equations shows that u and v can be calculated from (1) and (5), independently of the component w and that subsequent substitution in equation (3) yields to an equation for w alone. In fact, the solution for u and v is the same as in the corresponding two-dimensional case. The stream function of this flow is of the form

$$(7) \quad \begin{aligned} \psi = & \sin \theta \left(r - \frac{1}{r} \right) + \varepsilon 4 \sqrt{\frac{\nu' t}{\pi}} \sin \theta \left(-\frac{1}{r} + \exp(-r_1^2) - \sqrt{\pi} r_1 \operatorname{erfc} r_1 \right) \\ & + \varepsilon 24 \frac{\nu' t}{\sqrt{\pi}} \sin \theta \left[-\frac{\sqrt{\pi}}{4} (1 - \operatorname{erfc} r_1) + \frac{3\sqrt{\pi}}{2} r_1^2 \operatorname{erfc} r_1 - \frac{3}{2} r_1 \exp(-r_1^2) \right] \\ & + \varepsilon^2 8 \sqrt{\nu' t} \cdot t \sin \theta \cdot \cos \theta \int_0^{r_1} f(r_1) dr_1 + O(\varepsilon^3), \end{aligned}$$

where $r_1 = \frac{r-1}{2\varepsilon\sqrt{\nu' t}}$, $\operatorname{erfc} r_1 = \frac{2}{\sqrt{\pi}} \int_{r_1}^{\infty} e^{-\beta^2} d\beta$ and

$$\int_0^{r_1} f(r_1) dr_1 = \frac{11}{6\sqrt{\pi}} \exp(-r_1^2) \operatorname{erfc} r_1 - \frac{8}{3\sqrt{2\pi}} \operatorname{erfc}(\sqrt{2} r_1) + \frac{r_1^3}{3} \operatorname{erfc}^2 r_1$$

$$\begin{aligned}
& -\frac{2}{3\sqrt{\pi}} r_1^2 \exp(-r_1^2) \operatorname{erfc} r_1 + \frac{1}{3\pi} r_1 \exp(-2r_1^2) - \frac{r_1}{2} \operatorname{erfc}^2 r_1 + \frac{1}{\sqrt{\pi}} \left(\frac{4}{9\pi} - \frac{3}{2} \right) \exp(-r_1^2) \\
& - \left(1 + \frac{4}{9\pi} \right) r_1^3 \operatorname{erfc} r_1 + \frac{1}{\sqrt{\pi}} \left(1 + \frac{4}{9\pi} \right) r_1^2 \exp(-r_1^2) + \frac{2}{3\sqrt{\pi}} \operatorname{erfc} r_1 \\
& + \left(\frac{1}{2} - \frac{2}{3\pi} \right) r_1 \operatorname{erfc} r_1 + \frac{1}{\sqrt{\pi}} \left(\frac{8}{3\sqrt{2}} - \frac{4}{9\pi} - 1 \right).
\end{aligned}$$

Equation (3) shows that for the outer flow we have

$$W = W_1 + \varepsilon W_2 + \dots,$$

where

$$W_1 = W^0 \quad \text{and} \quad W_i = 0 \quad (i=2, 3, \dots).$$

For the inner region we set

$$(8) \quad w = w_1 + \varepsilon w_2 + \dots,$$

$$(9) \quad r = 1 + \varepsilon \zeta,$$

where ζ is a new stretched variable.

Substituting (8) into equation (3) and using that

$$\psi_1 = 4\sqrt{\nu' t} \sin \theta \left[r_1 \operatorname{erf} r_1 + \frac{1}{\sqrt{\pi}} (e^{-r_1^2} - 1) \right]$$

we obtain the following equations for w_1 and w_2

$$(10) \quad \frac{\partial^2 w_1}{\partial r_1^2} + 2r_1 \frac{\partial w_1}{\partial r_1} - 4t \frac{\partial w_1}{\partial t} = 0,$$

$$\begin{aligned}
(11) \quad & \frac{\partial^2 w_2}{\partial r_1^2} + 2r_1 \frac{\partial w_2}{\partial r_1} - 4t \frac{\partial w_2}{\partial t} = -4W^0 \left\{ \sqrt{\nu' t} \exp(-r_1^2) \right. \\
& \left. + t \left[4 \cos \theta r_1 \exp(-r_1^2) \operatorname{erf} r_1 + \frac{4 \cos \theta}{\sqrt{\pi}} \exp(-2r_1^2) - \frac{4 \cos \theta}{\sqrt{\pi}} \exp(-r_1^2) \right] \right\}.
\end{aligned}$$

$$\text{Here } r_1 = \frac{\zeta}{2\sqrt{\nu' t}} = \frac{r-1}{2\varepsilon\sqrt{\nu' t}} \quad \text{and} \quad \operatorname{erf} r_1 = \frac{2}{\sqrt{\pi}} \int_0^{r_1} e^{-\beta^2} d\beta = E.$$

Using conditions $w_1(0)=0$ and $w_1 \rightarrow W^0$ as $r_1 \rightarrow \infty$ we find

$$w_1 = W^0 \operatorname{erf} r_1.$$

The boundary and matching conditions for the function $w_2(r_1)$ are $w_2(r_1)=0$ on $r_1=0$ for all $t \geq 0$ and

$$w_1 + \varepsilon w_2 = W_1 + \varepsilon W_2$$

is correct to order ε in an intermediate region.

We assume that w_2 has the form

$$(12) \quad w_2 = \frac{-4W^0}{\sqrt{\pi}} \sqrt{\nu' t} F(r_1) - \frac{16W^0}{\sqrt{\pi}} \cos \theta \cdot t \cdot H(r_1).$$

For $F(r_1)$ and $H(r_1)$ we have equations

$$(13) \quad \frac{d^2 F}{dr_1^2} + 2r_1 \frac{dF}{dr_1} - 2F = \exp(-r_1^2),$$

$$(14) \quad \frac{d^2 H}{dr_1^2} + 2r_1 \frac{dH}{dr_1} - 4H = r_1 \exp(-r_1^2) \operatorname{erf} r_1 + \frac{1}{\sqrt{\pi}} \exp(-2r_1^2) - \frac{1}{\sqrt{\pi}} \exp(-r_1^2)$$

and boundary conditions

$$(15) \quad F(0) = F(\infty) = 0,$$

$$(16) \quad H(0) = H(\infty) = 0.$$

The general solutions of (13) and (14) have the form

$$(17) \quad F(r_1) = \alpha_1 p(r_1) + \beta_1 q(r_1) + F_1(r_1),$$

$$(18) \quad H(r_1) = \alpha_2 p(r_1) + \beta_2 q(r_1) + H_1(r_1),$$

where $p(r_1) = 2r_1^2 + 1$, $q(r_1) = (2r_1^2 + 1) \operatorname{erf} r_1 + \frac{2}{\sqrt{\pi}} r_1 \exp(-r_1^2) - 2r_1^2 - 1$,

$\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants and $F_1(r_1), H_1(r_1)$ are particular integrals of (13) and (14).

After some algebra we obtain

$$(19) \quad F_1(r_1) = -\frac{1}{4} \exp(-r_1^2),$$

$$(20) \quad H_1(r_1) = p \int \frac{qD(r_1)}{p'q - pq'} dr_1 - q \int \frac{pD(r_1)}{p'q - pq'} dr_1 \\ = \frac{\sqrt{\pi}}{16} (2r_1^2 + 1)E^2 + \frac{r_1}{8} e^{-r_1^2} E + \frac{1}{6\sqrt{\pi}} e^{-r_1^2} - \frac{1}{4\sqrt{\pi}} (2r_1^2 + 1),$$

where

$$(21) \quad D(r_1) = r_1 \exp(-r_1^2)E - \frac{1}{\sqrt{\pi}} \exp(-2r_1^2) - \frac{1}{\sqrt{\pi}} \exp(-r_1^2).$$

If we choose $\alpha_1 = 0$ and $\beta_1 = \frac{\sqrt{\pi}}{4}$ the function (17) holds conditions (15).

After choosing $\alpha_2 = \frac{4-\pi}{16\sqrt{\pi}}$ and $\beta_2 = \frac{8-3\pi}{48\sqrt{\pi}}$ we obtain function

$$(22) \quad H(r_1) = \frac{\sqrt{\pi}}{16} (2r_1^2 + 1)E^2 + \frac{r_1}{8} e^{-r_1^2} E + \frac{1}{6\sqrt{\pi}} e^{-r_1^2} \\ + \frac{8-3\pi}{48\sqrt{\pi}} \left[(2r_1^2 + 1)E + \frac{2}{\sqrt{\pi}} r_1 e^{-r_1^2} - 2r_1^2 - 1 \right] - \frac{\sqrt{\pi}}{16} (2r_1^2 + 1)$$

which holds conditions (16).

Therefore,

$$w = w_1 + \varepsilon w_2 + O(\varepsilon^2) = W^0 \operatorname{erf} r_1 \\ - \frac{4\varepsilon W^0}{\sqrt{\pi}} \sqrt{\nu} t \left\{ \frac{\sqrt{\pi}}{4} \left[r_1 \cdot E + \frac{1}{\sqrt{\pi}} \exp(-r_1^2) - r_1 \right] - \frac{1}{4} e^{-r_1^2} \right\}$$

$$\begin{aligned}
& -\frac{16\varepsilon W^0}{\sqrt{\pi}} \cos \theta \cdot t \left\{ \frac{\sqrt{\pi}}{16} (2r_1^2 + 1) E^2 + \frac{r_1}{8} e^{-r_1^2} E + \frac{1}{6\sqrt{\pi}} e^{-r_1^2} \right. \\
& \left. + \frac{8-3\pi}{48\sqrt{\pi}} \left[(\alpha r_1^2 + 1) E + \frac{2}{\sqrt{\pi}} r_1 e^{-r_1^2} - 2r_1^2 - 1 \right] - \frac{\sqrt{\pi}}{16} (2r_1^2 + 1) \right\} + O(\varepsilon^2).
\end{aligned}$$

The method of matched asymptotic expansions is systematic and we shall not go into higher orders in this paper.

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