

Механика на флуидите

Stochastic Model of Two Types Magnetic Fields

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The purpose of this work is the construction of a magnetic field stochastic model, or more precisely, the construction of a random magnetic vector, a sum of a random number local magnetic perturbations in random points of space. We will consider two types of perturbations: elementary direct currents and magnetic dipoles; and we will calculate the distribution of the resultant vector. We follow the scheme, given in (1).

Further, we will call the perturbation particles and we will consider them to be Poisson distributed. It means that in each compact set the number of particles follows the Poisson law; the numbers of particles in two disjointed sets are independent. Usually, that distribution is deduced making use of the following assumptions

1. independence of the number of particles in disjointed sets
2. the probability for availability of two or more particles in a set with decreasing volume, compared with that for only one particle is neglectable.

Further, we will suppose that the Poisson distribution describes adequately the physical nature of the phenomena studied namely, the availability of magnetic perturbations in an area of the space. The distribution of these perturbations satisfies the conditions mentioned above.

It is to the point to mention that the dimension of the object studied has no influence on the conclusions, i. e. these objects may model the magnetic field of elementary particles and the magnetic field of planets also.

We shall consider two different models. While the first one deals with short distances and times (micro level), the second one describes long ones (macro level).

Let us imagine a device which registers the magnetic field. Let us first suppose that an elementary particle moves in the space not far from our device, so for small time interval Δt depending on averaging capabilities of the device it covers a small distance Δs . Then the magnetic field to be registered follows the law of Bio-Savart. The number of particles in this case is random, as well as their speeds and directions.

The second model deals with magnetic dipoles situated randomly in the space. For example, the magnetic fields of stars and planets are said to be dipoles.

Any point in the space, where the magnetic field influences, may be characterized by its position and by the magnetic vector, which represents

the intensity of the field in the point. We suppose an isotropic magnetic field, which means spherical symmetry of distribution of magnetic field vector, or identity of all directions in the space, the magnetic field vector may follow.

I. We consider an elementary direct current, which creates a magnetic field. We suppose it is situated at the origin of the coordinate system. In general the dependence between the points of the space and the intensity of magnetic field, due to the characteristics of a given magnetic perturbation, is called function of influence. In that case it looks like that

$$\vec{n}(\vec{x}, \vec{M}) = \frac{|\vec{M}|}{|\vec{x}|^2} (\vec{e}_M \times \vec{e}_x),$$

where $\vec{x}$ is radius vector, $\vec{M}$ the current vector. The vectors $\vec{e}_x$ and $\vec{e}_M$ are unit vectors along $\vec{x}$ and $\vec{M}$ respectively. Let us examine the next random sum:

$$\vec{\xi} = \sum_{i=1}^{\infty} \vec{u}(\vec{x}_i, \vec{M}_i).$$

We calculate its characteristic function in the next way:

$$\begin{aligned} X(\vec{t}) &= E e^{i(\vec{t}, \vec{\xi})} = \lim_{|V| \rightarrow \infty} E e^{i(\vec{t}, \vec{\xi}_V)} = \\ &= \lim_{|V| \rightarrow \infty} X_V(\vec{t}) = \lim_{|V| \rightarrow \infty} \int \exp(i(\vec{t}, \vec{\xi}_V)) dP(\vec{\xi}_V), \end{aligned}$$

where $\vec{\xi}_V$ is the sum of the influence functions of the particles in the volume V . The situation, when in this volume there are exactly k particles we denote with ξ_V^k , and $\vec{\xi}_V^k$ is the sum of the influence functions of those particles. Following the assumption for Poisson distribution we obtain:

$$\Sigma P(\xi_V = \xi_V^k) = 1$$

and

$$P(\xi_V = \xi_V^k) = \exp(-\kappa |V|) \frac{(\kappa |V|)^k}{k!},$$

where ξ_V is the number of particles in the volume V , and κ is the Poisson process intensity. Using the formulae for the full probability we obtain:

$$\begin{aligned} X_V(\vec{t}) &= \int \exp(i(\vec{t}, \vec{\xi}_V)) dP(\vec{\xi}_V) = \exp(i(\vec{t}, \vec{\xi}_V)) \sum_{k=0}^{\infty} P(\xi_V = \xi_V^k) dP(\vec{\xi}_V^k) \\ &= \sum_{k=0}^{\infty} P(\xi_V = \xi_V^k) \int \exp(i(\vec{t}, \vec{\xi}_V)) dP(\vec{\xi}_V^k) \\ &= \sum_{k=0}^{\infty} \exp(-\kappa |V|) \cdot \frac{(\kappa |V|)^k}{k!} \underbrace{\int \exp(i(\vec{t}, \vec{\xi}_V^k)) \frac{dx_1 dx_2 \dots dx_k}{|V|^k} dP(M_1) \dots dP(M_k)}_{k} \end{aligned}$$

Then we use the identity and independence of particles to obtain:

$$\begin{aligned} X_V(\vec{t}) &= \exp(-\kappa |V|) \sum_{k=0}^{\infty} \frac{(\kappa |V|)^k}{k!} \left(\int \exp(i(\vec{t}, \vec{n})) \frac{d\vec{x}}{|V|} dP(\vec{M}) \right)^k \\ &= \exp(-\kappa |V|) \cdot \exp(\kappa \int \exp(i(\vec{t}, \vec{n})) d\vec{x} dP(\vec{M})). \end{aligned}$$

Hence:

$$\ln X_V(\vec{t}) = \kappa \left(\int \exp(i(\vec{t}, \vec{n})) d\vec{x} dP(\vec{M}) - |V| \right) = \kappa \int (\exp(i(\vec{t}, \vec{n})) - 1) d\vec{x} dP(\vec{M}).$$

This way we obtain:

$$\ln X(\vec{t}) = \lim_{|V| \rightarrow \infty} \ln X_V(\vec{t}) = \kappa \int (\exp(i(\vec{t}, \vec{n})) - 1) d\vec{x} dP(\vec{M}),$$

where the integral is on the whole space.

It is obvious for the influence function that:

$$\vec{n}(\vec{x}, \vec{M}) = -\vec{n}(\vec{x}, -\vec{M}).$$

Therefore:

$$\kappa \int i(\vec{t}, \vec{n}) d\vec{x} dP(\vec{M}) = 0.$$

Finally the logarithm of the characteristic function takes on the form:

$$\ln X(\vec{t}) = \kappa \int (\exp(i(\vec{t}, \vec{n})) - 1 - i(\vec{t}, \vec{n})) d\vec{x} dP(\vec{M}).$$

When the influence function is not odd, we obtain the above-mentioned form of the logarithm of the characteristic function, see (5).

Thus we have for examining the integral:

$$f(\vec{t}) = \kappa \int (\exp(i(\vec{t}, \vec{n})) - 1 - i(\vec{t}, \vec{n})) d\vec{x} dP(\vec{M}),$$

where $d\vec{x} = r^2 dr d\vec{e}_x$ and $dP(\vec{M}) = d\vec{e}_M P dM$, $\vec{x} = r\vec{e}_x$. If we take only the real part of the integral mentioned we obtain:

$$f(\vec{t}) = \kappa \int_{S_x} \int_{S_M} \int_0^\infty \cos\left((\vec{t}, \vec{a}) \frac{|\vec{M}|}{r^2} - 1\right) r^2 dr d\vec{e}_x d\vec{e}_M P dM,$$

where $\vec{a} = \vec{e}_M \times \vec{e}_x$ and S_x and S_M are unit spheres. After changing the variables we obtain:

$$f(\vec{t}) = \kappa \int_{S_x} \int_{S_M} [(\vec{t}, \vec{a}) |\vec{M}|]^{3/2} \left(- \int_0^\infty \frac{\cos \tau - 1}{2\tau^{5/2}} d\tau \right) d\vec{e}_x d\vec{e}_M P dM.$$

If we denote: $h = \int |\vec{M}| P dM$, we obtain:

$$f(\vec{t}) = -h\kappa \int_{S_x} \int_{S_M} |(\vec{t}, \vec{a})|^{3/2} d\vec{e}_x d\vec{e}_M = -h\kappa |\vec{t}|^{3/2} \int_{S_x} \int_{S_M} |(e_t, a)|^{3/2} d\vec{e}_x d\vec{e}_M.$$

Due to the spherical symmetry finally we get:

$$f(\vec{t}) = -\kappa h \text{const.} |\vec{t}|^{3/2}.$$

II. We consider a field, created by a magnetic dipole. In this case the function of influence looks like that:

$$\vec{u}(\vec{x}, \vec{M}) = \frac{|\vec{M}|}{|\vec{x}|^3} (3(\vec{e}_x, \vec{e}_M) \vec{e}_x - \vec{e}_M).$$

We transform the integral:

$$f(\vec{t}) = \kappa \int (\exp(i(\vec{t}, \vec{n})) - 1 - i(\vec{t}, \vec{n})) d\vec{x} dP(\vec{M})$$

as in the previous case and we obtain:

$$f(\vec{t}) = -h\kappa \int_{S_x} \int_{S_M} |(\vec{t}, \vec{a})| d\vec{e}_x d\vec{e}_M = -h\kappa |\vec{t}| \int_{S_x} \int_{S_M} |(\vec{e}_t, \vec{a})| d\vec{e}_x d\vec{e}_M,$$

where $\vec{a} = 3(\vec{e}_x, \vec{e}_M) \vec{e}_x - \vec{e}_M$.

In general, if we have an influence function of the form:

$$\vec{n}(\vec{x}, \vec{M}) = \frac{|\vec{M}|}{|\vec{x}|^\beta} \vec{g}(\vec{x}, \vec{M}), \quad \vec{g}(\vec{x}, \vec{M}) = -\vec{g}(\vec{x}, \vec{M}),$$

where $\vec{g}$ is a bounded function. Using transforms analogue to those for the logarithm of the characteristic function, we obtain:

$$f(\vec{t}) = -\kappa h \text{const.} |\vec{t}|^{3/\beta}.$$

Now, we will use the fact that the integral has the sense of a logarithm of the characteristic function, i. e. the characteristic functions in the first and the second case are:

$$\Psi_{3/2}(\vec{t}) = \exp(-\alpha |\vec{t}|^{3/2}), \quad \Psi_1(\vec{t}) = \exp(-\alpha |\vec{t}|).$$

These are stable distributions with powers 3/2 and 1 accordingly.

III. Densities and spherical functions of distributions. We obtain the corresponding densities, using the inverse Fourier transform:

$$\int_{R^3} \exp(-2\pi i(\vec{x}, \vec{t})) \Psi_i(\vec{t}) d\vec{t}, \quad i=1, 5; 1.$$

We examine both cases, using the function:

$$\Psi(\vec{t}) = \exp(-\alpha |\vec{t}|^r).$$

It is spherically symmetric and its Fourier image is the same. Denoting:

$$r = |\vec{t}|, \quad \rho = |\vec{x}|$$

we reduce the integral:

$$\int_{R^3} \exp(-2\pi i(\vec{x}, \vec{t})) \exp(a|\vec{t}|^\gamma) d\vec{t}$$

to (3):

$$\begin{aligned} I_\gamma(\rho) &= \frac{2}{\rho^{1/2}} \int_0^\infty \exp(-2\pi a r^\gamma) r^{3/2} J_{\frac{1}{2}}(2\pi \rho r) dr \\ &= \frac{2}{\rho} \int_0^\infty \exp(-2\pi a r^\gamma) r \sin(2\pi \rho r) dr. \end{aligned}$$

We decompose the function under integral in series and we obtain:

$$I_\gamma(\rho) = \frac{1}{2\pi\gamma\rho^3\beta^{2/\gamma}} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(\frac{2k+3}{\gamma}\right)}{(2k+1)! \beta^{\frac{2k+3}{\gamma}}},$$

where $\beta = 2\pi a \cdot (2\pi\rho)^{-\gamma}$

Now we may assume $\gamma=1$ and:

$$I_1(\rho) = \frac{a}{\pi^2(a^2 + \rho^2)^2},$$

which we can calculate directly. This is the well known Cauchy density.

We assume $\gamma=1,5$:

$$I_{\frac{3}{2}}(\rho) = \frac{1}{3\pi^2(2\pi)^{2/3}\rho^3} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(\frac{4k+6}{3}\right) \rho^{2k+1}}{(2k+1)! (a^{2k+1})^{2/3}},$$

which reminds the one-dimension stable distribution, considered by Feller (4), so called Holzmak distribution. For practical use we shall need the next probability:

$$\begin{aligned} P_{a,\gamma}(\rho \leq R) &= 4\pi \int_0^R I_\gamma(\rho) \rho^2 d\rho \\ &= \frac{1}{\pi^2\gamma(2\pi)^{\frac{3-4\gamma}{\gamma}}} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(\frac{2k+3}{\gamma}\right) R^{2k+3}}{(2k+1)! (2k+3) (2\pi)^{\frac{2k(1-\gamma)}{\gamma}} a^{\frac{2k+3}{\gamma}}}. \end{aligned}$$

If we denote $x_0 = R \cdot [a^{1/\gamma}(2\pi)^{\frac{1-\gamma}{\gamma}}]^{-1}$, finally we obtain for the function of distribution of $|\vec{x}|$:

$$P_{a,\gamma}(\rho \leq a^{1/\gamma}(2\pi)^{\frac{1-\gamma}{\gamma}} \cdot x_0) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(\frac{2k+3}{\gamma}\right)}{(2k+1)! (2k+3)} x_0^{2k+3}.$$

Particularly in the case $\gamma=1$ we obtain:

$$P_{\alpha,1}(\rho \leq \alpha x_0) = \frac{2}{\pi} \left(\operatorname{arctg} x_0 - \frac{x_0}{1+x_0^2} \right)$$

where $x_0 = R/\alpha$.

The conclusions of this work may be automatically transferred in hydrodynamics as in the classical analogy between magnetic and hydrodynamics objects.

As a matter of fact, we obtain the ability for evaluation which of the model gives a more adequate representation of the real space magnetic fields.

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