

The Initial Flow Past a Circular Cylinder which is Started with Unsteady Velocity

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The boundary-layer growth on a cylindrical body which is started from rest in a viscous incompressible fluid with velocity

$$(1) \quad U = U_0 \Omega(t) = U_0 t^\alpha (1 + A_1 t^n),$$

where $t = t'/T_0$ is the non-dimensional time from the start, T_0 , U_0 — characteristic time and velocity, A_1 — positive constant, α — real and positive, n — positive integer, has been studied in [1]. The boundary-layer equations have been solved by the method in a series of small unsteady parameters and the second approximation to the stream function has been calculated.

That solution may be improved especially for a circular cylinder if we take into consideration second-order effects as a curvature of body surface and a displacement effect of the boundary layer on the inviscid outer flow. In [2] this problem has been solved for cylinder velocity $U_0 t^\alpha$. The interaction between the boundary layer and the external flow has been analysed by the systematic application of the method of inner and outer expansions to the Navier-Stokes equations. This method, called also the method of matched asymptotic expansions, is used here for investigating the motion of the circular cylinder in the case, when the velocity is given by (1).

We consider only small times and assume that the Strouhal number $S = \frac{a}{U_0 T_0}$ and the Reynolds number $R = \frac{a U_0}{\nu'}$ are large and of the same order. Then we take in a small parameter

$$(2) \quad \varepsilon = \frac{1}{S} = \frac{1}{\nu R} \ll 1.$$

Here a is the radius of the cylinder, ν' is a kinematic viscosity and ν is a non-dimensional constant of order unity.

Let us refer the polar coordinates r' , θ' , the velocity components v'_r , v'_θ and the pressure p' to standard values by the expressions (ρ' is a fluid density)

$$(3) \quad r = \frac{r'}{a}, \quad \theta = \theta', \quad u = \frac{v'_r}{U_0}, \quad v = \frac{v'_\theta}{U_0}, \quad p = -\frac{p'}{\rho' U_0^2 S}.$$

We take the axes fixed in the body and the angle θ' measured from the direction of the body velocity. Let us define a stream function $\psi(r, \theta, t)$ satisfying the continuity equation by the transformations

$$(4) \quad u = -\frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad v = \frac{\partial \psi}{\partial r}.$$

Thus, the Navier-Stokes equations reduce to the following equation

$$(5) \quad \left(\frac{\partial}{\partial t} - \varepsilon^2 \nu \Delta \right) \Delta \psi = \frac{\varepsilon}{r} \left(\frac{\partial \psi}{\partial \theta} \frac{\partial}{\partial r} - \frac{\partial \psi}{\partial r} \frac{\partial}{\partial \theta} \right) \Delta \psi,$$

subject to the initial and boundary conditions

$$(6) \quad \psi \sim \left(r - \frac{1}{r} \right) \sin \theta \Omega(t) \quad \text{as } t \rightarrow 0^+, \quad r > 1,$$

$$(7) \quad \psi = \frac{\partial \psi}{\partial r} = 0 \quad \text{when } r = 1, \quad t > 0,$$

$$(8) \quad \psi \sim r \sin \theta \Omega(t) \quad \text{as } r \rightarrow \infty, \quad t > 0,$$

where

$$(9) \quad \Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r}$$

is the Laplacian operator.

In accordance to the method of matched asymptotic expansions the flow field is divided in two regions. In the outer field far from the body surface the stream function is expressed in a series of powers of ε

$$(10) \quad \psi = \Psi_0(r, \theta, t) + \varepsilon \Psi_1(r, \theta, t) + \dots$$

It was pointed out by Wang [3] that the external flow remains irrotational in the case of body motions from rest. Therefore, the functions Ψ_i ($i=0, 1, 2, \dots$) satisfy the Laplacian equation.

In the inner flow field which is spread in the vicinity of the cylinder, the function ψ can be represented by a series in ε

$$(11) \quad \psi = \varepsilon \psi_0(\zeta, \theta, t) + \varepsilon^2 \psi_1(\zeta, \theta, t) + \dots,$$

where $\zeta = (r-1)/\varepsilon$. Restricting our analysis to the second approximation, we have the equations

$$(12) \quad \left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial \zeta^2} \right) \frac{\partial^2 \psi_0}{\partial \zeta^2} = 0,$$

$$(13) \quad \left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial \zeta^2} \right) \frac{\partial^2 \psi_1}{\partial \zeta^2} = - \frac{\partial^2 \psi_0}{\partial t \partial \zeta} + \left[\left(\frac{\partial \psi_0}{\partial \theta} + 2\nu \right) \frac{\partial}{\partial \zeta} - \frac{\partial \psi_0}{\partial \zeta} \frac{\partial}{\partial \theta} \right] \frac{\partial^2 \psi_0}{\partial \zeta^2}.$$

The solutions of the Laplacian equation must satisfy conditions (6), (8) and the ones of equations (12), (13) — condition (7). Moreover, both expressions (10) and (11) must match in an intermediate overlap region to each approximation of ε .

First approximation. The first term of expression (10) is found to be

$$(14) \quad \Psi_0 = \left(r - \frac{1}{r} \right) \sin \theta \cdot \Omega(t),$$

which is the stream function of the inviscid potential flow past a circular cylinder.

In the inner field we set the function ψ_0 in the following form

$$(15) \quad \psi_0 = 4\sqrt{\nu t} \Omega(t) \sin \theta [F_0(\eta) + (q - a)F_n(\eta)],$$

where

$$(16) \quad \eta = \frac{\zeta}{2\sqrt{\nu t}}, \quad q = \frac{t}{\Omega} \frac{d\Omega}{dt} = a + \frac{nA_1 t^n}{1 + A_1 t^n},$$

as it was taken in [1]. There the functions $F_0(\eta)$ and $F_n(\eta)$ are given by the expressions

$$(17) \quad \begin{aligned} F_0(\eta) &= \eta + 2^{2a} \Gamma(a+1) g_{a+\frac{1}{2}}(\eta) - \frac{1}{2} \frac{\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}\right)}, \\ F_n(\eta) &= \frac{1}{n} \left[2^{2a+2n} \Gamma(a+n+1) g_{a+n+\frac{1}{2}}(\eta) - 2^{2a} \Gamma(a+1) g_{a+\frac{1}{2}}(\eta) \right. \\ &\quad \left. + \frac{1}{2} \frac{\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}\right)} - \frac{1}{2} \frac{\Gamma(a+n+1)}{\Gamma\left(a+n+\frac{3}{2}\right)} \right], \end{aligned}$$

where

$$(18) \quad g_a(\eta) = \frac{2}{\sqrt{\pi} \Gamma(2a+1)} \int_{\eta}^{\infty} (\gamma - \eta)^{2a} e^{-\gamma^2} d\gamma.$$

It is clear that the first approximation to the stream function in the inner flow field coincides with the corresponding approximation obtained on the base of the boundary-layer equations.

Second approximation. After matching $\Psi_0 + \varepsilon \Psi_1$ and $\varepsilon \psi_0$, using an intermediate variable

$$(19) \quad \bar{\eta} = \frac{r-1}{2\sqrt{\varepsilon \nu t}} = \sqrt{\varepsilon} \eta,$$

we obtain a boundary condition

$$(20) \quad \begin{aligned} \Psi_1|_{r=1} &= -2\sqrt{\nu t} \Omega(t) \sin \theta \cdot B_n(t; a), \\ B_n(t; a) &= \frac{\Gamma(a+1)}{\Gamma\left(a+\frac{2}{3}\right)} + \frac{A_1 t^n}{1 + A_1 t^n} \left(\frac{\Gamma(a+n+1)}{\Gamma\left(a+n+\frac{3}{2}\right)} - \frac{\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}\right)} \right). \end{aligned}$$

The solution of the Laplacian equation, subject to the boundary condition (20) and

$$(21) \quad \Psi_1 \rightarrow 0 \text{ as } t \rightarrow 0^+; \quad \Psi_1 \rightarrow 0 \text{ as } r \rightarrow \infty,$$

is found to be

$$(22) \quad \Psi_1 = -2\sqrt{\nu t} \Omega(t) B_n(t; a) \frac{\sin \theta}{r}.$$

The additional term Ψ_1 is caused by the displacement effect of the boundary layer on the outer flow. Its inverse influence over the inner flow can be seen

from the second approximation to the stream function (11). We seek the solution of equation (13) in the form

$$(23) \quad \begin{aligned} \psi_1 = & 4\sqrt{t}\Omega(t) \cdot \sin \theta [H_0(\eta) + (q-\alpha)H_n^{(1)}(\eta)] \\ & + 8\sqrt{t^{3/2}}\Omega^2(t) \sin \theta \cdot \cos \theta [\Phi_0(\eta) + (q-\alpha)\Phi_n^{(1)}(\eta) + (q-\alpha)^2\Phi_n^{(2)}(\eta)], \end{aligned}$$

where the functions depended on the variable η are governed by the following equations

$$(24) \quad H_0'''' + 2\eta H_0''' - 4\left(\alpha + \frac{1}{2}\right)H_0'' = 8a\eta - 2^{2\alpha+1}\Gamma(\alpha+1)g_{\alpha-\frac{1}{2}} - \frac{4\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)},$$

$$(25) \quad \begin{aligned} H_n^{(1)''''} + 2\eta H_n^{(1)'''} - 4\left(n + \alpha + \frac{1}{2}\right)H_n^{(1)''} = & 4H_0' + \frac{1}{n} \left[8n\eta + 2^{2\alpha+1}\Gamma(\alpha+1)g_{\alpha-\frac{1}{2}} \right. \\ & \left. - 2^{2\alpha+2n+1}\Gamma(\alpha+n+1)g_{\alpha+n-\frac{1}{2}} + 4\left(\frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)} - \frac{\Gamma(\alpha+n+1)}{\Gamma\left(\alpha+n+\frac{1}{2}\right)}\right) \right], \end{aligned}$$

$$(26) \quad \Phi_0'''' + 2\eta \Phi_0''' - 4(2\alpha+1)\Phi_0'' = -4(1+F_0F_0''-F_0'^2),$$

$$(27) \quad \Phi_n^{(1)''''} + 2\eta \Phi_n^{(1)'''} - 4(2\alpha+n+1)\Phi_n^{(1)''} = 4(2\Phi_0' - F_0F_n'' - F_nF_0'' + 2F_0'F_n'),$$

$$(28) \quad \Phi_n^{(2)''''} + 2\eta \Phi_n^{(2)'''} - 4(2\alpha+2n+1)\Phi_n^{(2)''} = 4(\Phi_n^{(1)'} - F_nF_n'' + F_n'^2),$$

subject to the boundary conditions on the cylinder surface

$$(29) \quad H_0(0) = H_0'(0) = H_n^{(1)}(0) = H_n^{(1)'}(0) = 0,$$

$$(30) \quad \Phi_0(0) = \Phi_0'(0) = \Phi_n^{(1)}(0) = \Phi_n^{(1)'}(0) = \Phi_n^{(2)}(0) = \Phi_n^{(2)'}(0) = 0.$$

Here a prime denotes differentiation with respect to η .

The procedure of solving of (24) has been described in [2] and the solution has been obtained in the following form

$$(31) \quad \begin{aligned} H_0(\eta) = & 2^{2\alpha}\Gamma(\alpha+1)\left[(2\alpha+3)g_{\alpha+1}(\eta) - \frac{1}{2}g_{\alpha}(\eta)\right] - \eta^2 \\ & + \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{3}{2}\right)}\eta - \frac{1}{4(\alpha+1)}. \end{aligned}$$

Here the same procedure is applied to (25). At first, we find a partial solution of the equation. Then, taking $H_n^{(1)}(\eta)$ as a sum of this solution and the general solution of the corresponding homogeneous equation and matching the expressions $\frac{\partial\psi_0}{\partial\zeta} + \varepsilon\frac{\partial\psi_1}{\partial\zeta}$ and $\frac{\partial\Psi_0}{\partial r} + \varepsilon\frac{\partial\Psi_1}{\partial r}$ in the intermediate region, we obtain a boundary condition for the general solution when $\eta \rightarrow \infty$. Finally, we have

$$(32) \quad \begin{aligned} H_n^{(1)}(\eta) = & \frac{1}{n} \left\{ 2^{2\alpha+2n}\Gamma(\alpha+n+1)\left[(2\alpha+2n+3)g_{\alpha+n+1}(\eta) - \frac{1}{2}g_{\alpha+n}(\eta)\right] \right. \\ & \left. - 2^{2\alpha}\Gamma(\alpha+1)\left[(2\alpha+3)g_{\alpha+1}(\eta) - \frac{1}{2}g_{\alpha}(\eta)\right] \right\} \end{aligned}$$

$$+ \left(\frac{\Gamma(\alpha+n+1)}{\Gamma\left(\alpha+n+\frac{3}{2}\right)} - \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{3}{2}\right)} \right) \eta + \frac{1}{4} \left(\frac{1}{\alpha+1} - \frac{1}{\alpha+n+1} \right) \Bigg\}.$$

Matching yields also boundary conditions for the other functions, namely

$$(33) \quad \Phi_0'(\infty) = \Phi_n^{(1)'}(\infty) = \Phi_n^{(2)'}(\infty) = 0.$$

The solutions of equations (26), (27), (28) with boundary conditions (30), (33) are given in [1].

We see that the coefficient of $\sin \theta \cdot \cos \theta$ in (23) is identical to the second approximation to the stream function of the boundary layer flow. The other term arises from the induced external flow and the curvature of the body surface.

A solution which is uniformly valid in the whole flow field, is obtained by adding the inner and the outer solutions and subtracting the common part. Thus, we have

$$(34) \quad \begin{aligned} \psi = & \Omega(t) \sin \theta \left(r - \frac{1}{r} \right) + \varepsilon \cdot 2\sqrt{\nu t} \Omega(t) \sin \theta \left\{ \left[-\frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{3}{2}\right)} \right. \right. \\ & + \frac{q-\alpha}{n} \left(\frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{3}{2}\right)} - \frac{\Gamma(\alpha+n+1)}{\Gamma\left(\alpha+n+\frac{3}{2}\right)} \right) \Bigg] \cdot \frac{1}{r} + 2^{2\alpha+1} \Gamma(\alpha+1) g_{\alpha+\frac{1}{2}}(\eta) \\ & + 2^{2\alpha+1} \left(\frac{q-\alpha}{n} \right) [2^{2n} \Gamma(\alpha+n+1) g_{\alpha+n+\frac{1}{2}}(\eta) - \Gamma(\alpha+1) g_{\alpha+\frac{1}{2}}(\eta)] \Bigg\} \\ & + \varepsilon^2 2\nu t \Omega(t) \sin \theta \left\{ 2^{2\alpha+1} \Gamma(\alpha+1) \left[(2\alpha+3) g_{\alpha+1}(\eta) - \frac{1}{2} g_{\alpha}(\eta) \right] - \frac{1}{2(\alpha+1)} \right. \\ & + 2^{2\alpha+1} \left(\frac{q-\alpha}{n} \right) [2^{2n} (2\alpha+2n+3) \Gamma(\alpha+n+1) g_{\alpha+n+1}(\eta) - (2\alpha+3) \Gamma(\alpha+1) g_{\alpha+1}(\eta)] \\ & + \frac{q-\alpha}{n} \left[-2^{2\alpha+2n} \Gamma(\alpha+n+1) g_{\alpha+n}(\eta) + 2^{2\alpha} \Gamma(\alpha+1) g_{\alpha}(\eta) + \frac{1}{2} \left(\frac{1}{\alpha+1} - \frac{1}{\alpha+n+1} \right) \right] \Bigg\} \\ & + \varepsilon^2 8\sqrt{\nu t}^{3/2} \Omega^2(t) \sin \theta \cdot \cos \theta [\Phi_0(\eta) + (q-\alpha) \Phi_n^{(1)}(\eta) + (q-\alpha)^2 \Phi_n^{(2)}(\eta)] + O(\varepsilon^3). \end{aligned}$$

Then the skin friction is found to be ($\mu' = \nu' \varrho'$)

$$(35) \quad \begin{aligned} \tau_{\theta} = & \mu' \left(\frac{\partial v'_{\theta}}{\partial r'} \right)_{r'=a} = 2 \left(\frac{\mu' \varrho'}{t'} \right)^{1/2} U \sin \theta \left\{ \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)} \right. \\ & + \frac{A_1 t^n}{1+A_1 t^n} \left[\frac{\Gamma(\alpha+n+1)}{\Gamma\left(\alpha+n+\frac{1}{2}\right)} - \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)} \right] + \varepsilon \sqrt{\nu t} \left(\frac{n A_1 t^n}{1+A_1 t^n} + \frac{1}{2} \right) \\ & + \varepsilon t \Omega(t) \cos \theta \left[\Phi_0''(0) + \frac{n A_1 t^n}{1+A_1 t^n} \Phi_n^{(1)''}(0) + \left(\frac{n A_1 t^n}{1+A_1 t^n} \right)^2 \Phi_n^{(2)''}(0) \right] \Bigg\}. \end{aligned}$$

Equating this expression to zero, we obtain an equation which determinates the position of the boundary layer separation at a given time. On the other hand, if we put $\theta = \pi$ in this equation, we shall find the time t_s required for

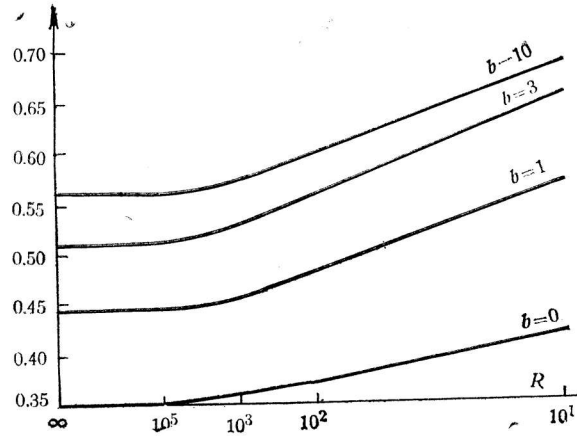


Fig. 1

separation at the rear stagnation point. It is convenient to consider the non-dimensional distance

$$(36) \quad \sigma = \frac{U_0}{a} \int_0^{t_s} \Omega(t) dt = \frac{1}{S} \frac{t_s^{a+1}}{a+1} \left(1 + \frac{a+1}{a+n+1} A_1 t_s^n \right),$$

which the cylinder moves before separation begins.

We apply formulae (35), (36) to the case of uniformly accelerated motion of the cylinder with initial velocity U_0 and accelerate $\frac{U_0 A_1}{T_0}$ ($\alpha=0$, $n=1$). The equation for determination of the separation time and the distance σ are found to be

$$(37) \quad 1 + \frac{bT_s}{1+bT_s} + \frac{\sqrt{\pi}}{2} \sqrt{\frac{T_s}{R}} \frac{1+3bT_s}{1+bT_s} - 2T_s \left[1.4244(1+bT_s) - 0.7182bT_s + 0.1460 \frac{b^2 T_s^2}{1+bT_s} \right] = 0,$$

$$(38) \quad \sigma = T_s \left(1 + \frac{1}{2} b T_s \right),$$

where

$$(39) \quad T_s = \frac{U_0 t_s'}{a}, \quad b = \frac{A_1 a}{U_0 T_0}.$$

Curves of σ for various values of b are illustrated in Fig. 1. Curve $b=0$ corresponds to the case of uniformly impulsive motion ($A_1=0$). As R tends to infinity the function σ decreases and approaches the corresponding value σ_0 , which is calculated in [1] on the base of the boundary layer equations. We see that the induced outer flow delays flow separation. For a given R this delay increases with b . If we plot curves of $\sigma - \sigma_0$ against the Reynolds number, curve $b=3$ will lay above the others. Therefore, the greatest deviation of σ from σ_0 occurs when b is equal to about 3.0.

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