

## The Dynamic Theory of Shakedown of Elasto-Plastic Structures

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### I. Introduction

The shakedown problem of elasto-plastic structures has an important role in the domain of design, with respect to the elasto-plastic system subjected to loads and temperatures variable with time.

The theorem of shakedown on quasi-static loadings has been proved by E. Melan [1]. W. Prager generalized afterwards Melan's theorem for the case of the system subjected both to loads and temperatures [2].

In [3] we extended E. Melan and W. Prager's theorems in quasi-static loadings, while at the same time we suggested the method of solving the shakedown problem by using the mathematical programming.

The theory of dynamic shakedown (i. e. the inertial forces included) has been proved by Hwa-Shan-Ho in 1970 [4]. In that work the author proved:

When taking the inertial force into account, i. e. in the problem of dynamic shakedown the elastic stress is the solution of the elasto-dynamic problem, then E. Melan's theorem is still true. At the same time, Hwa-Shan-Ho also introduced two approximation methods to solve the problem of dynamic shakedown, by bringing it to the statical problem, those methods however are only applied to the particular cases. In [5, 6] G. Ceradini and K. Gabarini also studied dynamic shakedown problem.

In the paper, we consider the following problems.

#### 1.1. Expressing and proving the extended theorem of dynamic shakedown and two consequences

The obtained theorem is more general than that of Hwa-Shan-Ho because it does not require the existence of a field of residual stress common for the total acting processes of external forces, but only for each portion of the time-axis, while the definition of the shakedown is unchanged. Thanks to that we can "stick" together the Hwa-Shan-Ho's shakedown domains in order to get a larger domain.

#### 1.2. Formulation of two kinds of shakedown problems

a) With a certain law of acting external force is the problem of control a shakedown problem? When does it occur, if it is shakedown?

b) With a given law on loading we try to find the largest set of similar loads with which the system is shakedown.

3. Suggesting the approximation method for the residual stress domain as a finite element method, from which we can solve approximately the problem of shakedown by the algorithms of mathematical programming.

## 2. Extension of the dynamic shakedown theorem

1. Expression of loadings in dynamic problem suppose the system subjected combined to the loads  $\vec{P}_i(t)$  ( $i=1, 2, \dots, n$ ) which change independently with time  $t$ . The temperature is considered as a particular case of load. Without loss of generality, we consider the direction of forces  $\vec{P}_i(t)$  as unchanged (otherwise we separate them into 3 forces having unchanged directions).

Thus the loads acting on the system are represented by the  $n$ -dimensional vector in the loading space  $\vec{P}=\{P_i(t)\}$ . The orbit of the loads (determined by the terminal end of the vector) in the  $n$ -dimensional loading space is a certain curve  $L$ , the time  $t$  is considered as a parameter representing the curve (here the representation may not be uniquevalued).

Corresponding to an orbit having infinite laws of motion, in the dynamic problem, then with the differential laws will lead to different stress states. Therefore, we only consider here a certain determined orbit of load and its similar set, when the law of load in it is given.

The following theorem is the sufficient condition that the system is shakedown on the orbit  $L$  with the law of loads given before.

2. The extended dynamic shakedown theorem. The elastic perfectly plastic system subjected to dynamic load  $\vec{P}(t)$  according to a certain given law, with respect to the orbit  $L$  in the loading space, is shakedown on  $L$  with  $\forall t \in [0, T]$  if:

At any point  $P^*(t^*) \in L$ ,  $t^* \in [0, T]$

- a) a vicinity  $\Delta L^* \in L$  of  $P^*$
- b) a time-independent residual stress

$$\rho_p^*(x)$$

c) The sum of the residual stress  $\vec{\rho}_p^*(x)$  with the dynamic ideal elastic stress  $\vec{\sigma}^{(e)}$ ,  $(x, t)$  is safety, i. e.

$$\begin{aligned} & \forall x \in V \\ & f(\vec{\rho}_p^* + \vec{\sigma}_p^{(e)}) < c \quad \forall p \in \Delta L^* \end{aligned}$$

when  $f=c$  is the yield condition.

3. Proof of the theorem. Before proving the extended theorem, we have a look back at E. Melan's proof of the quasi-static loadings and Hwa-Shan-Ho's proof of the dynamic loading.

E. Melan [1] had used the energy function  $A$ ,  $A \geq 0$ ,  $\forall t$  while Hwa-Shan-Ho — the function  $B(t) \geq 0 \forall t$  [4]. With their hypotheses they had proved:  $E(t) \leq 0$ ,  $B(t) \leq 0 \Rightarrow \exists t_0$  while  $\forall t \geq t_0 \Rightarrow B=E \Rightarrow \epsilon_j' = 0$ , i. e. the system is shakedown.

Otherwise speaking  $\forall t \leq t_0$ , the system is not yet plastic collapse.  
 $\forall t \geq t_0$ , the system is shakedown, i. e. it stops plastic deformation. Thus the system stops plastic deformation  $\forall t$  while  $B(t) \leq 0$ .

Now we return to the proof of the extended theorem.

According to the hypotheses of the extended theorem in the vicinity  $\Delta L$  of  $P$ , there are enough hypotheses of Hwa-Shan-Ho's theorem. Therefore there exists the field of residual stress  $\vec{\rho}_p(\vec{x})$  so that the system is shakedown at  $P$ . This occurs  $\forall P \in L$ , then the system is shakedown on  $L$ . The theorem is proved.

4. Consequences: The extended theorem which is more general than that of Hwa-Shan-Ho is however difficult to apply because at each point  $P \in L$ ,  $\exists \Delta L$  and  $\exists \vec{\rho}_p(\vec{x})$  are required.

To facilitate the application, we prove the two following important consequences:

Consequence 1: If the orbit of load  $L$  comprises a number of small closed segments, the system is shakedown on each segment according to Hwa-Shan-Ho's theorem, then the system is shakedown on  $L$ , corresponding to the law of load.

Proof: On the  $t$ -axis, it is divided into closed intervals of time  $\Delta t_i [t_i, t_{i+1}]$  corresponding to  $\Delta L_i$  on the orbit  $L$ .

We set up the following function  $C(t)$ :

$$C(t) = B_i(t) \quad \forall t \in \Delta t_i \quad i = 1, 2, \dots,$$

$$\text{where } B_i(t) = \int_V E(\eta_{kl} - \bar{\eta}_{kl}) dv + \int_V \frac{1}{2} m(u_i - u_i^{(e)})(u_i - u_i^{(e)}) dv \geq 0$$

$$B(0) = \int_V E(\bar{\eta}(\vec{x})) dv$$

$E$  is energy density of the elastic deformation,  $B$  is the residual deformation  $\eta$  is the velocity of the displacement.

With his hypotheses, Hwa-Shan-Ho proved:

$$\frac{d}{dt} B(t) = \dot{B}(t) = - \int_V (\sigma_{ij} - \sigma_{ij}^{(e)}) \epsilon_{ij}'' dv \leq 0.$$

Based on the Drucker's postulate, we deduce  $\exists t_0$  while  $\forall t \geq t_0$  then  $\epsilon_{ij}'' = 0$ , i. e. the system is shakedown. On each interval of time  $\Delta t_i$ , the function  $C(t)$  coincides with the Ho's function  $B(t)$ . Therefore with the hypotheses of the consequences, we deduce that the system is shakedown on  $\Delta t_i$  or  $\Delta L_i$ .

Consequence 2: The system is shakedown with load

$$\vec{P} = \vec{P}_0(\vec{x}) + \vec{\Phi}(\vec{x}, t)$$

and satisfying the condition

$$(1) \quad f(\vec{\rho} + \vec{\sigma}_{p_0}(\vec{x})^{(e)}) \leq C \quad \forall \vec{x} \in V,$$

then the system is shakedown with every loads  $\vec{P}_\lambda(t)$

$$\vec{P}_\lambda(t) = \vec{P}_0(\vec{x}) + \lambda \vec{\Phi}(\vec{x}, t), \quad \lambda \in [0, l]$$

where  $\vec{P}_0(\vec{x})$  is the time-independent load,  $\vec{\sigma}_p^{(e)}(\vec{x})$  is the infinite ideal elastic stress with respect to  $\vec{P}_0(\vec{x})$ . In fact, the system is shakedown with  $\vec{P}(t)$ , i. e.

$$(2) \quad \begin{aligned} \forall t \in [0, T] \Rightarrow \exists \vec{\rho}_p(\vec{x}), \exists \Delta L_p \Rightarrow \\ f[\vec{\rho}_p(\vec{x}) + \vec{\sigma}_p(\vec{x}, t)] \leq C \quad \forall \vec{x} \in V \\ \forall \vec{P} \in \Delta L_p \end{aligned}$$

at  $\vec{\rho}_\lambda(\vec{x}, t) = \vec{\rho}_0 + \lambda \vec{\Phi}(\vec{x}, t)$  we choose

$$\vec{\rho}_\lambda(\vec{x}) = \vec{\rho}(\vec{x}) \quad \forall \lambda \in [0, T]$$

and choose  $\Delta L$  with respect to  $\vec{P}(\vec{x}, t)$   
a) when  $P_0 = 0$  the elastic stress is

$$\lambda \vec{\sigma}_p^{(e)} \Rightarrow f(\vec{\rho} + \lambda \vec{\sigma}_p^{(e)}) \leq C,$$

b) when  $P_0 \neq 0$ , the elastic stress is

$$\vec{\sigma}_0^{(e)}(\vec{x}) + \vec{\sigma}_p^{(e)}(\vec{x}, t).$$

Due to the condition (1) and (2), with the hypotheses of the convex field surface, we have

$$f(\vec{\rho} + \vec{\sigma}_0^{(e)} + \vec{\sigma}_{\lambda\Phi}^{(e)}) \leq C.$$

Thus the consequence 2 is proved.

### 3. The method of solving the dynamic shakedown problem

1. The set problem. The external forces acting on the system  $\vec{P}(\vec{x}, t)$  are expressed as

$$\vec{P}(\vec{x}, t) = \vec{P}_0(\vec{x}) + \mu \vec{\Phi}(\vec{x}, t)$$

where  $\mu$  is the variable parameter,  $\vec{\Phi}(\vec{x}, t)$  is the given basic load.

If the system subjected to the temperature field  $\theta(\vec{x}, t)$ , we put:

$$\theta(\vec{x}, t) = \mu \theta_0(\vec{x}, t).$$

In the case that the dynamic coefficient has been determined, the problem becomes the statical one as Hwa-Shan-Ho [6], it is only a particular case of the above expression.

The problem of control according to the shakedown state is put as follows:

The law of loads acting on the system is given, then we examine whether the system is shakedown with loads?

To solve the set problem we can proceed in two ways: — Firstly: With the given law of acting loads, we examine whether at every point in a certain interval of time, the system is shakedown.

— Secondly: With the given law of variable loads, we determine the largest set of similar loads, so that the system is shakedown on it.

Thus, by the first method, we follow up determination of the shakedown portion on an orbit; by the second method we find the similar layer of loads.

As we know, the sufficient condition of shakedown is

$$\begin{aligned} \exists \vec{\rho}(x) \rightarrow f[\vec{\rho} + \vec{\sigma}^{(e)}] < C \\ \forall \vec{x} \in V, \forall t \in [0, T]. \end{aligned}$$

The condition  $f < c$  determines an open domain in the stress space. Often the programming problem determined on the open domain has no solutions.

To overcome this difficulty, we substitute the condition  $f < c$  by condition  $f \leq c - \varepsilon$ ,  $\varepsilon > 0$  which is sufficiently small.

If the solution of the problem is stable, then which a sufficiently small  $\varepsilon$ , it is led to small error.

The concept of stability of the linear programming problem and the method to find errors is already presented in [7].

2. The problem of dynamic shakedown according to the Hwa-Shan-Ho's theorem.

a) The problem of control by the orbit. On the given orbit of load we have the arc length  $S = S(t)$ . To check whether on the orbit it is shakedown we consider the problem

$$(I) \quad \begin{cases} S \rightarrow \max \\ \text{subject to} \begin{cases} L\vec{\rho} = 0 \quad \forall \vec{x} \in V, \notin S_p \\ N\vec{\rho} = 0 \quad \forall \vec{x} \in S_p \\ f[\vec{\rho} + \vec{\sigma}^{(e)}][S(t)] \leq C \quad \forall \vec{x} \in V \\ f[\vec{\rho} + \vec{\sigma}_0^{(e)}][S(t_0)] \leq C \quad \forall \vec{x} \in V, \forall t \in [0, t] \end{cases} \end{cases}$$

where  $\vec{\rho} = \vec{\rho}(x)$  is the residual stress field,  $\vec{\sigma}^{(e)}[S(t)]$  and  $\vec{\sigma}_0^{(e)}$  are the ideal elastic stress corresponding to the time  $t$  and  $t_0$  ( $t_0$  chosen before).

If the problem (1) has no solutions, then the system is not shakedown at  $t_0$ .

b) The problem of finding the layer of shakedown orbit

$$(II) \quad \begin{cases} \lambda \rightarrow \max \\ \text{subject to} \begin{cases} L\vec{\rho} = 0 \quad \forall \vec{x} \in V, \notin S_p \\ N\vec{\rho} = 0 \quad \forall \vec{x} \in S_p \\ f(\vec{\rho} + \vec{\sigma}_0^{(e)}) \leq C \quad \forall \vec{x} \in V \\ f(\vec{\rho} + \vec{\sigma}_0^{(e)} + \lambda \vec{\sigma}_\Phi^{(e)}) \leq C \quad \forall \vec{x} \in V \\ \lambda \geq 0 \quad \vec{\rho} = \vec{\rho}(x) \end{cases} \end{cases} \quad \forall t \in [0, T]$$

Solving the problem (II) we find  $\lambda_{\max} = \lambda^+$  and the field of residual stress and corresponds to  $\rho^+(x)$ .

According to the consequence 2, we have the loading layer, on it the system is shakedown are

$$\vec{P}_\lambda = \vec{P}_0 + \lambda \vec{\Phi}(x, t) \quad \forall \lambda \in [0, 1].$$

3. The shakedown problem based on the extended dynamic shakedown theorem. a) The problem of control with respect to orbit. Solving the problem (I) we find  $S_{\max} = S_l$  (with respect to the point  $A_l$  on the orbit), corresponding to the elastic stress  $\vec{\sigma}_1^{(e)}$ .

Thereafter we solve the extended problem

$$(III) \quad \begin{cases} S \rightarrow \max \\ \text{subject to} \begin{cases} L\vec{\rho} = 0 \quad \forall x \in V, \notin S_p \\ N\vec{\rho} = 0 \quad \forall x \in S_p \\ f(\vec{\rho} + \vec{\sigma}_1^{(e)}) \leq C \quad \forall x \in V \\ f(\vec{\rho} + \vec{\sigma}_1^{(e)}) \leq C \quad \forall x \in V, \forall t \in [0, T] \\ S \geq 0 \quad \vec{\rho} = \vec{\rho}(x) \end{cases} \end{cases}$$

We proceed similarly until the problem has only one plan and we stop.

b) The problem of finding the layer of Shakedown orbit.

$$(IV) \quad \begin{cases} \lambda_1 \rightarrow \max \\ \text{subject to} \begin{cases} L\vec{\rho} = 0 \quad \forall x \in V, \notin S_p \\ N\vec{\rho} = 0 \quad \forall x \in S_p \\ f(\vec{\rho} + \vec{\sigma}_0^{(e)}) \leq C \quad \forall x \in V \\ f(\vec{\rho} + \vec{\sigma}_0^{(e)} + \lambda_1 \vec{\sigma}_0^{(e)}(x, t)) \leq C \\ \forall x \in V, \forall t \in [0, T] \end{cases} \end{cases}$$

In the problem IV, the field of residual stress  $\vec{\rho} = \vec{\rho}(x, t)$  depends on  $t$ . Therefore we must prove that the solution of the problem (IV) satisfies the extended theorem.

At first, we see that according to the conditions of the extended theorem, with any point on the orbit, there exists the field of residual stress, which does not depend on time, with respect to vicinity of the point of the orbit and which is able to change from this vicinity to the next.

Now we prove that: although the problem (IV) gives us  $\vec{\rho} = \vec{\rho}(x, t)$ , the above requirements of the extended theorem are still satisfied.

When using the conditions  $f \leq c - \varepsilon$  to deduce the points on the boundary of the domain  $f \leq c - \varepsilon$ , there exists a sufficiently small vicinity lying entirely in the domain  $f \leq c$ .

Therefore at any  $t = t^*$  on the orbit corresponds  $\vec{\rho} = \vec{\rho}(x, t^*)$

$$\exists \Delta t^* \Rightarrow \exists \text{ vicinity } \Delta L^* \Rightarrow$$

$$f\{\vec{\rho}^* + \vec{\sigma}_0 + \lambda \vec{\sigma}_\Phi^{(e)}\} \leq C$$

$$\forall \vec{x} \in V, \forall t \in \Delta t^*, \forall \vec{P} \in \Delta P^*,$$

i. e. the hypotheses of the extended theorem are satisfied.

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