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A Personal Computer Program for the Elastoplastic Analysis of Frames*

1. Introduction

In this paper, a personal computer program for the elastoplastic analysis of frames is presented. This program is based upon a method converging to the exact solution and can easily cooperate with the existing elastic analysis programs.

Methods that are using linear, quadratic or nonlinear Mathematical Optimization give results converging to the exact solution. Furthermore, by the use of these methods, the engineer can get a better understanding of the behavior of the structures through the study of the optimization problems to which the elastoplastic analysis problems are equivalent.

2. Elastoplastic analysis

2.1. Incremental elastoplastic analysis. First the basic laws of incremental elastoplastic analysis are formulated under the following assumptions:

1. The continuum is discretized into n Finite Elements.
2. The displacements are large.
3. The deformations are small.
4. The existence of physical instabilizing effects is accepted.
5. The stress and strain fields of each element are described by means of "natural" generalized stresses and strains [1-5, 19].
6. The generalized strains are not affected by the rigid body motion of the element, the generalized stresses are self-equilibrated in each element.
7. The external loads are applied to the nodes. Further, the continuum is described by means of the nodal displacement vector u referred to a fixed Cartesian coordinate system and to a given reference configuration.

Under these assumptions the problem is described [20, 24-29, 10, 11, 18] by relations

$$(1) \quad \dot{e} = \dot{e}_0 + \dot{e}_E + \dot{e}_P;$$

$$(2), (3) \quad \dot{e}_E = F_0 \dot{s}, \quad \dot{e}_P = V \dot{\lambda};$$

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$$(4) \quad \dot{F} = N^T \dot{s} - H \dot{\lambda};$$

$$(5), (6), (7) \quad \dot{\lambda} \geq 0, \quad \dot{F} \leq 0, \quad \dot{F}^T \dot{\lambda} = 0$$

for the assembled structure. Further, the equilibrium and compatibility equations

$$(8), (9) \quad G \dot{s} + K_G \dot{u} = \dot{p}, \quad \dot{e} = G^T \dot{u}$$

are valid. As it is well known, relations (5)-(7) form a Linear Complementarity Problem (L. C. P.) [28, 29, 11, 13]. Using the same notation as in [41, 8], it can be proved, by the use of variational inequalities [38, 41, 39] that this L. C. P. is equivalent to the Quadratic Optimization Problem (primal minimization problem)

$$(10) \quad \Pi(x) = \min \left\{ \Pi(x_1) = \frac{1}{2} x_1^T M x_1 + q^T x_1 \mid x_1 \in K \right\}.$$

For matrix M the assumption is made that it is symmetric and positive semidefinite. The former assumption is valid iff $V=N$ (normality) and there exists reciprocity of the interaction of the yield modes ($H=H^T$). The sufficient condition for matrix M to be positive semidefinite is ensured by the positive semidefiniteness of matrices H (non-existence of softening) and K_G (geometrical effects non-instabilizing). Under this assumption $x^T M x$ is convex.

Expressed in terms of the displacements $\dot{u}$ and the plastic multipliers $\dot{\lambda}$, (10) takes the form of relations

$$(11) \quad \min \left\{ \Pi(\dot{u}, \dot{\lambda}) = \frac{1}{2} \dot{u}^T K \dot{u} + \frac{1}{2} \dot{\lambda}^T H \dot{\lambda} - \dot{u}^T G K_0 N \dot{\lambda} + \frac{1}{2} \dot{\lambda}^T N^T K_0 N \dot{\lambda} + \dot{e}_0^T K_0 N \dot{\lambda} - \dot{e}_0^T K_0 G^T \dot{u} - \dot{p}^T \dot{u} \mid \dot{\lambda} \geq 0 \right\}$$

or

$$(12) \quad \min \left\{ \Pi(\dot{u}, \dot{\lambda}) = \frac{1}{2} \dot{u}^T K \dot{u} + \frac{1}{2} \dot{\lambda}^T H \dot{\lambda} - \dot{u}^T G K_0 N \dot{\lambda} + \frac{1}{2} \dot{\lambda}^T N^T K_0 N \dot{\lambda} + \dot{e}_0^T K_0 N \dot{\lambda} - \dot{e}_0^T K_0 G^T \dot{u} - \dot{p}^T \dot{u} \mid \dot{\lambda} \geq 0 \right\}.$$

In a similar way, under the assumption again that M is symmetric and positive semidefinite, the dual minimization problem is obtained

$$(13) \quad \Pi^c(x) = \min \left\{ \Pi^c(x_1) = \frac{1}{2} x_1^T M x_1 \mid x_1 \in K^c \right\}.$$

Expressed in terms of the displacements $\dot{u}$ and the plastic multipliers $\dot{\lambda}$, (13) takes the form of

$$(14) \quad \min \left\{ \Pi^c(\dot{u}, \dot{\lambda}) = \frac{1}{2} \dot{u}^T K \dot{u} + \frac{1}{2} \dot{\lambda}^T H \dot{\lambda} - \dot{u}^T G K_0 N \dot{\lambda} + \frac{1}{2} \dot{\lambda}^T N^T K_0 N \dot{\lambda} \mid K \dot{u} - G K_0 N \dot{\lambda} - (\dot{p} + G K_0 \dot{e}_0) = 0, \quad N^T K_0 G^T \dot{u} - (H + N^T K_0 N) \dot{\lambda} - N^T K_0 \dot{e}_0 \leq 0 \right\}.$$

From (14), (2), (1) and (9) and by the use of $K_E = GK_0N^T$, the dual minimization problem takes the form

$$(15) \quad \min \left\{ \Pi^c(\dot{u}, \dot{\lambda}, \dot{s}) = \frac{1}{2} \dot{u}^T K_G \dot{u} + \frac{1}{2} \dot{\lambda}^T H \dot{\lambda} + \frac{1}{2} \dot{s}^T F_0 \dot{s} + \dot{s}^T \dot{e}_0 + \frac{1}{2} \dot{e}_0^T K_0 \dot{e}_0 \mid \right. \\ \left. | G^T \dot{u} - V \dot{\lambda} = F_0 \dot{s} + \dot{e}_0, G \dot{s} + K_G \dot{u} = \dot{p}, N^T \dot{s} - H \dot{\lambda} \leq 0 \right\}.$$

Relation (15), in the case that $K_G = 0$, and under the condition that H is positive semidefinite (non-softening), takes the form

$$(16) \quad \min \left\{ \Pi^c(\dot{s}, \dot{\lambda}) = \frac{1}{2} \dot{\lambda}^T H \dot{\lambda} + \frac{1}{2} \dot{s}^T F_0 \dot{s} + \dot{s}^T \dot{e}_0 + \frac{1}{2} \dot{e}_0^T K_0 \dot{e}_0 \mid G \dot{s} = \dot{p}, N^T \dot{s} - H \dot{\lambda} \leq 0 \right\}.$$

When the initial assumption, that matrix M is symmetric and positive semidefinite, is valid, then the set of solutions of the primal problem (10) is convex. In this case any solution of the primal problem (1)-(9), or equivalently of (10), is also a solution of the dual problem (13) but not conversely [23]. Furthermore, the primal problem has a solution iff there exists a solution for the dual problem [12, 27].

Under the assumption that matrix K is nonsingular, another family of variational inequalities is formulated, expressed with respect to the plastic multipliers $\dot{\lambda}$. Through the substitutions

$$(17) (18) \quad Z = K_0 G^T K^{-1} G K_0 - K_0, \quad \dot{s}^E = K_0 G^T K^{-1} \dot{p} + Z \dot{e}_0$$

relation

$$(19) \quad \dot{F} = N^T \dot{s}^E - (H - N^T Z V) \dot{\lambda}$$

is obtained. This relation together with (5)-(7) constitute a L.C.P. If matrix $D = H - N^T Z V$ is symmetric ($N = V$, $H = H^T$) and positive semidefinite, then by the use again of variational inequalities, the minimization problem

$$(20) \quad R(\dot{\lambda}) = \min \left\{ R(\dot{\lambda}_1) = \frac{1}{2} \dot{\lambda}_1^T D \dot{\lambda}_1 - (N^T \dot{s}^E)^T \dot{\lambda}_1 \mid \dot{\lambda}_1 \geq 0 \right\}$$

is obtained. The dual of (20) is

$$(21) \quad R^c(\dot{\lambda}) = \min \left\{ R^c(\dot{\lambda}_1) = \frac{1}{2} \dot{\lambda}_1^T D \dot{\lambda}_1 \mid N^T \dot{s}^E - D \dot{\lambda}_1 \leq 0 \right\}.$$

It must be noted that for the relations between the solutions of the primal (rel. (20)) and the dual (rel. (21)) problem and depending on the nature of matrix D the various cases that have been examined for matrix M are valid.

2.2. Holonomic elastoplastic analysis. Unlike the incremental elastoplastic analysis, in the holonomic case, only the final loading condition is important and not the loading history. Furthermore, after the complete unloading, there are no permanent deformations. The assumptions and relations that are valid for the structure are analogous to (1)-(7), with the difference that the Y -vector of the positive constants k is present in the relation of the yield functions (cf rel. (4)). Furthermore, the stress and strain

vectors are finite and not incremental. The holonomic minimization problems are similar to the incremental ones.

3. The optimization method

The method that has been used for the solution of the problem [21] is a generalization of the the methods of Dantzig [15] and of Van de Panne and Whinston [43] for the convex problems of quadratic optimization. This method is considered [11], together with the methods of Beale and Wolfe, as one of the most prominent ones for the solution of quadratic programming problems. As it is based on the exchange between a dependent and an independent variable of a system of equations, the method is a direct one [13]. Direct methods are applied in L.C.P.s where they generate a sequence of transformations of $W=q+Mz$. The method is not an iterative one, in the sense that the methods of Hildreth or Cryer are [13]. That is, it does not produce a sequence of iterations, possibly infinite in number, (trial solutions) converging to a solution. Unlike the iterative methods which modify the trial solutions, this method modifies the problem data and uses first-order information from the transformation of the objective function in order to guide its steps. Furthermore, most convex quadratic optimization algorithms require positive semidefinite matrices and unlike the method used here, do not satisfy the necessary conditions in the nonconvex case. An exception is the algorithm of Beale [43, 9, 11], which however, in most cases, needs a number of steps at least two times as many as the method studied here. Furthermore, in Beale's method the tableaux of the problems are larger.

3.1. The general Q. O. problem and the equivalent tableau

The quadratic optimization problem

$$(22) \quad \min \left[\frac{1}{2} u^T D u + c^T u + \beta \mid A u \geq b \text{ and } u > 0 \right]$$

is considered, where D is a $m \times m$ symmetric matrix, A is a $p \times m$ matrix, b and c are respectively p - and m -vectors and β a constant. As it is very well known, the solution of the Kuhn-Tucker ($K-T$) conditions of the problem is a special case of the L.C.P. [28, 11, 41 etc.]

$$(23) \quad W = q + Mz$$

$$(24) \quad W \geq 0, z \geq 0, W^T z = 0,$$

where

$$(25) \quad M = \begin{bmatrix} 0 & A \\ -A^T & D \end{bmatrix}, \quad q = \begin{bmatrix} -b \\ c \end{bmatrix}, \quad W = \begin{bmatrix} s \\ q \end{bmatrix}, \quad z = \begin{bmatrix} u \\ v \end{bmatrix}.$$

It can be proved [11, 7, 8] that rel. (22) is equivalent to

$$(26) \quad \min \left[f(x_J) = \frac{1}{2} x_J^T M_{JJ} x_J + q_J^T x_J + \beta \mid M_{JJ} x_J \geq -q_1 \text{ and } x_J > 0 \right].$$

and the problem can take the form of the following tableau

$$(27) \quad \begin{array}{c|c|c|c} y_I^T & x_J^T & 1 & \\ \hline M_{II} & M_{IJ} & q_I & = x_I \\ \hline -M_{IJ}^T & M_{JJ} & q_J & = y_J \\ \hline -q_I^T & q_J^T & a & = F(x, y) - x^T y, \end{array}$$

where $M_{II}=0$ and $a=2\beta$.

In the general case, where M_{II} need not be equal to zero, the following proposition can be formulated:

Proposition 3.1. Let P be the above tableau, where matrices M_{II} and M_{JJ} are symmetric. Then, by a finite number of 1×1 or 2×2 pivots, it is possible to obtain a new tableau P^* from which one of the following can be concluded:

3.1.a. The K - T conditions of the problem are satisfied. In this case vector q^* is non-negative.

3.1.b. The objective function is unbounded from below in the feasible region.

3.1.c. The feasible region is empty.

In all three cases matrix $M_{1,1}^*$ is positive semidefinite.

3.2. The algorithm

The whole procedure takes the form of the following algorithm [21]:

Step 0. Let $c=1$.

Step 1. If $q_c^c \geq 0$ is valid, then the procedure has been completed. If not go to the second step.

Step 2. A $\tau \in J^c$ is chosen such that $q_\tau^c < 0$ is valid. Go to the third step.

Step 3. If $m_{\tau\tau}^c \leq 0$ is valid, then go to the fourth step. If on the other hand, $m_{\tau\tau}^c > 0$ is valid, then find, if possible, a k such that relation

$$\frac{q_k^c}{m_{k\tau}^c} = \max \left[\frac{q_i^c}{m_{i\tau}^c} \mid m_{i\tau}^c < 0 \ i \in I^c \right]$$

is valid also:

(3.a.) If either relation $\frac{q_\tau^c}{m_{\tau\tau}^c} \geq \frac{q_k^c}{m_{k\tau}^c}$ is true or the set of indices i for which $m_{i\tau}^c < 0, i \in I^c$, is empty, then perform a 1×1 pivot on τ and return to the first step replacing c by $c+1$.

(3.b.) If $\frac{q_\tau^c}{m_{\tau\tau}^c} < \frac{q_k^c}{m_{k\tau}^c}$ and if $m_{kk}^c \neq 0$, then perform a 1×1 pivot on k and return to the third step replacing c by $c+1$.

(3.c.) If $\frac{q_\tau^c}{m_{\tau\tau}^c} < \frac{q_k^c}{m_{k\tau}^c}$ and if $m_{kk}^c = 0$, then perform a 2×2 pivot on (k, τ) and return to

the first step replacing c by $c+1$.

Step 4. If $m_{\tau\tau}^c < 0$ is valid, then find, if possible, a k such that relation

$$\frac{q_k^c}{m_{k\tau}^c} = \max \left[\frac{q_i^c}{m_{i\tau}^c} \mid m_{i\tau}^c < 0, i \in I^c \right]$$

is valid also:

- (4.a.) If the set of indices i for which $m_{i\tau}^c < 0, i \in I^c$, is empty, then case 3.1.b. has been reached.
- (4.β.) If $m_{kk}^c \neq 0$, then perform a 1×1 pivot on k and return to the third step replacing c by $c+1$.
- (4.γ.) If $m_{kk}^c = 0$, then perform a 2×2 pivot on (k, τ) and return to the first step replacing c by $c+1$.

4. The computer program

The general steps of a computer program using this algorithm in the form of a subprogram is further presented [7, 8]. As it can be seen, most of the necessary software is usually available.

1st Step. Member Information (Nodal Coordinates, Moments of Inertia, Plastic Moments of the Sections etc).

The use of a Worksheet or Text Editor is possible for the preparation of the input data.

2nd Step. Analysis of the Statically Determinate Structure

- a. For the Unknowns
- b. For the External Loading.

Necessary Software: Any available program for the elastic analysis of frames giving the results in Text File form.

3rd Step. Automatic Generation of the Matrices that will be used in the next Step.

Necessary Software: Simple programs of matrix analysis or the equivalent library subroutines.

4th Step. a. Automatic Creation of the Problem Tableau:

- b. Application of the Algorithm for the given Load/Load Factor.

Necessary Software: The SYSFORSIM subprogram.

The numerical results (Plastic Multipliers λ , Rotations θ , Plastic Moments s^P and Total Moments s) can be read from the screen, printed by a printer or stored in Text (ASCII Code) Files for further use.

5. The numerical applications

5.1. General remarks

In the following, some applications of the method presented above, in the problems of the theory of plasticity, will be given (for applications in other classes of problems equivalent to L.C.P., see [6]). All the matrices or vectors are compu-

ted during the elastic analysis of the structure or are obtained directly from this analysis. This results to the easy implementation of the method.

The method can be used for a number of unknowns at least equal to the one that the available elastic analysis method can accept. It must be pointed out that the algorithm is characterized as a physical one, in the sense that there is a direct relationship between the steps of the method and the successive equilibrium phases of the structure, up to collapse.

5.2. Further assumptions

The following assumptions are valid in addition to the ones made at the beginning of the second chapter:

- a). The material law is elastic-perfectly plastic.
- b). The inelastic deformations concentrate in single sections causing rotations θ .
- c). The plastic moment-rotation θ relationship is depicted in Fig. 1.
- d). The moment-rotation law is reversible (holonomic).

5.3. Formulation to be used

For the solution of the problem the following formulation will be used (see also rel. (20)).

$$(28) \quad \min \left\{ R(\lambda) = \frac{1}{2} \lambda^T D \lambda - (N^T s^E - k)^T \lambda \mid \lambda \geq 0 \right\}.$$

5.4. Numerical examples

First, the simple portal frame of Fig. 2 is studied [39, 8]. The frame is of uniform cross section $EJ = 10.0 \text{ kNm}^2$ with a plastic moment $M_p = 65.73 \text{ kNm}$.

As it is well known, the possible critical sections of the structure are the 8. Since the work-hardening matrix H is zero, the relation giving the 16×16 matrix D takes the form

$$(29) \quad D = N^T Z N,$$

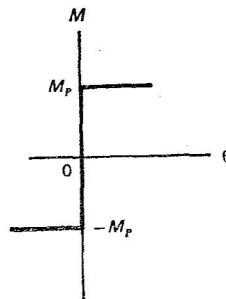


Fig. 1

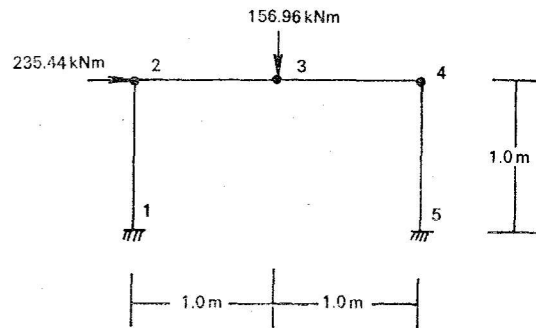


Fig. 2

where the 8×8 matrix Z is obtained from rel. (17) and N is the 8×16 diagonal matrix of [1-1]. The 8×1 vector s^E is obtained from relation

$$(30) \quad s^E = K_0 G^T K^{-1} p$$

and k is the 16×1 vector of the plastic moments of the sections.

From the elastic analysis of the statically determinate structure, the elastic moment vector s_0^T is obtained

$$s_0^T = [0, 39.24, 39.24, 0, 0, -196.2, -196.2, 0].$$

Furthermore, the 8×8 flexibility and stiffness matrices F_0 and K_0 , the 5×8 equilibrium matrix G and the 5×5 matrix K can be easily computed [34]. For

$$(31) \quad d_1 = -(N^T s^E - k)$$

the K - T conditions of the problem take the form

$$(32) \quad D\lambda + d_1 = y$$

$$(33) \quad (34) \quad (35) \quad \lambda \geq 0, \quad y \geq 0, \quad \lambda^T y = 0.$$

The 17×17 tableau of the problem has the form

$$\begin{array}{c|c|c} \lambda^T & 1 & \\ \hline D & d_1 & = y \\ \hline d_1^T & 0 & = P - \lambda^T y. \end{array}$$

The solution, i. e. the values of the plastic multipliers, has been obtained in three pivotal steps in elements 15/15, 12/12 and 2/2

$$\lambda^T = [0, 17.422, 0, 0, 0, 0, 0, 0, 0, 0, 0, 25.048, 0, 0, 20.162, 0] \text{ rad.}$$

Since [16] $\theta = N\lambda$, $s^P = Z\theta$ and $s = s^E + s^P$, the vector

$$s^T = [-65.73, 38.247, 38.247, 64.738, 64.738, -65.73, -65.73, 65.73] \text{ kNm}$$

of the moments can be easily obtained. The occurrence of a forth hinge is caused by a rise in the values of the loads, to 236.63 kN and 157.75 kN, respectively. The solution has been obtained again in three pivotal steps and the bending moments are given by

$$s^T = [-65.73, 39.438, 39.438, 65.73, 65.73, -65.73, -65.73, 65.73] \text{ kNm}$$

(cf. the results in [33]). A further increase in the values of the loads, to 236.64 kN and 157.76 kN respectively, causes the collapse of the structure after an unsuccessful attempt to redistribute the loading. This attempt is equivalent to the forth step of the algorithm. Thus the program has to optimize an unbounded from below potential energy function, but the structure is now unstable and the procedure is interrupted.

The structure of the second example is the three bay-ten storey steel frame depicted in Fig. 3.

In this case the critical sections of the structure are 200. The workhardening matrix H is zero again. The structure must be analyzed for combined gravity and

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wind loads. Following the same procedure but taking into consideration the fact that in this case the loading is continuous [42], the 401×401 tableau of the problem is formulated. The structure has been analyzed for different values of the load factor α

Data for the Frame of Fig. 3

Storey Height:	3.0 m	Columns	
Bay Width:	6.0 m	Bars	Section
E:	200.8 GPa	1 to 20	8W40
Roof Load:	20.43 kN/m	21 to 40	8W35
Floor Load:	23.94 kN/m	41 to 60	8W28
Wind Load:	5.84 kN/m height	61 to 80	8W20
		81 to 100	8W13
		Beams	
		Bars	Section
		2 to 19	16W26
		22 to 59	14W26
		62 to 79	14W22
		82 to 99	12W22

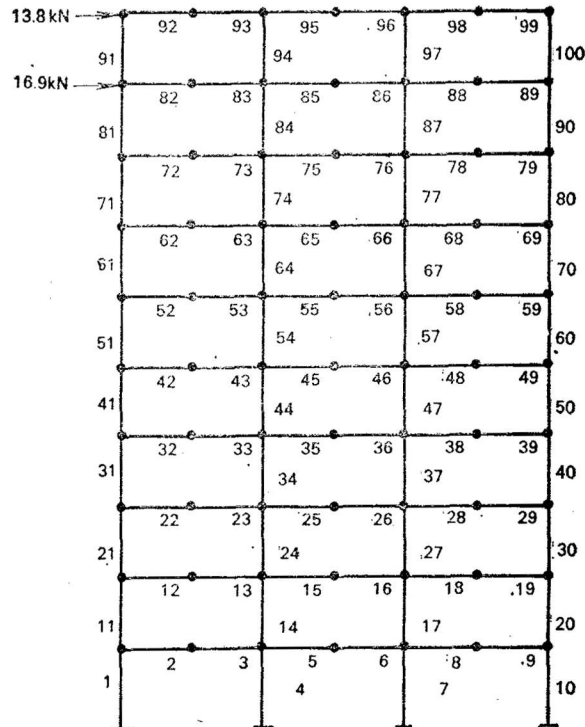


Fig. 3

starting from 1.24, in which appears the first hinge, up to 2.21, in which the structure collapses fully. The change of the values of the moments in the sections in which a hinge appears, for the different load factors, has been obtained in graphs, for each of the ten storeys separately. Some of these graphs are included here. Fig. 4 to Fig. 8 show the moments for values of α from 1.24 up to 2.09. For this value of the load factor, 66 hinges have been formed. If the load factor takes the value 2.10, the structure begins to collapse gradually. This can be seen in Fig. 5 to Fig. 9. It must be

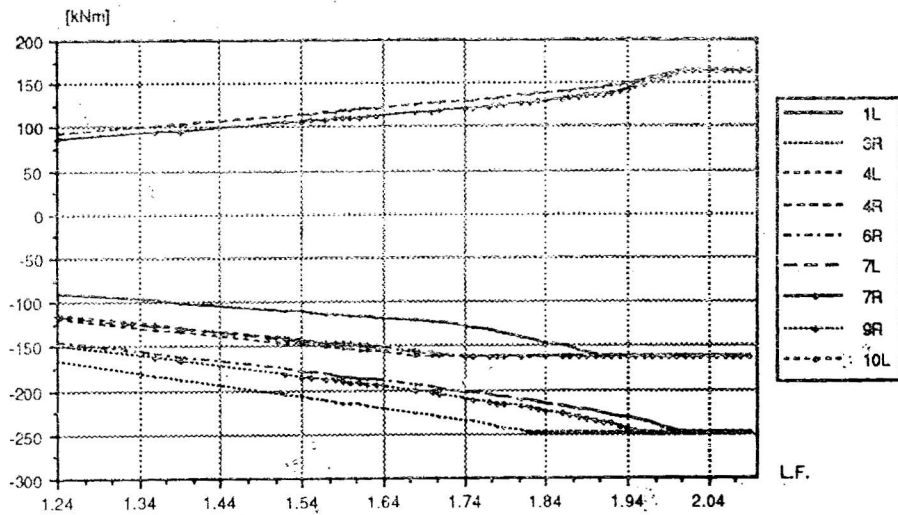


Fig. 4. First storey-plastic

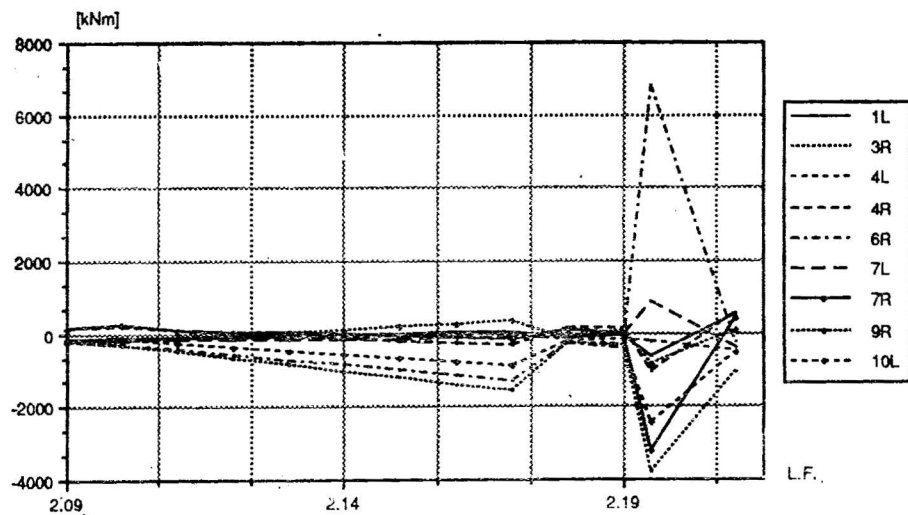


Fig. 5. First storey-collapse

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pointed out that, as it can be seen in Fig. 10 and Fig. 11 unloading phenomena in the plastic range appear in bar 20L for α equal to 2.03.

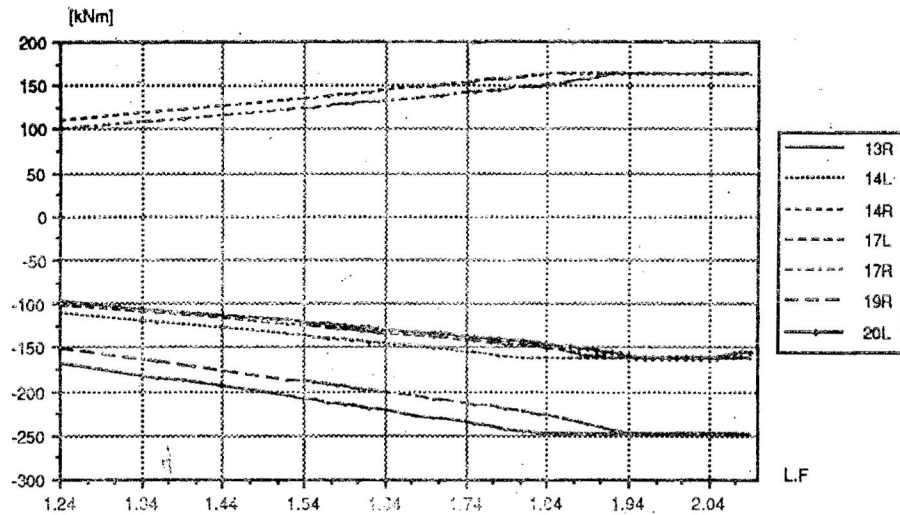


Fig. 6. Second storey-plastic

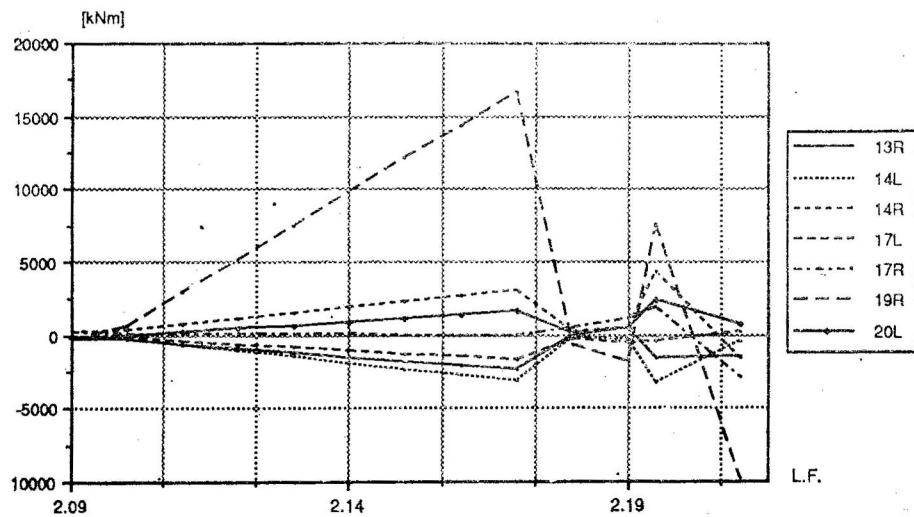


Fig. 7. Second storey-collapse

The only data the program needs are the member information for the flexibility matrix of the structure, the initial loading, the results of the elastic analysis of the statically determinate structure for this loading and the plastic moments of the sections, all stored in text (ASCII code) files. The numerical results, i. e. the plastic multipliers λ , the rotations θ , and the moments s , can be read from the screen, printed by

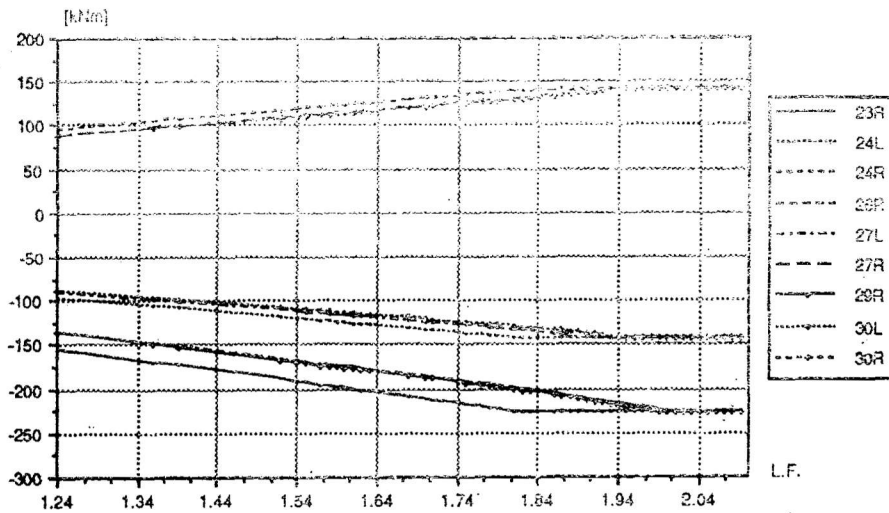


Fig. 8. Third storey-plastic

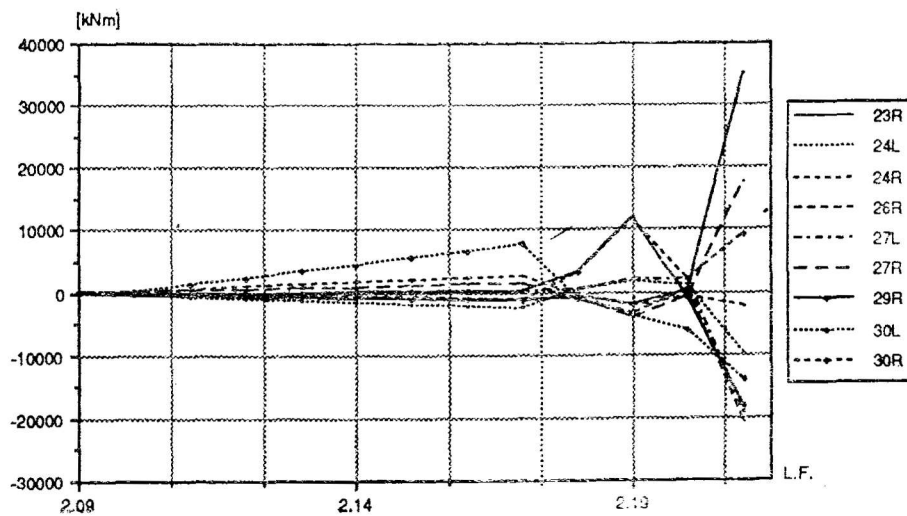


Fig. 9. Third storey-collapse

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a printer or stored in text files so as to be further processed easily by any of the popular Spreadsheet, Data-base or Graphics programs.

The implementation of the applications has been done in an Apple-Macintosh SE personal computer with 1 MB RAM-256 ROM memory (20 MB Hard Disc), in com-

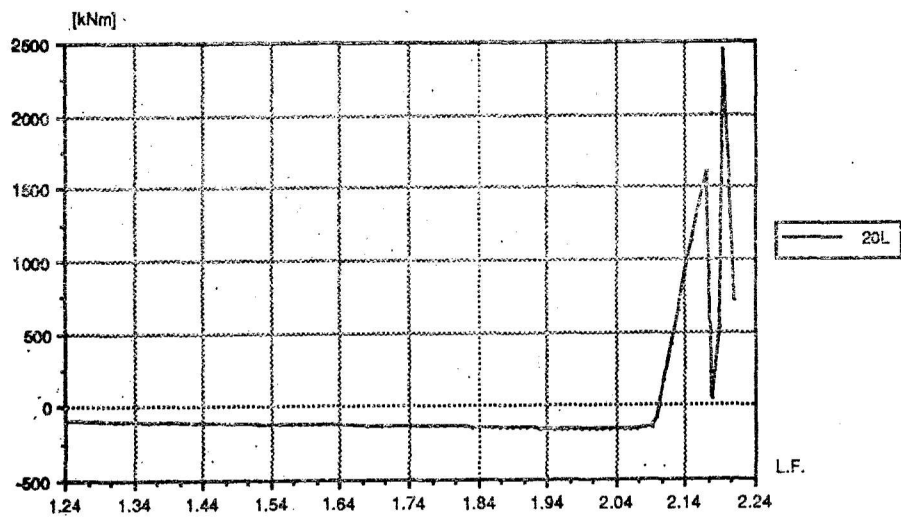


Fig. 10. Bar 20 L

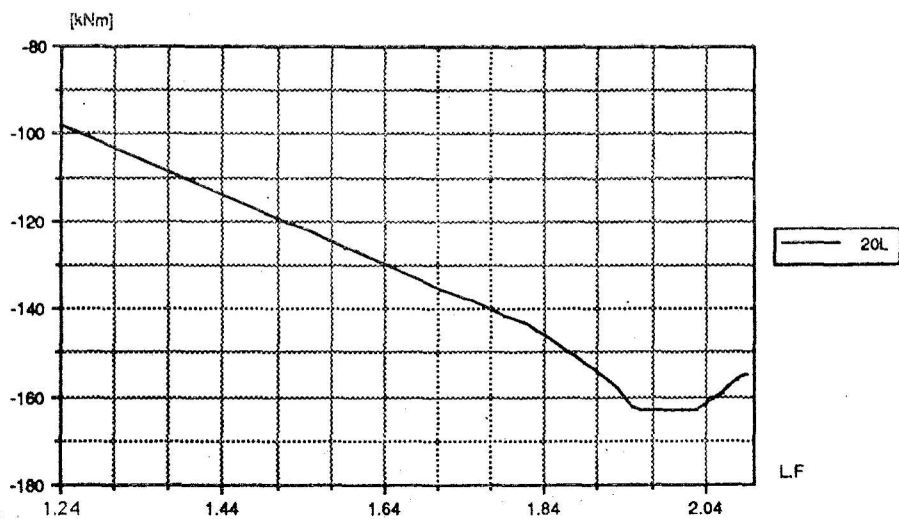


Fig. 11. Bar 20L-plastic

piled Basic and with single precision variables. A Bland-type [14] technique has been used to avoid degeneracy for certain values of the load factor.

List of Symbols

- $\dot{\epsilon}$ — the strain vector,
- $\dot{\epsilon}_0$ — the initial strain vector,
- $\dot{\epsilon}_E$ — the elastic strain vector,
- $\dot{\epsilon}_P$ — the plastic strain vector,
- $\dot{F}_0$ — the natural flexibility matrix,
- $\dot{s}$ — the stress vector,
- $\dot{F}$ — the yield functions vector,
- N — the gradient of the yield functions which are zero,
- V — the gradient of the corresponding plastic potentials,
- H — the workhardening matrix,
- $\dot{\lambda}$ — the plastic multipliers vector,
- G — the equilibrium matrix,
- K_G — the geometric stiffness matrix,
- $\dot{u}$ — the displacement vector,
- $\dot{p}$ — the load vector

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