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Numerical treatment of dynamic viscoelastic problem for 4-parameter media

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1. Introduction

The material's behaviour under high rates of deformation has recentty been a subject of considerable attention. Numerous investigations have been made of plane wave propagation in viscoelastic bodies. Commonly used is the assumption that three-dimensional effects (described as radial or transverse inertia effects) are negligible and consequently the one-dimensional analysis is valid with sufficient degree of accuracy. Solutions have been published both for semi-infinite and finite rods [1, 2]. Laplace transform method is widely used, but inverse transformation difficulties appear considerably restricting it's application.

The purpose of the present paper is to investigate the one-dimensional dynamic viscoelastic problem for 4-parameter media numerically.

II. Problem statement

The viscoelastic media is described by the 4-parameter model of Burgers that approximates closely actual behaviour of amorphos polymers (Fig. 1). Constitutive equations for this model are given by [3]

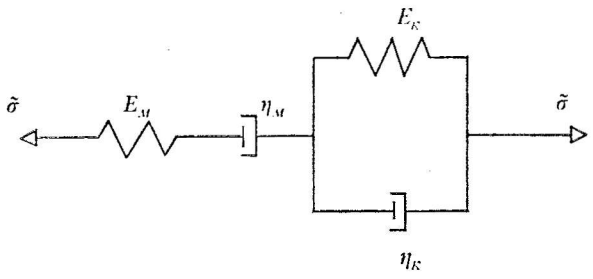


Fig. 1

$$\left|\begin{array}{l} \tilde{\sigma}=E_M\tilde{\varepsilon}_1 \\ \tilde{\sigma}=\eta_M\dfrac{\partial\tilde{\varepsilon}_2}{\partial\tilde{t}} \\ \tilde{\sigma}=E_K\tilde{\varepsilon}_3+\eta_K\dfrac{\partial\tilde{\varepsilon}_3}{\partial\tilde{t}} \\ \tilde{\varepsilon}=\tilde{\varepsilon}_1+\tilde{\varepsilon}_2+\tilde{\varepsilon}_3 \end{array}\right. \quad (1)$$

where the notation mean: $\tilde{\sigma}$ — applied stress; $\tilde{t}$ — time; $\tilde{\varepsilon}$ — strain of the model; E_M — modulus of instantaneous elasticity; E_K — modulus of retarded elasticity; η_M — viscosity; η_K — viscosity of retarded elasticity.

If the stress is rapidly applied along the x-axis, the dynamic equilibrium is valid

$$\frac{\partial\tilde{\sigma}}{\partial\tilde{x}}=\rho\frac{\partial^2\tilde{u}}{\partial\tilde{t}^2}.$$

Taking into account the definitions of $\tilde{v}$, $\tilde{\varepsilon}$

$$\tilde{v}=\frac{\partial\tilde{u}}{\partial\tilde{t}},\quad \tilde{\varepsilon}=\frac{\partial\tilde{u}}{\partial\tilde{x}},$$

the system (1) can be represented in a relative (dimensionless) form, according to the approach, used in [4]:

$$(2)\qquad \left|\begin{array}{l} \dfrac{\partial\varepsilon}{\partial t}-\dfrac{\partial v}{\partial x}=0 \\ \varkappa\dfrac{\partial\sigma}{\partial t}-\dfrac{\partial\varepsilon}{\partial t}+\dfrac{\partial\varepsilon_3}{\partial t}+\eta\sigma=0 \\ \dfrac{\partial\varepsilon_3}{\partial t}-\sigma+\varepsilon_3=0 \\ \dfrac{\partial v}{\partial t}-\dfrac{\partial\sigma}{\partial x}=0 \end{array}\right.$$

Here the following dimensionless parameters are introduced:

$$x=\frac{\sqrt{E_K\rho}}{\eta_K}\tilde{x},\quad v=\frac{\sqrt{E_K\rho}}{E_M}\tilde{v},\quad \sigma=\frac{\tilde{\sigma}}{E_M}$$

$$t=\frac{E_K}{\eta_K}\tilde{t},\ \varepsilon_3=\frac{E_K}{E_M}\tilde{\varepsilon}_3,\ \varkappa=\frac{E_K}{E_M},\ \eta=\frac{\eta_K}{\eta_M}.$$

It can be shown, that the characteristics of the system (2) are real ones, that define it as a hyperbolic one. The properties of this class of systems determine the basic problems, that could be considered:

— Initial problem;

— Initial boundary value problem.

In the second problem, the system (2) is investigated in the area $0\leqq x\leqq L$, $0\leqq t\leqq T$. The solution is determined under following initial and boundary conditions:

Initial conditions:

$$(3)\qquad \begin{array}{ll} \sigma(x,\ 0)=\varphi_1(x), & \varepsilon(x,\ 0)=\varphi_2(x),\quad 0\leq x\leq L \\ \varepsilon_3(x,\ 0)=\varphi_3(x), & v(x,\ 0)=\varphi_4(x),\quad 0\leqq x\leqq L \end{array}$$

Boundary conditions:

— on the left boundary: $v(0,\ t)=\psi_1(t)$, $0\leqq t\leqq T$

— on the right boundary: $\sigma(L,\ t)=\psi_2(t)$, $0\leqq t\leqq T$

It is well known that the initial boundary value problem is well-posed when the boundary conditions are dissipative ones. According to definition in [5], that means non-negativity imposed on the right boundary and non-positivity of the left boundary of the quadratic form $(w,\ Bw)$, i. e. (see [4])

$$(w,\ Bw)\leqq 0\quad x=0$$

$$(w,\ Bw)\geqq 0\quad x=L.$$

The initial data must also satisfy the conformity conditions

$$\varphi_1(L)=\psi_2(0), \quad \varphi_4(0)=\psi_1(0).$$

III. Method of Finite Differences (MFD)

The finite difference technique will be applied to the problem (2), (3). Consider in the area $\bar{G}=\{0\leqq x\leqq L,\ 0\leqq t\leqq T\}$ a mesh $\bar{\omega}$ of points with coordinates $x_k=kh$, $t_j=j\tau$, where k, j are integers and τ and h are the time step and the step along the x-axis. More precisely, $\bar{\omega}=\omega_h x\bar{\omega}_\tau$, where ω_h is the mesh in the x — direction and $\bar{\omega}_\tau$ is the time mesh, defined by

$$\omega_h=\{x_k=kh,\ k=0,\ 1\ldots,\ N_1,\quad h=L/N_1\},$$

$$\bar{\omega}_\tau=\{t_j=j\tau,\ j=0,\ 1,\ldots,\ N_2,\quad \tau=T/N_2\}.$$

The common nonsubscript finite difference notations and formulae will be used [6]. Applying the MFD the following difference scheme approximating the problem (2), (3) is obtained:

$$\left|\begin{array}{l}\varepsilon_t-\hat{v}_{\bar{x}}=0\\ \varkappa\sigma_t-\varepsilon_t+\varepsilon_{3,t}+\eta\hat{\sigma}=0\\ \varepsilon_{3,t}-\sigma+\varepsilon_3=0\\ v_t-\sigma_x=0\end{array}\right. \tag{4}$$

$$\sigma_k^0=\varphi_{1,k},\ \varepsilon_k^0=\varphi_{2,k},\ \varepsilon_{3,k}^0=\varphi_{3,k},\ v_k^0=\varphi_{4,k},\quad k=\overline{0,\ N_1} \tag{5}$$

$$v_0^j=\psi_{1,0}^j,\quad \sigma_{N_1}^j=\psi_{2,N_1}^j. \tag{6}$$

If the solution of differential problem (2), (3) is sufficiently differentiable, the scheme approximates this problem with an error $O(h+\tau)$.

The method of harmonics will be used to investigate the stability of the scheme [7] Consider a particular solution in the form of difference harmonics:

$$\begin{array}{l}\sigma_k^j=Pq^j\zeta^k,\quad \varepsilon_k^j=Qq^j\zeta^k,\\ \varepsilon_\varepsilon^j=Rq^j\zeta^k,\quad v_k^j=Sq^j\zeta^k,\end{array} \tag{7}$$

where $\xi=e^{i\varphi}$; P, Q, R, S are the amplitudes of the harmonics; i — the imaginary unit ($i^2=-1$); φ — an arbitrary real number; $q(\varphi)$ — complex number, which should be determined so, that the harmonics satisfy the difference scheme.

Inserting relations (7) into equations (4) renders the following system of algebraic equations:

$$\left|\begin{array}{l}\zeta(q-1)Q-\gamma q(\zeta-1)S=0\\ [\varkappa(q-1)+\tau\eta q]P-(q-1)Q+(q-1)R=0\\ q-1+\tau)R-\tau P=0\\ (q-1)S-\gamma(\zeta-1)P=0,\end{array}\right.$$

where $\gamma=\tau/h$.

For existence of nontrivial solution, it is necessary and sufficient that the determinant of this system vanishes, that means

$$(8)\qquad (q-1)\left\{\gamma^2\frac{(\zeta-1)^2}{\zeta}q(q-1+\tau)-(q-1)[\varkappa(q-1)(q-1+\tau)+\tau\eta q(q-1-\tau)+\tau(q-1)]\right\}=0.$$

Note that $\zeta=\cos\varphi+i\sin\varphi$ and $(\zeta-1)^2/\zeta=-4\sin^2(\varphi/2)$.

It can be easily seen that $q=1$ satisfies equation (8). The second rooth can be obtained expanding $q(\varphi)$ in polynomial series in the vicinity of $q=1$

$$(9)\qquad q=1+C_1\tau+\ldots$$

Substituting (9) into the equation

$$4\gamma^2\sin^2(\varphi/2)q[q-(1-\tau)]+(q-1)\{\varkappa(q-1)[q-(1-\tau)]+\tau\eta q[q-(1-\tau)]+\tau(q-1)\}=0,$$

we obtain $C_1=-1$, i. e. $q=1-\tau+\ldots$

The rest two values of q must satisfy the equation

$$aq^2-bq+c=0.$$

The notations $a=\tau\eta$, $c=\varkappa-\tau$, $b=a+c-4\gamma^2\sin^2(\varphi/2)$ are used further for brevity. Then the roots can be expressed as

$$q_\pm=b\pm\sqrt{b^2-4ac}.$$

If

$$(10)\qquad \Delta=b^2-4ac<0,$$

it can be easily shown that $|q_\pm|<1$, i. e. the harmonics damping. That proves the stability of the scheme (4), (5), (6) on particular solutions. The inequality (10) gives the following necessary conditions for stability of the scheme

$$a+c-2\sqrt{ac}\leqq 4\gamma^2\leqq a+c+2\sqrt{ac},$$

$$(11)\qquad 2\varkappa+\tau(\eta-1)-2\sqrt{(\varkappa+\tau\eta)(\varkappa-\tau)}\leqq 4\gamma^2\leqq 2\varkappa+\tau(\eta-1)+2\sqrt{(\varkappa+\tau\eta)(\varkappa-\tau)}.$$

If the inequality (11) does not hold true, i. e. $\Delta>0$, then $|q_-|>1$, that means instability of the scheme.

IV. Numerical results

To illustrate the results obtained, numerical evaluations of the functions σ, ε, V, ε_3 are carried out. The example on Fig. 2 shows that the approximate solution σ_a is in fair agreement with the exact solution, obtained in [4]. Therefore, the

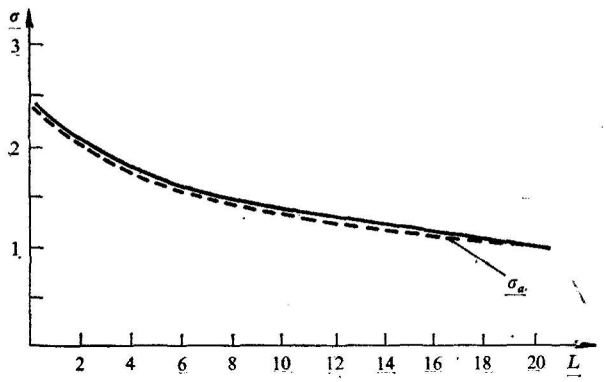


Fig. 2

scheme proposed could be recommended for numerical treatment of practical viscoelastic problems.

It was observed that the violation of the stability condition (11) yields an error increasing. This suggests that the use of the difference scheme (4), (5), (6) should be restricted to mesh steps τ and h, satisfying condition (11).

V. Conclusion

The dynamic viscoelastic problem for 4-parameter media was investigated. The initial problem and initial boundary value problem were posed. The finite difference technique was applied to the secont problem and the method of harmonics was used to investigate the stability of the constructed scheme. A condition for stability was formulated and the numerically obtained results were in good agreement with the exact solution of the problem. The scheme proposed can be useful for 2-dimensional and nonlinear problems.

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