

The Axisymmetric Boussinesq Problem for a Thick Elastic Plate under a Punch of Arbitrary Profile

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1 Introduction

Boussinesq [1] first referred to a problem determining the stress and displacement distribution in the elastic half space when its free surface is indented by a rigid punch. Several alternative solutions have been derived by Harding and Sneddon [2], Sneddon [3], Galin [4], Segedin [5] and Muki [6] following Boussinesq's solution. Subsequently the axisymmetric Boussinesq's problem for a half-space under a punch of arbitrary profile has been considered by Sneddon [7] and obtained simple formulae for the depth of penetration of the tip of the punch. The expressions for total load, that must be applied to the punch to achieve this penetration, the distribution of pressure under the punch, and the shape of the free surface have been derived.

The punch problem of an elastic layer is of a more complicated nature and has been considered by Lebedev and Ufliand [8], Vorovich and Ustinov [9], Florence [10], England [11], Dhaliwal [12]. These authors have solved the problem by using the integral transform technique. It is worth mentioning that a systematic treatment of the punch problems for a half-space and a layer are available from Sneddon [13] and Sneddon and Lowengrub [14] books. Subsequently Dhaliwal and Rau [15] have also derived analytical results for the Boussinesq problem of a punch of arbitrary profile on an elastic layer lying over an elastic half-space. They reduced the problem into dual integral equations also considered earlier by a number of workers. The particular results have been obtained by Dhaliwal and Rau [15] for cylindrical, conical, spherical, paraboloidal and ellipsoidal punch shapes. Recently Lal et. al. [16], solved the axisymmetric punch problem in an infinite elastic thick plate and this problem was reduced into dual integral equations having been solved through series method adopting Legendre orthogonal polynomials.

In the present communication, the axisymmetric Boussinesq problem for a thick elastic plate under a punch of arbitrary profile have taken up when the faces of punch are slightly concave. When the indented load is small, the contact area will be annulus, the outer circumference of which coincides with the edges of the punch. The inner circumference will shrink with an increase of load and there will be a critical load above which the entire punch face makes contact with the plate. The contact problem for high loads can, there-

fore, be treated on the basis of the theory of elasticity. Simple expressions have been derived for the normal displacement and stress under the parabolic and conical concave punch and the indented load 'P' required on the punch for a given depth of penetration.

2. Basic equations and boundary conditions

The stresses and displacements in an axisymmetric problem can be derived from the Love function ψ satisfying

$$(2.1) \quad \nabla^2 \nabla^2 \psi = 0,$$

where

$$(2.2) \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

The displacement and stress fields are

$$(2.3) \quad u = -\frac{\partial^2 \psi}{\partial r \partial z}; \quad w = 2(1-\eta) \Delta^2 \psi - \frac{\partial^2 \psi}{\partial z^2},$$

$$(2.4) \quad \sigma_{zz} = 2G \frac{\partial}{\partial z} \left[(2-\eta) \nabla^2 - \frac{\partial^2}{\partial z^2} \right] \psi,$$

$$(2.5) \quad \sigma_{rz} = 2G \frac{\partial}{\partial r} \left[(1-\eta) \nabla^2 - \frac{\partial^2}{\partial z^2} \right] \psi,$$

where u and w are axial and radial displacements and G is the modulus of rigidity and η is Poisson's ratio, respectively.

Now we consider an infinite, isotropic, homogeneous thick elastic plate bounded by the planes $z = \pm h$. The free surfaces of the plate are indented over a circular area by a rigid punches of arbitrary profiles. The stress and displacement fields in the elastic plate satisfy equations (2.3)–(2.5) and our problem is now to determine the solution satisfying the boundary conditions

$$(2.6) \quad w(r, \pm h) = -\delta - u g(r); \quad 0 \leq r \leq a,$$

$$(2.7) \quad \sigma_{zz}(r, \pm h) = 0; \quad r > a,$$

$$(2.8) \quad \sigma_{rz}(r, \pm h) = 0; \quad r \geq 0,$$

where the function $g(r)$ is prescribed by the fact that, referred to the tip as origin, the punch has the equation $z = g(r)$, so that $g(0) = 0$ δ is a parameter (as yet unspecified). The depth to which the tip of the punch penetrates the elastic plate is the physical significance of δ .

3. Solution of the problem

A solution of the equation (2.1) can be assumed in the form

$$(3.1) \quad \psi = \int_0^\infty \xi^{-3} [A(\xi) \cos h(\xi z) + B(\xi) \xi z \sinh(\xi z)] J_0(\xi r) d\xi,$$

where A and B are functions of ξ only.

The expressions for displacement and stresses by using the equations (2.3)–(2.5) and (3.1) are as follows:

$$(3.2) \quad u = \int_0^{\infty} \xi^{-1} [\{A(\xi) + B(\xi)\} \sin h(\xi z) + B(\xi) \xi z \cos h(\xi z)] J_1(\xi r) d\xi,$$

$$(3.3) \quad w = \int_0^{\infty} \xi^{-1} [\{2B(\xi)(1-2\eta) - A(\xi)\} \cos h(\xi z) - B(\xi) \xi z \sin h(\xi z)] J_0(\xi r) d\xi,$$

$$(3.4) \quad \sigma_{zz} = 2G \int_0^{\infty} [\{B(\xi)(1-2\eta) - A(\xi)\} \sin h(\xi z) - B(\xi) \xi z \cos h(\xi z)] J_0(\xi r) d\xi,$$

$$(3.5) \quad \sigma_{rz} = 2G \int_0^{\infty} [\{A(\xi) + 2\eta B(\xi)\} \cos h(\xi z) + B(\xi) \xi z \sin h(\xi z)] J_1(\xi r) d\xi.$$

Now from the boundary condition (2.8) that the shearing stress is zero on both the faces of the plate, therefore,

$$(3.6) \quad A(\xi) = -[2\eta + \xi h \tan h(\xi h)] B(\xi).$$

By putting the value of $A(\xi)$ in the equations (3.3) and (3.4) and using the boundary conditions (2.6) and (2.7), these equations [(3.3) and (3.4)] are reduced to dual integral equations

$$(3.7) \quad \left. \begin{aligned} \int_0^{\infty} M(\xi) J_0(\xi r) d\xi &= -\frac{1}{2(1-\eta)} [\delta - u g(r)]; \quad 0 \leq r \leq a, \\ \int_0^{\infty} \xi M(\xi) J_0(\xi r) d\xi &= 0, \quad r > a, \end{aligned} \right\}$$

where

$$M(\xi) = \xi^{-1} B(\xi) \cos h(\xi h).$$

By using the Abel's transform method of Lal [17], equation (3.7) is equivalent to the following equation:

$$(3.8) \quad \int_0^{\infty} M(\xi) \cos(\xi x) d\xi = \frac{1}{2(1-\eta)} \left[c_0 + x \int_0^{\infty} \frac{u g'(t) dt}{v(x^2 - t^2)} \right] H(a - x),$$

where $H(x)$ is Heaviside's step function and c_0 is an unknown constant. Therefore, the function $M(\xi)$ is obtained by Fourier inverse transform as follows:

$$(3.9) \quad M(\xi) = \frac{1}{\pi(1-\eta)} \left[c_0 \xi^{-1} \sin(\xi a) + \int_0^a x \cos(\xi x) dx \int_0^{\infty} \frac{u g'(t) dt}{v(x^2 - t^2)} \right].$$

Now we study two parabolic and conical concave punch cases.

(a) The case of the parabolic concave punch

When the punch faces of the plate are concave, we denote $u=u_1$, and $g(r)=1-(r/a)^2$. Then, the normal displacement $w(r, \pm h)$ in the contact region is

$$(3.10) \quad w(r, \pm h) = \delta - u_1 [1 - (r/a)^2], \quad 0 \leq r \leq a.$$

The function $M(\xi)$ is

$$(3.11) \quad M(\xi) = \frac{1}{\pi(1-\eta)} [\xi^{-1}(c_0 - 2u_1) \sin(\xi a) - \{4u_1 \xi a \cos(\xi a) - \sin(\xi a)\} a^2 \xi^{-3}].$$

Substituting equation (3.11) into equation (3.7), $w(r, \pm h)$ and $\sigma_{zz}(r, \pm h)$ are

$$(3.12) \quad w(r, \pm h) = -\frac{2}{\pi} \left[c_0 \begin{cases} \frac{\pi}{2}, & a \geq r \geq 0 \\ \sin^{-1}(a/r), & a \leq r \end{cases} - \frac{2u_1}{a^2} \int_0^a \frac{x^2 H(r-x) dx}{v(r^2-x^2)} \right],$$

$$\sigma_{zz}(r, \pm h) = \frac{1}{\pi(1-\eta)} \left[\frac{c_0^{-2} 4_1}{v(a^2-r^2)} - \frac{4u_1}{a^2} v(a^2-r^2) \right] H(a-r).$$

Since $w(r, \pm h)$ in equation (3.12) must be coincided with the equation (3.10) in $0 < r < a$, we obtain

$$c_0 = u_1 - \delta.$$

Then the normal displacement $w(r, \pm h)$ on the free surfaces ($r \geq a$) are

$$(3.13) \quad w(r, \pm h) = \frac{2}{\pi} [\{\delta - u_1(1-r^2a^{-2})^{1/2}\} \xi \sin^{-1}(a/r) - u_1 r a^{-1} v(1-a^2r^{-2})], \quad r \geq a,$$

and the normal stresses $\sigma_{zz}(r, \pm h)$ in the contact regions ($0 \leq r \leq a$) are

$$(3.14) \quad \sigma_{zz}(r, \pm h) = -\frac{1}{a\pi(1-\eta)} [\{\sigma - 3u_1 + 4u_1(r/a)^2\} \{1 - (r/a)^2\}^{-1/2}], \quad 0 \leq r \leq a.$$

The condition that the entire punch surfaces of the plate make contact with the half-space are $\sigma_{zz}(r, \pm h) \leq 0$ in $0 \leq r < a$. Then we obtain from equation

$$(3.14)$$

$$(3.15) \quad \delta \geq 3u_1,$$

and the indented load 'P' is

$$(3.16) \quad P = -2\pi \int_0^a \sigma_{zz}(r, \pm h) r dr = \frac{2aG}{(1-\eta)} (\delta - u_1/3), \quad \left[\geq P_0 \equiv \frac{16}{3} \frac{au_1}{(1-\eta)} \right],$$

where the critical load 'P' means the minimum indented load for the entire face to contact. If $P \geq P_0$, 'P' increases with the increase of the indented depth δ . If $P < P_0$ or $\delta < 3u_1$, $\sigma_{zz}(r, \pm h)$ becomes tension in the neighbourhood of $r=0$, and it denotes that the contact region becomes annular physically.

Additionally, if $u_1 < 0$ and $\delta \geq -u_1$, the punch faces are convex and the stresses $\sigma_{zz}(r, \pm h)$ are always compressive in the contact regions $0 \leq r \leq a$. If $u_1 < 0$ and $\delta < -u_1$, the stresses $\sigma_{zz}(r, \pm h)$ are compressive without the neighbourhood of $r=a$.

(b) The case of the conical concave punch

When the punch faces are conical concave, we denote $u = u_2$ and $g(r) = 1 - (r/a)$. Then, the normal displacements $w(r, \pm h)$ in the contact regions are

$$w(r, \pm h) = \delta - u_2 [1 - (r/a)], \quad 0 \leq r \leq a,$$

and the function $M(\xi)$ is

$$(3.18) \quad M(\xi) = \frac{1}{\pi(1-\eta)} [(c_0 - \pi u_0/2) \xi^{-1} \sin(\xi a) + \{\pi u_2 (1 - \cos(\xi a)) / 2a \xi^2\}].$$

Substituting equation (3.18) into equation (3.7) and using equation (3.13) for normal displacement, we obtain $c_0 = u_2 - \delta$. Then, the normal displacements $w(r, \pm h)$ in the free surfaces of the plate and the normal stresses $\sigma_{zz}(r, \pm h)$ in the contact regions are

$$(3.19) \quad w(r, \pm h) = \left[\{2(\delta - u_2) \sin^{-1}(a/r)\} \pi + u_2 r a^{-1} \{1 - (1 - a^2 r^{-2})^{1/2}\} \right] a < r,$$

$$\sigma_{zz}(r, \pm h) = -\frac{1}{\pi(1-\eta)} \left[\left\{ \delta + \left(\frac{\pi}{2} - 1 \right) u_2 \right\} (a^2 - r^2)^{-1/2} - \{ \pi u_2 \cos h^{-1}(a/r) \} / 2 \right] H(a - r).$$

If $u_2 > 0$, the stresses $\sigma_{zz}(r, \pm h)$ are always tension in the neighbourhood of $r = 0$. Therefore, the contact regions in the case of the conical concave punch are always annular, or the critical load is infinite.

Additionally, if $u_2 < 0$ and $\delta \geq -(\pi/2 - 1)u_2$, the punch faces are conical convex and $\sigma_{zz}(r, \pm h)$ are always in the state of compression in the boundary condition $0 \leq r \leq a$. If $u_2 < 0$ and $\delta < -(\pi/2 - 1)u_2$, the stresses $\sigma_{zz}(r, \pm h)$ are in compression without the neighbourhood of $r = a$ and thus the results obtained by Love [18] are confirmed.

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