

SOLID MECHANICS

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Identification of the Parameters of the Plated Element of the Netlike Medium

The paper shows the method and identifies the elastic plated elements parameters of the netlike medium. Those elements along with the ones in paper [1, 2] constitute the basis of a netlike model structure for stereomechanical analysis of elastic objects.

Let us consider the elastic layer [3] of local constant thickness whose central surface is a plane. The example of netlike model geometry of such layer is shown in Fig. 1.

Let us consider the case of load in model nodes of only perpendicular direction towards the central surface. And let us assume that the surface distortion depends on displacement of the central surface nodes₀ in the perpendicular direction to that surface and on the connector rotation being perpendicular to the central surface.

Hence,

$$(1) \quad u_1^k = u_2^k = 0,$$

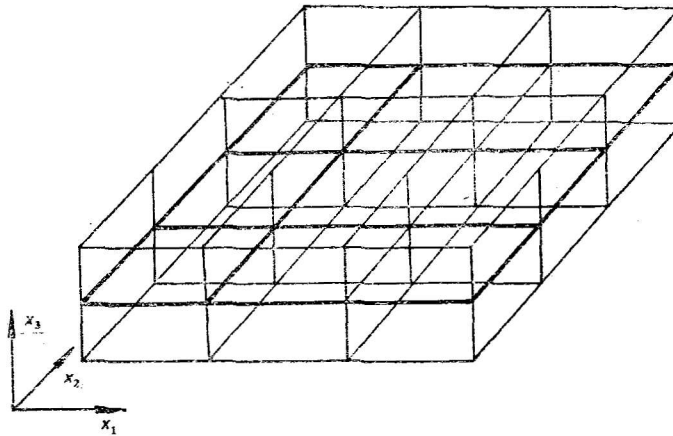


Fig. 1. Netlike model of elastic layer of flat central surface and local constant thickness

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for each node of central surface. Elastic properties of the layer in its netlike mode of the central surface is shown in Fig. 2. The model of that type is called plated netlike medium.

Let u^k determine displacement of the node e^k of central surface in the perpendicular direction and u^{jk} — difference of the displacements of the kinematic pairs

$$(2) \quad u^{jk} = u^k - u^j.$$

Vector $u_{,i}^k$, $i=1, 2$ satisfying the equation

$$(3) \quad u^{jk} = u_{,i}^{jk} t_i^{jk} r^{jk},$$

where r^{jk} is a vector beginning in the node e^j and ending in the node e^k , while t_i^{jk} is a versor of direction and vector r^{jk} sense — is called the gradient of displacement of connector d^{jk} . If there is a vector satisfying equation (3) for each connector of the netlike element then it is called the gradient of displacements of netlike element. We can determine the gradient of displacements of dual netlike element in the same way. If there are vector $u_{,i}^k$ and tensor $u_{,ij}^k$ satisfying the equation

$$(4) \quad u^{kl} u_{,i}^k t_i^{kl} r^{kl} = \frac{1}{2} u_{,ij}^k t_i^{kl} t_j^{kl} (r^{kl})^2,$$

$$i, j=1, 2,$$

for each connector of dual element with node e^k then vector $u_{,i}^k$ is called the gradient of displacements and tensor $u_{,ij}^k$ is the tensor of deflection that is determined in the node e^k . If the gradients of displacements and tensors of deflection are determined in the nodes e^k and e^l then they satisfy the relations

$$(5) \quad u_{,i}^{kl} = u_{,i}^l - u_{,i}^k = u_{,ij}^k r^{kl} t_j^{kl},$$

$$u_{,ij}^k = u_{,ij}^l.$$

If the gradient displacements and tensor of deflection are determined in the node e^k then the angle of rotation of the connector d^{jk} can be put as

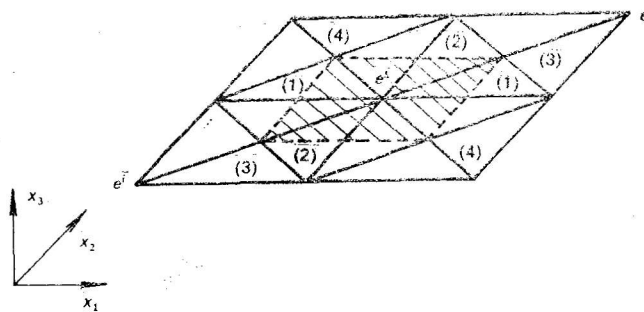


Fig. 2. Model of plated netlike medium

$$(6) \quad \theta_m^{jk} = \frac{-u^{jk}}{r^{jk}} t_m^{jk} = \frac{u^{kj}}{r^{jk}} t_m^{jk},$$

$$\theta_m^{jk} = \frac{1}{r^{jk}} \left[u_{,i}^k t_i^{kj} r^{kj} + \frac{1}{2} u_{,ij}^k t_i^{kj} t_j^{kj} (r^{kj})^2 \right] t_m^{jk},$$

or replacing versor t^{kj} by versor $(-t^{jk})$

$$(7) \quad \theta_m^{jk} = \left(-u_{,i}^k t_i^{jk} + \frac{1}{2} u_{,ij}^k t_i^{jk} t_j^{jk} r^{jk} \right) t_m^{jk},$$

$$i, j, m = 1, 2.$$

Connector d^{kl} corresponds with central-symmetric to it in relation to node e^k connector d^{jk} in the dual element of the node e^k . The angle of rotation of this element is

$$(8) \quad \theta_m^{kl} = \frac{u^{kl}}{r^{kl}} \tilde{t}_m^{kl} = - \left(u_{,i}^k t_i^{jk} + \frac{1}{2} u_{,ij}^k t_i^{jk} t_j^{jk} r^{jk} \right) \tilde{t}_m^{kl},$$

where t^{kl} is a versor turned by $\pi/2$ in relation to t^{jk} . There were also taken into consideration the dependences $t^{kl} = t^{jk}$, $\tilde{t}^{kl} = \tilde{t}^{jk}$, $r^{kl} = r^{jk}$.

Let us assume that the node of the plated element e^k is turned according to the deformation state of the dual netlike element

$$(9) \quad \varphi^k = \varphi^k(u_{,i}^k; u_{,ij}^k)$$

and that for each connector d^{kl} of the dual element the following dependence is satisfied

$$(10) \quad \varphi_m^{kl} = \varphi_m^l - \varphi_m^k = \omega_{m,i}^k u_{,i}^{kl} + \omega_{m,ij}^k u_{,ij}^{kl},$$

$$i, j, m = 1, 2,$$

where $u_{,ij}^{kl} = u_{,ij}^l - u_{,ij}^k$, $\omega_{m,i}^k$, $\omega_{m,ij}^k$ are numeral coefficients.

Because on the basis of (4)

$$(11) \quad u_{,ij}^{kl} = 0,$$

assumption (10) is as follows

$$(12) \quad \varphi_m^{kl} = \omega_{m,i}^k u_{,i}^{kl}.$$

Differences of the rotation angles of the connector d^{jk} and node e^i and e^k are

$$(13) \quad \psi_m^{kj} = \theta_m^{jk} - \varphi_m^j,$$

$$\psi_m^{jk} = \theta_m^{jk} - \varphi_m^k,$$

and respectively for the connector d^{kl} being central-symmetric in relation to the node e^k to the connector d^{jk} the angles are

$$(14) \quad \psi_m^{lk} = \theta_m^{kl} - \varphi_m^k,$$

$$\psi_m^{kl} = \theta_m^{kl} - \varphi_m^l.$$

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Angle components determined by (13) and (14) of versor sense $\tilde{v}^k = \tilde{t}^{kl}$, i. e. perpendicular to the connector displacement plane are equal

$$(15) \quad \begin{aligned} \tilde{\psi}_m^{kj} &= (\theta_m^{jk} - \varphi_m^j) \tilde{t}_m^{jk}, & \tilde{\psi}_m^{lk} &= (\theta_m^{kl} - \varphi_m^k) \tilde{t}_m^{lk}, \\ \tilde{\psi}_m^{jk} &= (\varphi_m^{jk} - \varphi_m^k) \tilde{t}_m^{jk}, & \tilde{\psi}_m^{kl} &= (\varphi_m^{kl} - \varphi_m^l) \tilde{t}_m^{kl}. \end{aligned}$$

Connector moment interactions into the node e^k are

$$(16) \quad \begin{aligned} \tilde{M}^{jk} &= -G^{jk}(2\tilde{\psi}^{jk} + \tilde{\psi}^{kj}), \\ \tilde{M}^{lk} &= -G^{kl}(2\tilde{\psi}^{lk} + \tilde{\psi}^{kl}), \end{aligned}$$

where G^{jk} is the coefficient of flexural rigidity of connector d^{jk} . The moments have versor $\tilde{v}^{jk}$ direction. Marking the type of considered connector pair by g we can put the sum of the interacting moments of that pair into the node e^k as

$$(17) \quad \begin{aligned} \tilde{M}^g &= \tilde{M}^{jk} + \tilde{M}^{kl}, \\ \tilde{M}^g &= -G^g(2\tilde{\psi}^{jk} + \tilde{\psi}^{kj} + 2\tilde{\psi}^{lk} + \tilde{\psi}^{kl}), \end{aligned}$$

and substituting the relations (15), we get

$$(18) \quad \tilde{M}^g = -G^g(3\theta_m^{jk} - 2\varphi_m^k - \varphi_m^j + 3\theta_m^{lk} - 2\varphi_m^k - \varphi_m^l) \tilde{t}_m^g.$$

Determining the coordinates of the angles of rotation φ^j and φ^l by φ^k on the basis of (10) and (5) we get

$$(19) \quad \tilde{M}^g = -G^g(3\theta_m^{jk} - 3\theta_m^{kl} - G\varphi_m^k + \omega_{m,i}^k u_{i,j}^k r_j^g t_j^g - \omega_{m,i}^k u_{i,j}^k r_j^g t_j^g) \tilde{t}_m^g,$$

and after reducing the components of opposite sign and substituting the dependences (5) and (8) that determine the connector angles of rotation we get

$$(20) \quad \tilde{M}^g = -G^g \left[2\varphi_m^k + \left(u_{i,i}^k t_i^g + \frac{1}{2} u_{i,j}^k t_i^g t_j^g r_j^g + u_{i,i}^k t_i^g + \frac{1}{2} u_{i,j}^k t_i^g t_j^g r_j^g \right) t_m^g \right] \tilde{t}_m^g,$$

and hence,

$$(21) \quad \tilde{M}^g = 3G^g(2\varphi_m^k t_m^g + 2u_{i,i}^k t_i^g).$$

Coordinates of versor t_m^g and t_i^g are linked by tensor of rotation

$$(22) \quad \begin{aligned} \tilde{t}_m &= a_{mi} t_i, \\ a_{11} &= a_{22} = 0, \quad a_{21} = -a_{12} = 1. \end{aligned}$$

Using this dependence we can get the coordinates of moment vector

$$(23) \quad \begin{aligned} M_n^g &= M^g t_n^g, \\ M_n^g &= 6G^g(\varphi_m^k a_{mi} + u_{i,i}^k) t_i^g t_n^g. \end{aligned}$$

Summing up the moments M^g for all incidental connector pairs in the node e^k (the sum must be zero in the state of node balance) we get

$$(24) \quad \sum_g \tilde{M}_n^g = 6(\varphi_m^n a_{mi} + u_{,i}^k) \sum_g G^g t_i^g \tilde{t}_n^g = 0.$$

Because of general case $\sum_g G^g t_i^g \tilde{t}_n^g \neq 0$ we can get

$$(25) \quad \varphi_m^k = -a_{mi}^{-1} u_{,i}^k = -a_{mi} u_{,i}^k,$$

or by means of permutating symbol

$$(26) \quad \varphi_m^k = \varepsilon_{mi3} u_{,i}^k.$$

The relation (26) might have more general form determining it in threedimensional space. Taking into account the conditions (1) and assuming that the connector rotation of axis direction x_3 — perpendicular to central surface of elastic layer in the node e^k equals the rotation of the node e^k that means

$$(27) \quad u_{3,1} = -u_{1,3}, \quad u_{3,2} = -u_{2,3},$$

the relation (26) can be put as

$$(28) \quad \varphi^k = \frac{1}{2} \text{rot } u^k.$$

In the theory of continual medium that condition is defined by the so-called Cosserats' pseudocontinuum. Netlike plate may be considered as the example of netlike Cosserats' pseudocontinuum. Similar relation to (26) determining the node angle of rotation can be formed for each node in which objects $u_{,i}$ and $u_{,ij}$ are determined. In this case the difference of angles of rotation (12) of nodes e^l and e^k can be as follows:

$$(29) \quad \varphi_m^{kl} = \varphi_m^l - \varphi_m^k = \varepsilon_{mi3} (u_{,i}^l + u_{,i}^k),$$

and substituting (5) we can get nodes e^l and e^k rotations

$$(30) \quad \begin{aligned} \varphi_m^l &= \varphi_m^k + \varepsilon_{mi3} u_{,ij}^k r^{kl} t_j^{kl} = \varphi_m^k + \varepsilon_{mi3} u_{,ij}^k r^{kl} t_j^{kl}, \\ \varphi^j &= \varphi_m^k + \varepsilon_{mi3} u_{,ij}^k r^{kl} t_j^{kl} = \varphi_m^k + \varepsilon_{mi3} u_{,ij}^k r^{kl} t_j^{kl}. \end{aligned}$$

Comparing the formulas to the assumed dependence (12) we get

$$(31) \quad \omega_{m,i}^k = \varepsilon_{mi3}.$$

Substituting formulas (4) and (8) and (26) and (30) by formulas (15) we can determine components perpendicular to the plane of displacement of connectors d^{jk} and d^{kl} of differences of connector angles of rotation and nodes for nodes e^i, e^k, e^l

$$(32) \quad \begin{aligned} \tilde{\Psi}_m^{kj} &= \left(\frac{1}{2} u_{,ij}^k t_i^{jk} t_j^{jk} + \varepsilon_{mi3} u_{,ij}^k t_j^{jk} \tilde{t}^{jk} \right) r^{jk}, \\ \tilde{\Psi}^{jk} &= \frac{1}{2} u_{,ij}^k t_i^{jk} t_j^{jk} r^{jk}, \\ \tilde{\Psi}^{kl} &= -\frac{1}{2} u_{,ij}^k t_i^{lk} t_j^{jk} r^{jk}, \end{aligned}$$

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$$\tilde{\Phi}_m^{lk} = \left(-\frac{1}{2} u_{,ij}^k t_i^{jk} t_j^{lk} - \varepsilon_{mi3} u_{,ij}^k t_j^{lk} \tilde{t}^{jk} \right) r^{jk}.$$

Connector bending energies are respectively

$$(33) \quad U_g^{jk} = G^{jk} (\tilde{\Psi}_m^{jk^2} + \tilde{\Psi}_m^{kj} \tilde{\Psi}_m^{lk} + \tilde{\Psi}_m^{lk^2}),$$

$$U_g^{jk} = G^{jk} r^{jk^2} \left(\frac{3}{4} t_i^{jk} t_j^{lk} t_r^{jk} t_s^{lk} + \frac{3}{2} \varepsilon_{mr3} t_i^{jk} t_j^{lk} t_s^{jk} \tilde{t}_m^{lk} + \varepsilon_{nr3} \varepsilon_{mi3} t_j^{lk} \tilde{t}_m^{jk} t_s^{lk} \tilde{t}_n^{jk} \right) u_{,ij}^k u_{,rs}^k,$$

$$(34) \quad U_g^{kl} = G^{kl} (\tilde{\Psi}_m^{kl^2} + \tilde{\Psi}_m^{kl} \tilde{\Psi}_m^{lk} + \tilde{\Psi}_m^{lk^2}),$$

$$U_g^{kl} = U_g^{jk}.$$

Connector bending energies of the same type being central-symmetric in the netlike dual element are equal to each other for the assumed state of deformation. Bending energy of dual netlike element is

$$(35) \quad U_g^k = \frac{1}{2} \sum_{jk} U_g^{jk}.$$

On the basis of (26) connector d^{jk} angle of torsion can be determined as follows

$$(36) \quad \theta^{jk} = (\varphi_m^k - \varphi_m^j) t_m^{jk},$$

and substituting (30)

$$(37) \quad \theta^{jk} = \varepsilon_{mi3} u_{,ij}^k t_m^{jk} t_j^{lk} r^{jk}.$$

For the connector d^{kl} the angle is

$$(38) \quad \bar{\theta}^{jk} = (\varphi_m^l - \varphi_m^k) t_m^{kl},$$

$$\bar{\theta}^{jk} = \varepsilon_{mi3} u_{,ij}^k t_m^{kl} t_j^{lk} r^{kl}.$$

So for the central-symmetric connectors those angles are equal. Torsional energy of the connector is

$$(39) \quad U_s^{jk} = \left(\frac{1}{2} S^{jk} r^{jk} \varepsilon_{nr3} \varepsilon_{mi3} t_j^{lk} t_m^{jk} t_s^{lk} t_r^{jk} \right) u_{,ij}^k u_{,rs}^k.$$

Torsional energy of the dual netlike element is

$$(40) \quad U_s^k = \frac{1}{2} \sum_{jk} U_s^{jk}.$$

Energy density of dual netlike element distorsion is

$$(41) \quad u^k = \frac{U^k}{V^k} = \frac{1}{V^k} (U_G^k + U_s^k).$$

where V^k is the volume of the netlike element area.

Substituting the determined dependences that define the components of distorsion energy (35) and (40), we get

$$(42) \quad u^k = P_{ijrs}^k u_{,ij}^k u_{,rs}^k,$$

where

$$(43) \quad P_{ijrs}^k = \frac{1}{2V^k} \sum_{jk} \left[G^{jk} r^{jk} \left(\frac{3}{4} t_i^{jk} t_j^{jk} t_r^{jk} t_s^{jk} + \frac{3}{2} \varepsilon_{mr3} t_i^{jk} t_j^{jk} t_s^{jk} t_m^{jk} + \varepsilon_{nr3} \varepsilon_{mi3} t_j^{jk} t_m^{jk} t_s^{jk} t_n^{jk} \right) \right. \\ \left. + \frac{1}{2} S^{jk} r^{jk} \varepsilon_{nr3} \varepsilon_{mi3} t_j^{jk} t_m^{jk} t_s^{jk} t_n^{jk} \right].$$

Tensor P_{ijrs} is called the tensor of torsional-flexural rigidity of the netlike element. Comparison of element tensor of torsional-flexural rigidity P_{ijrs}^k of netlike element with rigidity values experimentally determined allows physical parameter choice of the netlike model. E. g. for the netlike medium of element structure shown in Fig. 3a — Fig. 3b shows the area of dual element of node e^k of that model density energy of distortion determined on the basis of dependence (43) is

$$(44) \quad u^k = \left(\frac{1}{2d_1d_2} G^1 d_1^2 + G^3 \frac{d_1^2 + d_2^2}{2d_1d_2} \cos^4 \alpha \right) u_{,11}^2 + \left(\frac{1}{2d_1d_2} G^2 d_2^2 + G^3 \frac{d_1^2 + d_2^2}{2d_1d_2} \sin^4 \alpha \right) u_{,22}^2 \\ + G^3 \frac{d_1^2 + d_2^2}{2d_1d_2} \sin^2 \alpha \cos^2 \alpha + u_{,11} u_{,22} + \left(2G^3 \frac{d_1^2 + d_2^2}{d_1d_2} \sin^2 \alpha \cos^2 \alpha + \frac{d_1 + d_2}{d_1d_2} S \right) u_{,12}^2.$$

Comparing it to the density energy for the case of pure bending of isotropic medium elastic layer

$$(45) \quad u = \frac{1}{2} D (u_{,11}^2 + u_{,22}^2 + 2\nu u_{,11} u_{,22}) + D(1-\nu) u_{,12}^2, \\ D = \frac{Eh^3}{12(1-\nu^2)},$$

we can get parameter values of netlike model from the equations

$$(46) \quad \begin{aligned} G^1 d_1^2 + G^3(d_1^2 + d_2^2) \cos^4 \alpha &= D d_1 d_2, \\ G^2 d_2^2 + G^3(d_1^2 + d_2^2) \sin^4 \alpha &= D d_1 d_2, \\ G^3(d_1^2 + d_2^2) \sin^2 \alpha \cos^2 \alpha &= D \nu d_1 d_2, \end{aligned}$$

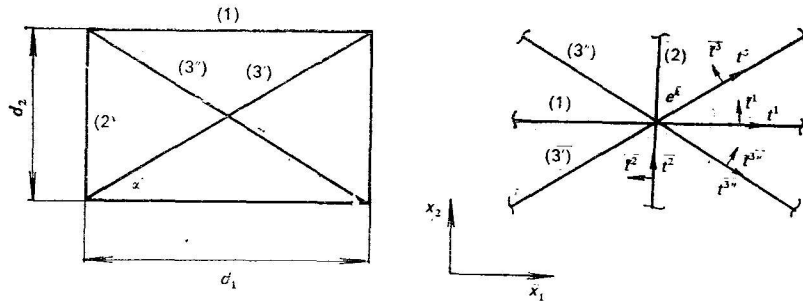


Fig. 3. Netlike element and dual netlike element of plate netlike medium

$$2G^3(d_1^2 + d_2^2) \sin^2 \alpha \cos^2 \alpha + S(d_1 + d_2) = D(1 - \nu) d_1 d_2,$$

hence,

$$(47) \quad \begin{aligned} G^1 &= D \kappa \left(1 - \frac{\nu}{\kappa^2}\right), \quad G^2 = D \frac{1}{\kappa} (1 - \nu \kappa^2), \\ G^3 &= D \nu \left(\kappa + \frac{1}{\kappa}\right), \quad S = D \frac{1 - 3\nu}{\kappa + 1/\kappa}, \end{aligned}$$

where $\kappa = d_1/d_2 = \tan \alpha$.

Assuming that $d = 0.1$ m, $\nu = 0.3$, $d_1/d_2 = \kappa = 1$, $E = 2 \cdot 10^{11}$ Pa, we get

$$D = 18.315 \times 10^6 \text{ Nm}, \quad G^1 = G^2 = 12.82 \times 10^6 \text{ Nm},$$

$$G^3 = 10.99 \times 10^6 \text{ Nm}, \quad S = 3.663 \times 10^6 \text{ Nm}.$$

Those values have been used in the analytic model of the plate whose continual model is shown in Fig. 4. Deformed discrete model is shown in Fig. 5. Theoretical values of continual model deflection is 0.22 m while calculated is 0.215 m. A few per cent difference results from a small number of netlike elements and qualitative difference of the model.

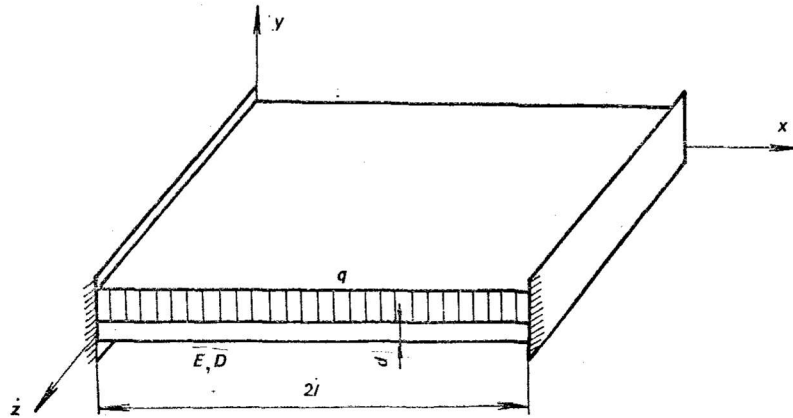


Fig. 4. Plate continual model

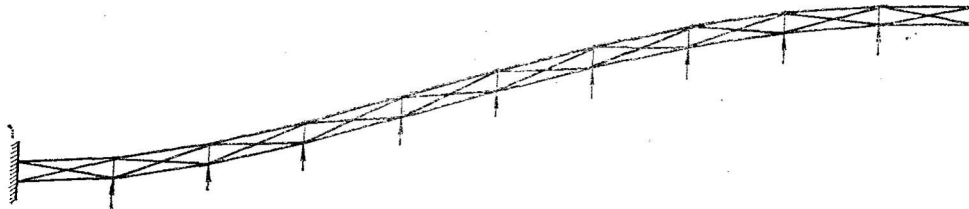


Fig. 5. Discrete deformed model of plate

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