

Обща механика

Can the reputation of the Lagrangean dynamical equations be saved?

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My name has been mentioned in the article [1] of T. Boyadjiev published in this journal. Since this quotation has been made in the utmost inadequate manner, I find it *ad nec plus ultra* necessary to put things right in this connection.

The article [1] begins with the following statement:

„На семинаре сектора аналитической механики проф. Ив. Чобановым были предложены две задачи об исследовании движения твердых тел при наличии связей определенного вида. В первой задаче рассматривается вопрос, возможно ли движение в сфере тяжелого стержня, длина которого равна диаметру сферы. Во второй задаче аналогичный вопрос ставится для системы из двух тяжелых однородных шаров в сфере, диаметр которой равен сумме диаметров шаров.“

I have done nothing of the kind.

Before explaining what namely I have done, I find it relevant to make the following, though non-mathematical, remark.

The article [1] is dated 31. 10. 1983. Not a word in connection with the dynamical problems formulated in the above fragment from [1] has been published by me until 1987, when the articles [2—4] have seen the light of day. The basic ideas developed in [2—4] have been reported by me in the seminar on analytical mechanics not later than 1982. Now I suppose it goes without saying that any participant in a scientific seminar should, for elementary ethical reasons, refrain from printing any results engendered by some report in this seminar before the author of the report has published the latter himself, not to say anything about misrepresenting the content of the report in question.

Alas, my report in the seminar on analytical dynamics has been intolerably misrepresented in the article [1].

First of all, it is not true that „проф. Ив. Чобановым были предложены две задачи об исследовании движения твердых тел“, as T. Boyadjiev states. The home truth is that I have reported the *solutions* of the problems in question rather than the *formulation* of these problems only.

Second, it is not true that I have formulated the problems in question „при наличии связей определенного вида“. The home truth is that I have emphasized that all constraints in these dynamical problems are *smooth*.

Third, it is not true that in these dynamical problems „рассматривается вопрос, возможно ли движение...“ The home truth is that I have proved that *these two pro-*

blems are in possession of entirely different, completely inconsistent, solutions when they are solved by the Eulerian dynamical axioms (the laws or principles of momentum and of moment of momentum (kinetical moment) of a rigid body), on the one hand, and by the Lagrangean dynamical equations, on the other hand. This is the core of my observation which I have reported in the seminar on analytical mechanics. Such dynamical problems have been called by me *would-be dynamical paradoxes* (dynamical "paradoxes") or *dynamical antinomies* (see [2—4], as well as [5—9]). The aims of all the publications [2—9] is to disclose the mechanism of this mathematical phenomenon, namely the existence of holonomic, as well as non-holonomic, dynamical problems concerning rigid bodies, the solutions of which obtained by the Eulerian and by the Lagrangean dynamical traditions contradict each other.

I have explained this mechanism in the seminar on analytical mechanics mentioned in [1]. The one, only, and unique reason for the contradiction of the results obtained by means of the Eulerian approach, on the one hand, and by the Lagrangean one, on the other hand, is rooted in the fact that the Lagrangean dynamical equations are necessary but by no means sufficient conditions for the motion of a rigid body; that the Lagrangean dynamical equations are immediate corollaries from the Eulerian dynamical axioms but the vice versa is not true: the Eulerian dynamical axioms are underrivable on the basis of the Lagrangean dynamical equations; that the Lagrangean dynamical equations are not in the position to answer the question about the existence of a solution of a dynamical problem concerning rigid bodies; at last, that any dynamical "paradox" represents a dynamical problem, the conditions of which are inconsistent, i. e. which does not possess a solution; at that, the Eulerian dynamical axioms "catch" this fact, while the Lagrangean dynamical equations are "blind" at its face.

Not a word in connection with these circumstances has been said in the article [1] although its author is familiar with them, these ideas being not once nor twice set forth in the seminar on analytical mechanics. This is not a fair play, in my opinion. Euler's and Lagrange's names are never mentioned in [1] in this context. As a matter of fact, Euler's name is mentioned twice, and Lagrange's once, in [1]. The corresponding texts read as follows (p. 13):

„Вместе с соответствующими кинематическими уравнениями, уравнение (10) образует замкнутую систему, эквивалентную уравнениям Эйлера для тела с неподвижной точкой.“

„Сюда добавим кинематические уравнения Эйлера...“

„Исключая величины ω_i из уравнений (14) и (15), при помощи (16) получаем уравнения сферического физического маятника — систему из двух квазилинейных уравнений второго порядка, эквивалентную уравнению Лагранжа второго рода для стержня.“

Why such a line of conduct?

The answer is a very simple one. The author of the article [1] is striving for saving the reputation of the Lagrangean dynamical equations, in particular, and of the Lagrangean dynamical tradition, in general. (He has already engaged his name with these equations and with this tradition in his previous writings.) Faced with particular cases of dynamical antinomies or "paradoxes", i. e. with cases when the Lagrangean dynamical equations enter in irreconcilable contradictions with the Eulerian dynamical axioms, he seeks an exoneration of Lagrange's approach towards dynamical problems in a slight reformulation of the conditions of the problems. In other words, instead of *smooth* constraints, as the original formulation proposed by me reads, he substitutes it by a convenient for his cause surrogate, proposing *ideal* constraints, and he states that in such a case no contradiction between Euler's and Lagrange's solutions takes place. Now what of it? This is an entirely different problem. The behaviour of the author of [1] may be compared with that of a person who turns down Euler's proof that the Diophantine equation $a^3 + b^3 = c^3$ has no solution in whole numbers a, b, c by means of the objection that this equation has solutions in real numbers a, b, c .

Can, however, the reputation of the Lagrangean dynamical equations be saved?

Before proceeding to the answer of this question let us note that examples of dynamical problems concerning rigid bodies possessing no solutions have been known long ago. So for instance, the author of [10] has written:

„On sait que lorsqu'on schématise, selon l'habitude des problèmes mécaniques concrets physiquement possibles, on obtient parfois des problèmes mathématiques impossibles... Exemples avec frottement (Painlevé), exemple connu de la barre pesante diamétrale glissant sans frottement dans une sphère“ (p. 7).

To all appearances, however, the traditional mechanicians have paid little attention to such exotic mechanical extravagances and they have not considered it worthy to dedicate them a closer attention.

For my part, I have sought such solutionless dynamical problems with premeditation. It will not be superfluous to expose the reasons for such a scientific policy.

Still in my student time I have noticed that in all textbooks, treatises, and monographs on analytical mechanics I had at my disposal the Lagrangean dynamical equations were not deduced but for mechanical systems consisting of a finite number of mass-points only. The first years of my assistantship in the chair of analytical mechanics in the Mathematical Faculty at the University of Sofia only confirmed this impression. Now rigid balls, ellipsoids, cylinders, prisms, pyramids, etc. are by no means systems of a finite number of mass-points. Nevertheless, the dynamical problems I was compelled to solve on the blackboard by means of the Lagrangean dynamical equations, by virtue of the syllabus and of the established mechanical tradition, involved those namely balls, ellipsoids, cylinders, prisms, pyramids, etc., for which these Lagrangean dynamical equations have been never proved. I have described in detail the psychological atmosphere I was compelled to work at in my lecture [6]. Now, it was quite natural to try and find a proof of the Lagrangean dynamical equations for rigid bodies.

I did not manage to find such a proof in the course of three clear decades, till 1982, when at last I happened to find one [11]. (Here and later, saying “I”, I mean “we”: the names of the collaborators get clear from the literature cited.) The analysis of this proof unambiguously manifested the following mathematical facts:

1. The Lagrangean dynamical equations are necessary corollaries from the Eulerian dynamical axioms.
2. The Lagrangean dynamical equations are not equivalent with the Eulerian dynamical axioms (in other words, the first do not imply the latter) save in the case of a free from geometrical constraints rigid body.
3. The reactions of the geometrical constraints being expelled from them, the Lagrangean dynamical equations cannot answer the question why (under the action of which forces) does a rigid body move.
4. The Lagrangean dynamical equations cannot answer the question of the consistency of the conditions of the dynamical problem under consideration (i. e. of the existence of a solution of the problem).
5. The most the Lagrangean dynamical equations can accomplish is to answer the question how does a rigid body move, if it moves at all.
6. No dynamical problem concerning a rigid body subjected to any geometrical constraints can be solved by the aid of the Lagrangean dynamical equations.
7. The so-called Lagrangean mechanics (i. e. analytical mechanics based on the Lagrangean dynamical equations) reduces to nonsense.

Even God needs tolling bells, as the saying goes. Euler does not. All the analytical dynamics is based on, and is developed from, his famous dynamical laws. Let us state them explicitly here in a modern redaction.

Ax 1 E (*first Eulerian dynamical axiom or law (principle) of momentum of a rigid body*). There exists such a Cartesian system of reference S that, all derivatives being taken with respect to S , for any rigid body B and for any system of forces F

acting on B , the derivative with respect to the time of the momentum of B equals the basis of F .

Df 1 E. Any system of reference satisfying Ax 1 E is called *inertial according to Euler*.

Ax 2 E (*second Eulerian dynamical axiom or law (principle) of moment of momentum (kinetical moment) of a rigid body*). If S is an inertial according to Euler system of reference and all derivatives are taken with respect to S , then for any rigid body B and for any system of forces F acting on B , the derivative with respect to the time of the moment of momentum of B equals the moment of F ; both moments being taken with respect to the origin of S .

Things being as they are, after the results published in [11] it became clear that when seeking a strict mathematical proof of the Lagrangean dynamical equations for rigid bodies subjected to any geometrical constraints and to the action of any active forces I have been following the traces of a phantom. In the same time a suspicion has come into being that maybe there exist dynamical problems with no solutions (according to Euler's dynamical axioms) for which the Lagrangean dynamical equations should describe a motion. As the articles [2—9] unequivocally display, such problems exist indeed. The first of them, together with the explanations given above, have been reported in the already mentioned seminar on analytical mechanics.

Things did not, however, come to a standstill here.

About the end of 1981 it has been observed that, for a free rigid body at least, the Lagrangean dynamical equations are projections of the Eulerian dynamical axioms on appropriate axes [12], see also [13]. This fact brought forth the suspicion that one has to do with a more general mathematical phenomenon, in other words, that the Lagrangean dynamical equations are projections of the Eulerian dynamical axioms even in the general case of non-free rigid bodies, i.e. of rigid bodies subjected to any geometrical constraints. This idea turned out to be a rather fruitful one. The corresponding results are reflected in the articles [14—16]. Moreover, it turned out that this mathematical phenomenon holds in the non-holonomic case too [17]. For the sake of brevity, I shall restrict my following exposition to the holonomic case only.

In order to fix the ideas, let B be a rigid body subjected to certain geometrical constraints restricting its possible positions in space and reducing its degrees of freedom from 6 to l ; let $q_\lambda (\lambda = 1, \dots, l)$ be mutually independent parameters of B determining any of its positions in space compatible with the geometrical constraints in question; the derivatives $\dot{q}_\lambda$ of $q_\lambda (\lambda = 1, \dots, l)$ with respect to the time t are mutually independent quantities (holonomic dynamical problem), inasmuch as the contrary is not hypothesized explicitly in the conditions of the dynamical problem under consideration (non-holonomic dynamical problem). One of the aims of any dynamical problem concerning rigid bodies consists in determining of the parameters of B as functions $q_\lambda = q_\lambda(t) (\lambda = 1, \dots, l)$ of the time, the initial conditions $q_{\lambda 0} = q_\lambda(0)$ and $\dot{q}_{\lambda 0} = \dot{q}_\lambda(0)$ of the dynamical problem being prescribed, dots denoting, as traditional for analytical mechanics, derivatives with respect to the time. At that, it is supposed that B is under the action of the *active forces* (i.e. forces *wholly determined* in the conditions of the dynamical problem under consideration for any possible position of B).

$$(1) \quad \vec{F}_\mu = (F_\mu, M_\mu) \quad (\mu = 1, \dots, m)$$

where

$$(2) \quad F_\mu \neq 0, F_\mu M_\mu = 0 \quad (\mu = 1, \dots, m),$$

all moments $M_\mu (\mu = 1, \dots, m)$ being taken with respect to the origin of a fixed inertial according to Euler orthonormal right-hand orientated Cartesian system of reference $Oxyz$.

Let the geometrical constraints imposed on B generate the *passive forces* or *reactions of the constraints*

$$(3) \quad \vec{R}_v = (R_v, N_v) \quad (v = 1, \dots, n)$$

where

$$(4) \quad R_v \neq 0, R_v N_v = 0 \quad (v = 1, \dots, n),$$

all moments $N_v (v = 1, \dots, n)$ being also taken with respect to O . The quantities (3) are *unknown*, and another of the aims of the dynamical problem under consideration is their determination.

For the sake of brevity let

$$(5) \quad \vec{F}_{m+v} = \vec{R}_v \quad (v = 1, \dots, n),$$

i. e.

$$(6) \quad F_{m+v} = R_v, M_{m+v} = N_v \quad (v = 1, \dots, n)$$

provided

$$(7) \quad \vec{F}_{m+v} = (F_{m+v}, M_{m+v}) \quad (v = 1, \dots, n)$$

and let

$$(8) \quad \mathbf{F} = \sum_{v=1}^{m+n} \mathbf{F}_v, \mathbf{M} = \sum_{v=1}^{m+n} \mathbf{M}_v.$$

Let P denote any point of B , $\mathbf{r} = \mathbf{OP}$, $\mathbf{v} = \dot{\mathbf{z}}$ (with respect to $Oxyz$). Besides, let dm denote elementary mass of B . Then by definition

$$(9) \quad \mathbf{K} = \int \mathbf{v} dm, \mathbf{L} = \int \mathbf{r} \times \mathbf{v} dm, T = \frac{1}{2} \int v^2 dm$$

(the integrals being taken over the space occupied by B) are the momentum, the moment of momentum, and the kinetic energy of B respectively with regard to $Oxyz$. Now the Eulerian dynamical axioms have the following mathematical expressions

$$(10) \quad \frac{d\mathbf{K}}{dt} = \mathbf{F}, \quad \frac{d\mathbf{L}}{dt} = \mathbf{M}.$$

Let $G\xi\eta\zeta$ be an orthonormal right-hand orientated Cartesian system of reference invariably connected with B , G denoting the mass—center of B . Let $\bar{\omega}$ be the instantaneous angular velocity of $G\xi\eta\zeta$ with respect to $Oxyz$, and let by definition

$$(11) \quad \mathbf{L}_G = \int \bar{\rho} \times (\bar{\omega} \times \bar{\rho}) dm - m\mathbf{r}_G \times \mathbf{v}_G,$$

m denoting the mass of B . If

$$(12) \quad \mathbf{M}_G = \mathbf{M} + \mathbf{F} \times \mathbf{r}_G,$$

then it is proved that the Eulerian dynamical axioms (10) may be written in the form

$$(13) \quad \dot{\mathbf{K}} - \mathbf{F} = \mathbf{0}, \quad \dot{\mathbf{L}}_G - \mathbf{M}_G = \mathbf{0}.$$

At last, let $P_v (v = 1, \dots, m+n)$ be points of B incident in the moment of time t with the directrices of the forces $\vec{F}_v (v = 1, \dots, m+n)$ respectively. In other words, if $\mathbf{r}_v = \mathbf{OP}_v (v = 1, \dots, m+n)$, then

$$(14) \quad \mathbf{r}_v \times \mathbf{F}_v = \mathbf{M}_v \quad (v = 1, \dots, m+n).$$

Besides, let

$$(15) \quad Q_\lambda = \sum_{\mu=1}^m \mathbf{F}_\mu \frac{\partial \mathbf{r}_\mu}{\partial q_\lambda}, \quad Q_\lambda^* = \sum_{v=1}^n \mathbf{R}_v \frac{\partial \mathbf{r}_{m+v}}{\partial q_\lambda}$$

($\lambda = 1, \dots, l$) denote the so-called *active* and *passive generalized forces* acting on B respectively.

Under these conditions and notations the following relations are proved

$$(16) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda - Q_\lambda^* = (\dot{\mathbf{K}} - \mathbf{F}) \frac{\partial \mathbf{r}_G}{\partial q_\lambda} + (\dot{\mathbf{L}}_G - \mathbf{M}_G) \frac{\partial \bar{\omega}}{\partial q_\lambda} \quad (\lambda = 1, \dots, l).$$

The relations (16) are *identities*, i. e. they are derived by means of identical transformations accomplished on the definitions of the mechanical entities therein involved. No *dynamical hypothesis* has been made in the process of deriving these relations. In other words, the relations (16) are *tolerant* to any such hypothesis or, should we say, to any dynamical principle whatever.

The relations (16) have been called *fundamental identities of Lagrangean formalism* in the article [15]. The reasons for this denomination will be explained immediately.

The choice of the dynamical principles being wholly at our disposal inasmuch as the relations (16) are concerned, let us hypothesize the Eulerian dynamical axioms (13). Then (16) imply

$$(17) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda - Q_\lambda^* = 0 \quad (\lambda = 1, \dots, l).$$

The relations (17) represent the Lagrangean dynamical equations in the presence of any reactions of the geometrical constraints. If

$$(18) \quad \sum_{\lambda=1}^l Q_\lambda^{*2} \neq 0,$$

then the equations (17) are useless: according to the definition (15) of the passive generalized forces Q_λ^* ($\lambda = 1, \dots, l$) acting on B , the number of the unknown quantities participating in (17) exceeds the number l of these equations. Therefrom, a cardinal hypothesis of the Lagrangean dynamical tradition consists in the acceptance that the mechanical nature of the geometrical constraints imposed on B is such that the conditions

$$(19) \quad Q_\lambda^* = 0 \quad (\lambda = 1, \dots, l)$$

are satisfied. In the light of the definition (15) the relations (19) can be written in the form

$$(20) \quad \sum_{v=1}^n \mathbf{R}_v \frac{\partial \mathbf{r}_{m+v}}{\partial q_\lambda} = 0 \quad (\lambda = 1, \dots, l).$$

Geometrical constraints for which the conditions (20) hold are called *ideal*, and the conditions (20) themselves are called the *postulate of ideal constraints*.

Accepting the postulate of ideal constraints, one obtains from (17) the traditional form of the Lagrangean dynamical equations

$$(21) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda = 0 \quad (\lambda = 1, \dots, l).$$

It is immediately seen that the Lagrangean equations (21) are immediate corollaries from the Eulerian dynamical axioms. Moreover, (16) and (19) imply

$$(22) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda = (\dot{\mathbf{K}} - \mathbf{F}) \frac{\partial \mathbf{r}_G}{\partial q_\lambda} + (\dot{\mathbf{L}}_G - \mathbf{M}_G) \frac{\partial \bar{\omega}}{\partial \dot{q}_\lambda}$$

($\lambda = 1, \dots, l$). In other words, the left-hand sides of the Lagrangean dynamical equations (21) are, simply and purely, linear combinations of the projections of the left-hand sides of the Eulerian dynamical axioms (13) on appropriate axes defined by the vectors $\frac{\partial \mathbf{r}_G}{\partial q_\lambda}$ and $\frac{\partial \bar{\omega}}{\partial \dot{q}_\lambda}$ ($\lambda = 1, \dots, l$). This fact throws a new light on Lagrange's performances in analytical dynamics.

If the Lagrangean dynamical equations are only projections of the Eulerian dynamical axioms on appropriate axes, as the identities (22) provided (19) doubtless manifest, then what value can be attributed to these equations? What value can be ascribed to a mathematical result (Lagrange's equations) which is obtainable by a mere projection on appropriate axes of somebody else's result (Euler's axioms)? Lagrange has not seen this fact? This is Lagrange's affair. Why did not Euler himself see it? There are many reasons. First of all, Lagrange's dynamical equations have become popular after 1788, the year when his famous treatise [18] has been printed, and Euler died in 1783. It is true that the equations have been proposed by Lagrange not later than 1780 [19], but the publication has seen the light of day in 1782, a year before Euler's death. On the other hand, Euler has been completely blind more than a dozen of years ago. At last, Lagrange's mechanical expositions do not exactly justify his own appraisal that they *ne laissera rien à désirer*. For instance, apropos of Lagrange's article [20] where one reads "Je considère les corps proposés comme l'assemblage d'une infinité de corpuscules ou points massifs unis ensemble de manière qu'il gardent toujours nécessairement entre eux les mêmes distances..." (quoted by [21], p. 590), Euler who refused to accept the picture of a rigid body as composed of infinitely many mass-points, has written in the introduction of his memoir [22] with ill-concealed if latent irony: "But when I tried with greatest avidity to follow his extremely profound thoughts, truly I could not get myself to go through all his calculations. Even the first lemma so deterred me that on account of my blindness I could not hope in any way to check through all the analytical devices he used" (quoted by [23], p. 260, see also [24]). In the same time, Euler's blindness did not deter him from discovering and publishing in this same memoir [22], the starry flash in his scientific mechanical life, his greatest dynamical discovery, his immortal dynamical axioms—a mathematical scheme "applicable to every part of every body, whether punctual or space-filling, whether rigid or deformable... scheme that remained general enough for all mechanics for at least 100 years" ([23], p. 128, 173).

The fundamental identities of Lagrangean formalism (22) most clearly disclose the mathematical mechanism of the dynamical "paradoxes". Let us suppose that the conditions of a dynamical problem are inconsistent, i. e. that

$$(23) \quad (\dot{\mathbf{K}} - \mathbf{F})^2 + (\dot{\mathbf{L}}_G - \mathbf{M}_G)^2 \neq 0.$$

Then, in general, the equations (21) will be violated. If now one solves such a problem by the aid of the Lagrangean dynamical equations (21), then he is accomplishing an illegal mathematical act. The penalty for such an unallowed deed is a mathematical error: a dynamical "paradox".

As it is well known, the Lagrangean dynamical equations have various numerous non-holonomic variants. Much could be said in connection with these versions which

are incomparably more pointless than the original holonomic Lagrangean equations themselves [25]; the space-limitations of this journal, however, do not permit us to enter in just any details whatever in this connection.

Although contradictory to the *status quo* of nowadays mechanical establishment (a phenomenon often observed in mankind's history), the picture of the relation Eulerian dynamical axioms — Lagrangean dynamical equations I have cursory sketched here is nevertheless the *nuda veritas*, notwithstanding does Somebody like it or not. To oppose it is a hopeless task. To oppose it means to oppose the logical spirit of the Twentieth-Century mathematics. Someone has said that every century has its Middle Ages. The Middle Ages of analytical mechanics, however, have lasted two clear centuries due to Lagrange's dynamical ideas.

Lagrangean dynamical equations?

Old sins have long shadows.

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