

The Determination and Analysis of Links Position Functions of a Spatial Linkage

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Introduction. In this article are suggested the methods for the determination: a) of links position functions; b) of special links positions in which the position functions are not one-value by a local consideration; c) of conditions when a linkage will have one or two cranks or will have none.

Matrix Methods for the Determination of Links Position Functions. Such methods that are based on matrices with real or dual numbers were used by researchers: in USSR by S. G. Kislizyn [3], F. M. Dimentberg [1], Ju. F. Moroshkin [7], P. A. Lebedev [4], Cshan Zy-Sjan [8], F. L. Litvin [5] and others; b) in USA by Denavit and Hartenberg [12], Sheth and J. Uicker [15], Soni, Pamidi [16], Yan, Freudenstein, Woo [17]; c) in Germany by Beyer [10]; d) in Rumania by B. Mangeron and C. Dragen [14]; e) in Czechoslovakia by V. Brat [11]; in Bulgaria by G. S. Kalizyn [2].

Many of those researchers have used in their works the matrix equation of a contour closure. The determination of such equation is based upon the coordinate transformation when translation on the contour from link m to link n is going in two opposite directions (Fig. 1). By this is composed such a matrix equation

$$(1) \quad M_{n(n+1)} M_{(n+1)(n+2)} \dots M_{(m-1)m} \\ = M_{n(n-1)} M_{(n-1)(n-2)} \dots M_{(m+1)m}.$$

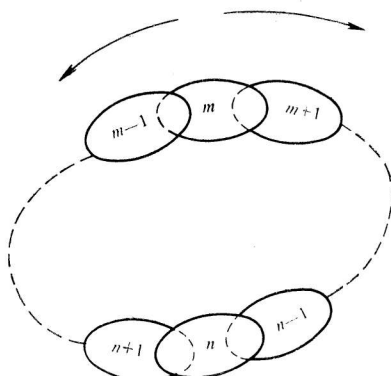


Fig. 1

The lower indices in symbol M design the links numbers. The elements of matrices contain the parameters of relative motions, parameters of geometry and mounting of the links.

Let us denote by M_{nm} and $\tilde{M}_{nm}$ the product of matrices of the left and right parts of equation (1) and the elements of M_{nm} and $\tilde{M}_{nm}$ by a_{kl} and $\tilde{a}_{kl}$. On the ground of (1)

$$(2) \quad M_{nm} = \tilde{M}_{nm}, \quad a_{kl} = \tilde{a}_{kl},$$

where $k=1, 2, 3, 4$, $l=1, 2, 3, 4$ in the matrices of order 4 with real numbers; $k=1, 2, 3$, $l=1, 2, 3$ in the matrices of order 3 with dual numbers.

On the ground of (2) we can compose 12 and 18 equations by using matrices with real and dual numbers, but only 6 of those equations are independent. A system with such 6 equations we are writing in such a form:

$$(3) \quad f_i(x_1, \dots, x_m) = 0 \quad (i=1, 2, \dots, 6),$$

where $x_1, \dots, x_m$ are the independent parameters of motion; $m=i+1$ for a mechanism with one degree of freedom.

The determination from equations (3) of the function communicating positions of the driven and driving links is connected with some difficulties: a) it is necessary previously to select 6 independent equations out of the system (2) or to test all the equations; b) for the determination of the function $x_m(x_1)$ it is necessary to solve a system which contains no less than 6 nonlinear equations and that brings many difficulties in computing operations.

Conventional Division of a Mechanism into Unclosed Kinematic Chains.

The author of the paper has suggested [6] a simplified method for the determination of the mechanism position function which is based upon the imaginary unclosure of the mechanism by excluding from it: a) a kinematic pair; b) a link with two kinematic pairs connecting the removed link to the other links. Such method permits to compose a system of nonlinear equations with the number of equations $i < 6$.

The mechanism unclosure method by dividing it in a kinematic pair was used by Jenkins, Crossley, Hunt [16], Brat [10].

To define equations of correlation of motion parameters remaining after the mechanism unclosure it is necessary to coordinate motions of the two open chains of the disintegrated mechanism. The number i of equations (3) is dependent on how the mechanism unclosure was obtained: either by a link removal out of it or by its disconnection in a kinematic pair.

Let us study a case when a link number k with turning pairs A_m and A_n is removed from a mechanism. To correlate motions of both chains the following equations have to be used

$$(4) \quad \mathbf{R}^{(O_n)} - \mathbf{R}^{(O_m)} = h_k \frac{\mathbf{a}^{(A_m)} \times \mathbf{a}^{(A_n)}}{|\mathbf{a}^{(A_m)} \times \mathbf{a}^{(A_n)}|},$$

$$(5) \quad \mathbf{a}^{(A_m)} \mathbf{a}^{(A_n)} = \cos \alpha_k.$$

In these equations $\mathbf{a}^{(A_m)}$ and $\mathbf{a}^{(A_n)}$ (Fig. 2) are the unit vectors of the axis of the kinematic pairs A_m and A_n ; O_m and O_n are the points of intersection of the shortest distance line between $\mathbf{a}^{(A_m)}$ and $\mathbf{a}^{(A_n)}$ with these vectors; $\mathbf{R}^{(O_m)}$ and $\mathbf{R}^{(O_n)}$ are the radius-vectors of the points O_m and O_n ; α_k is the angle of skew of $\mathbf{a}^{(A_m)}$ and $\mathbf{a}^{(A_n)}$. The unit vector $(\mathbf{a}^{(A_m)} \times \mathbf{a}^{(A_n)}) / |\mathbf{a}^{(A_m)} \times \mathbf{a}^{(A_n)}|$ is directed along the line of the shortest distance between $\mathbf{a}^{(A_m)}$ and $\mathbf{a}^{(A_n)}$ from the point O_m to O_n .

On the base of equations (4) and (5) we can compose a system which contains only 4 nonlinear equations instead of 6 equations in (3). Those equations contain all motion parameters of the mechanism without the motion parameters of the removed link k .

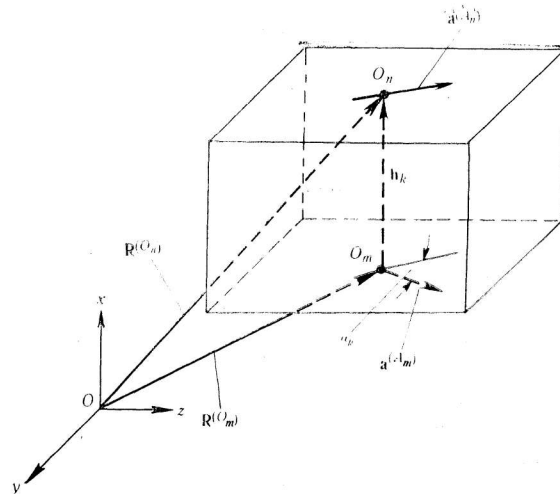


Fig. 2

From the equation (5) follows (suggested by the author and S. E. Karpovitch)

$$(6) \quad (\mathbf{R}^{(M_n)} - \mathbf{R}^{(M_m)})(\mathbf{a}^{(A_n)} \times \mathbf{a}^{(A_m)}) = h_k \sin \alpha_k,$$

where M_m and M_n —points, which positions on the axes A_m and A_n are known.

The obtaining of equation (6) is based on this:

$$\begin{aligned} \mathbf{R}^{(M_m)} &= \mathbf{R}^{(O_m)} + \lambda_m \mathbf{a}^{(A_m)}, \quad \mathbf{R}^{(M_n)} = \mathbf{R}^{(O_n)} + \lambda_n \mathbf{a}^{(A_n)}, \\ (\mathbf{R}^{(O_n)} - \mathbf{R}^{(O_m)})(\mathbf{a}^{(A_n)} \times \mathbf{a}^{(A_m)}) &= h_k (\mathbf{a}^{(A_n)} \times \mathbf{a}^{(A_m)})^2 : |\mathbf{a}^{(A_n)} \times \mathbf{a}^{(A_m)}| = h_k \sin \alpha_k. \end{aligned}$$

The system of equations (4) and (5) ought to be used for analysis of such mechanism, for which the number of motion parameters is more than 3 after removing of one link. As an example may be considered the analysis of six-joint linkage with six revolute and one cylindric pair. For such mechanism the equations (4) and (5) will have five motion parameters after removing the link $k=3$ (the frame has number 0); these parameters will be connected by 4 equations. But by the analysis of a five-joint linkage with 3 revolute and 2 cylindrical pairs (RRCCR) it is better to use the system of equations (6) and (4). After removing the link $k=3$ these equations will contain only 3 motion parameters which are connected by two equations.

We have considered only the case when from one-contour-mechanism is removed a link with two revolute pairs. Other cases of mechanism dividing were described in the author's article [6]. The method of a conventional division of a mechanism is also effective by using of matrices of 3-order with dual numbers.

We are noting that the motion parameters which are excepted by the first dividing can be found by the second dividing of the mechanism in an another kinematic pair or by removing another link.

The formalising of the operations of the matrices. The matrices used by us may be divided into two groups: a) with symbol G for description of

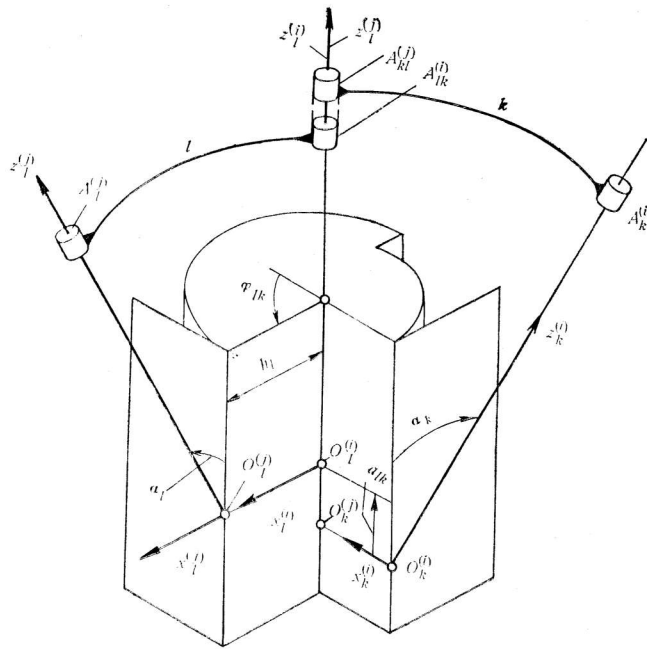


Fig. 3

the link's geometry; b) with symbol M for description of the links motion. Analogous dividing of the matrices was accomplished by Sheth and Uicker [15], but we are using other matrices and symbols.

In Fig. 3 are represented links k and l , which are connected by a revolute pair $A_{kl}^{(j)} = A_{lk}^{(i)}$. Every link has elements of two revolute pairs. Two coordinate systems are rigidly connected with every link and z -axes of these systems are the axes of the revolute pairs. So systems $S_l^{(j)}$ and $S_l^{(i)}$ are rigidly connected with link k ; axes $z_l^{(i)}$ and $z_l^{(j)}$ are coincided with the axes of elements of pairs $A_{lk}^{(i)}$ and $A_{lk}^{(j)}$ with the skew angle α_l . The origins $O_l^{(i)}$ and $O_l^{(j)}$ of systems $S_l^{(i)}$ and $S_l^{(j)}$ coincide with the intersection points of the shortest distance line between the kinematic pair axes with the mentioned above axes. Axis $x_l^{(i)}$ is directed along the vector $\mathbf{h}_l = \overline{O_l^{(i)}O_l^{(j)}}$. Analogous coordinate systems $S_k^{(i)}$ and $S_k^{(j)}$ are rigidly connected with the link k .

For the transformation of points coordinate from system $S_k^{(i)}$ to system $S_l^{(i)}$ there have to be used the matrix product $M_{lk}G_{kk}^{(ji)}$:

$$(7) \quad G_{kk}^{(ji)} = \begin{bmatrix} 1 & 0 & 0 & -h_k \\ 0 & \cos \alpha_k & \sin \alpha_k & 0 \\ 0 & -\sin \alpha_k & \cos \alpha_k & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(8) \quad M_{lk} = \begin{bmatrix} \cos \varphi_{lk} & \sin \varphi_{lk} & 0 & 0 \\ -\sin \varphi_{lk} & \cos \varphi_{lk} & 0 & 0 \\ 0 & 0 & 1 & -a_{lk} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where algebraic value $a_{lk} > 0$ of drawing in Fig. 3.

Let us imagine now that the link l is connected with the link m also with a revolute pair. For the transformation of points coordinate from system $S_k^{(i)}$ to system $S_m^{(i)}$ a product of matrices $M_{ml}G_{ll}^{(ji)}M_{lk}G_{kk}^{(ii)}$ is used. The structure of matrices with symbols G and M is defined by equations (7) and (8); in the matrix elements are changed the lower indices only.

The Determination Links Special Positions and the Conditions of Cranks Existence in Linkages. Let us remind that the links position functions are defined by a nonlinear equations system

$$(9) \quad f_i(x_1, x_2, \dots, x_m) = 0,$$

where $i=m-1$ for a mechanism with one degree of freedom (m is the number of independent motion parameters). The number i of nonlinear equations and the number m of motion parameters are dependent on the way how the system (9) is obtained.

The special positions of the links are such links positions for which the functions $x_k(x_1)$ ($k=2, \dots, m$) are not one-value in the neighbourhood of the point $(x_k^{(0)}, x_1^{(0)})$, i. e. they are not one-value in local sense. In special links positions the infinitesimal displacements of the link k are not in one-to-one correspondence to the infinitesimal displacement of link 1. It does ensue that in special positions the links k may change the direction of their motion while the link 1 keeps the same one.

To define the special links positions we'll use the theorem of the existence of an implicit functions set. When link 1 is a driving one then positions of links k will be special ones and functions $x_k(x_1)$ will be not one-value if according to referred theorem a Jacobian

$$(10) \quad \frac{D(f_1, f_2, \dots, f_i)}{D(x_2, x_3, \dots, x_m)} = 0 \quad (i=m-1).$$

In the case when a driving link is the link m the special positions of the links k ($k=1, 2, \dots, m-1$) will appear if the Jacobian

$$(11) \quad \frac{D(f_1, f_2, \dots, f_i)}{D(x_1, x_2, \dots, x_{m-1})} = 0.$$

For example by a slider-crank mechanism if the link 1 is the driving one the functions $x_2(x_1)$, $x_3(x_1)$ are not one-value in the local sense by $x_2=\pi/2$ (Fig. 4,a); when $x_2=\pi/2$

$$(12) \quad \frac{D(f_1, f_2)}{D(x_2, x_3)} = 0$$

and the links positions are special ones.

When in the referred mechanism the link 3 is the driving one the special positions will have place when $x_1=0$ and $x_1=\pi$ (Fig. 4, b, c); for these values of x_1

$$(13) \quad \frac{D(f_1, f_2)}{D(x_1, x_2)} = 0.$$

The conditions of existence of a crank in a linkage are also based on the theorem of the existence of an implicit functions set. Let the links 1 and m be connected with the frame by the revolute pairs. If the links 1 and m are not cranks the functions $x_m(x_1)$ and $x_1(x_m)$ are like a function $x_r(x_s)$ which

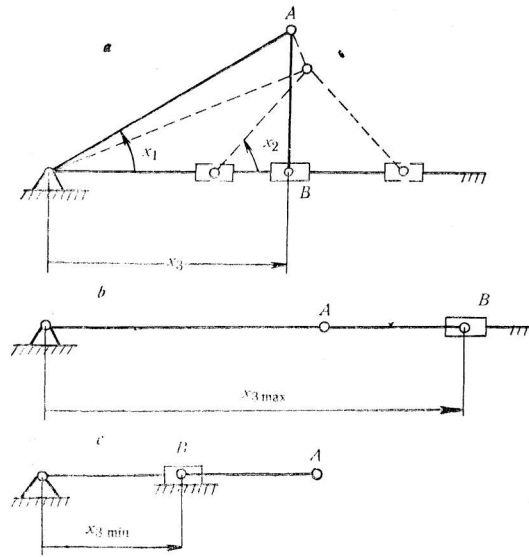


Fig. 4

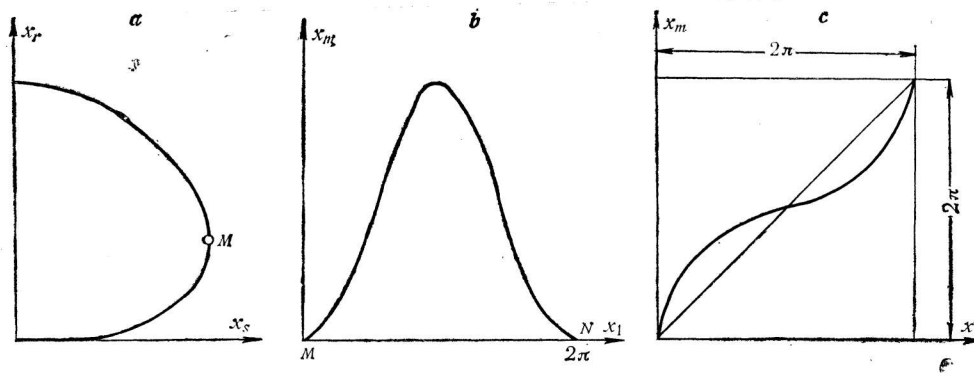


Fig. 5

is illustrated in Fig. 5, *a*; this function is not one-value in the local sense in the point *M*. In the case when link 1 is a crank the function $x_m(x_1)$ must be one-value in local sense for all values $0 \leq x_1 \leq 2\pi$. For existence a crank 1 is sufficiently that

$$(14) \quad \frac{D(f_1, f_2, \dots, f_i)}{D(x_2, x_3, \dots, x_m)} \neq 0 \quad \text{by} \quad 0 \leq x_1 \leq 2\pi.$$

If there is not imposed a demand that link *m* must be also a crank, then it is possible that in the mentioned interval for x_1 the Jacobian

$$(15) \quad \frac{D(f_1, f_2, \dots, f_i)}{D(x_1, x_2, \dots, x_{m-1})} = 0,$$

and the function $x_1(x_m)$ are not one-value in the points *M* and *N* for which is true the equation (15) (Fig. 5, *b*).

A linkage will have two cranks when the functions $x_m(x_1)$ and $x_1(x_m)$ are one-value in local sense for all values $0 \leq x_1 \leq 2\pi$ and $0 \leq x_m \leq 2\pi$ (Fig. 5, c). For existence of two cranks is sufficiently the observing of inequality (14) and inequality

$$(16) \quad \frac{D(f_1, f_2, \dots, f_i)}{D(x_1, x_2, \dots, x_{m-1})} \neq 0 \quad \text{by} \quad 0 \leq x_m \leq 2\pi.$$

Conclusion. The article's content permits: a) to simplify the analysis of a spatial linkage by dividing it in two open kinematic chains; b) to simplify the transformation of coordinate by using of the links geometry and relative motion matrices; c) to determine the special links positions and the conditions of existence of one and two cranks in a linkage.

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