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ASYMPTOTIC RECONSTRUCTION OF THE FOURIER EXPANSION OF AN EXTERNAL PERIODIC FORCE

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ABSTRACT: The problem under consideration involves identifying the coefficients of a partial sum of a Fourier series used to approximate the effect of an external periodic force on a Liénard oscillator. A method for constructing a nonlinear identifier is proposed, enabling the real-time asymptotical estimation of the oscillator velocity and coefficients of a partial sum of a Fourier series by means of displacement measurements. This method is based on the synthesis of invariant relations that make it possible to interrelate the variables of a special extended dynamic system and determine unknown variables as functions of known ones. The asymptotic convergence of estimates of unknowns to their true values is proven. The results of numerical simulation of the proposed method for the Duffing oscillator are presented.

KEY WORDS: Identification, Liénard oscillator, Fourier series, external perturbation, invariant relations.

1 INTRODUCTION

Many mechanical structures are subject to external periodic forces. Therefore, it is natural for theoretical and engineering problems to arise, wherein estimating the magnitude of external influence becomes important. Such problems are typical, for example, for compensation of harmonic disturbances in automatic control algorithms caused by feedback delay effects, in vibration isolation of structures, for adaptive filtering in signal processing, etc. In principle, least squares, Fourier analysis and Laplace transform provide potential solutions to related problems. However, these methods may not be suitable, for example, for control algorithms with real-time data processing.

In this regard, methods for online estimation of periodic load on a dynamic system, as well as methods for real-time estimation of unknown parameters of a sinusoidal signal, have been studied by many authors. In particular, the algorithm of the nonlinear representation of a sinusoidal signal with the subsequent application of a

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nonlinear Luenberger observer was considered in work [1]. The authors of this work built a rather complicated construction of the corresponding observer on the basis of the nonlinear observation general theory. In [2], after presenting the sinusoidal signal as a linearly parameterized form, several adaptive laws were developed and applied to estimate the sinusoid parameters using the methods of the theory of the sliding mode control. In [3], an estimator for a multi-sinusoidal signal was constructed using the standard gradient algorithm. For a biased multi-sinusoidal signal an estimator is designed to identify the frequency, amplitude and phase for each of sinusoids as well as the overall offset. Methods for estimating non-stationary parameters of a sinusoidal signal are proposed in works [4, 5].

Natural are the applied problems of determining the parameters of external influences on a dynamic system based on its response. In [6], a method for estimating a periodic signal presented as a partial sum of a Fourier series acting on a linear non-stationary system was proposed. A similar approach for a linear oscillator was considered in [7]. At the first stage, the solvability of the problem of estimating the coefficients of the Fourier series at a given frequency was proven and a scheme for determining these coefficients was proposed. At the second stage, based on the results obtained, an observer is constructed for the unknown signal frequency.

The aim of this work is to extend the method of invariant relationships over the problems of determining parameters of external influence on mechanical systems. This method [8] was developed in analytical mechanics and was intended, in particular, to find particular solutions (dependencies between variables) in problems of the dynamics of a rigid body with a fixed point. Its modification to problems in observation and identification theory allowed for the synthesis of additional connections between known and unknown quantities of the original system that arise on the trajectories of its extended dynamic model [9, 10]. Previously, the authors [11, 12] have already considered the problem of determining the parameters of external influence using this method, assuming knowledge of information such as the position and speed of oscillations, the order of the Fourier series, and the main angular frequency. In the present article we assume that the position of the oscillator is measured, but the speed of its movement is unknown and must be determine as a function of time together with the parameters of the external force.

2 The Problem of Identifying an External Periodic Force

As a model of self-oscillating system, let us consider the equation of the forced motion of the Liénard oscillator, the following second-order differential equation that describes the oscillations of a material point [13]:

$$\ddot{x}(t) + f(x(t))\,\dot{x}(t) + g(x(t)) = u(t)\,. \tag{1}$$

Here, $x(t)$ is the displacement of the point from the coordinate origin; $\dot{x}(t)$ is the velocity of this displacement; and the multiplier $f(x(t))$ and the function $g(x(t))$ describe the dissipation law and potential forces, respectively. An external periodic force $u(t)$ acting on the oscillator will be further represented as a partial sum of the Fourier series

$$u(t) = \sum_{k=1}^{n} (a_{2k-1} \cos k\omega t + a_{2k} \sin k\omega t) .$$

Remark. *Currently, in the qualitative theory of differential equations, many problems related to equation (1) regarding boundedness of oscillations, the existence of periodic solutions, the presence of limit cycles, etc. have been studied. Without considering certain qualitative properties of the Liénard equation, it is assumed that the proposed method is applicable only to those oscillatory systems whose phase vector components are bounded functions of time within* $[0, \infty)$.

Let us write the original equation (1) in the form of the Liénard system [13] by changing the variables as follows:

$$x_1(t) = x(t) , \qquad x_2(t) = \dot{x}(t) + F(x(t)) ,$$

where $F(x(t)) = \int_0^x f(\sigma)\, d\sigma$. As a result, we get

$$\dot{x}_1(t) = x_2(t) - F(x_1(t)) , \quad \dot{x}_2(t) = -g(x_1(t)) + u(t) . \tag{2}$$

Taking into account the boundness of $x(t)$, $\dot{x}(t)$, we will further assume that the phase vector of the system (2) belongs to a certain bounded domain $S_x = \{(x_1(t), x_2(t))t \geq 0\} \subset R^2$. Also suppose that the first component $x_1(t)$ of a phase vector measured during the oscillations, i.e., it is known as function of time. The component $x_2(t)$ and coefficients a_i, $i = \overline{1, 2n}$ of the Fourier series segment are unknown and must be determined from the data on the oscillator movement. Such information is the output $x_1(t)$ and those quantities that can only be obtained from the output. In particular, based on the number of unknowns, we introduce into consideration the vector $\xi(t) = (\xi_0(t), \xi_1(t), \cdots, \xi_{2n}(t))^T \in R^{2n+1}$, which we define as a solution to the Cauchy problem for some system of differential equations

$$\dot{\xi}(t) = U(t, \xi(t), x_1(t)) , \quad \xi(0) \in R^{2n+1} . \tag{3}$$

The idea of the method of invariant relations is to determine $U(t, \xi(t), x_1(t))$ and additional connections that arise on some trajectories of the extended system of differential equations (2), (3) between the known quantities $x_1(t)$, $\xi(t)$ and the unknowns $x_2(t)$, a_i, $i = \overline{0, 2n}$. In what follows, this method is used to solve the **Problem:**

Find asymptotically accurate estimates of the phase vector component $x_2(t)$ and the coefficients a_i, $i = \overline{1,2n}$ based on the output data about $x_1(t)$ for system (2).

3 Derivation of a Family of Invariant Relations

Let us find the additional relations on the trajectories of the extended system of differential equations (2), (3) in the form

$$x_2 = \Phi_0(t, x_1) + \xi_0\,, \qquad a_i = \Phi_i(t, x_1) + \xi_i\,, \quad i = \overline{1,2n}\,. \tag{4}$$

Similar to the right side $U\,(t, \xi(t), x_1(t))$ of the differential equations (3), the functions $\Phi_i(t, x_1)$, $i = \overline{0,2n}$ are still undefined. It is assumed that these functions are continuously differentiable with respect to their arguments. Let us also introduce into consideration the variables ε_i, $i = \overline{0,2n}$ that characterize deviations from relations (4):

$$\varepsilon_0 = x_2 - \Phi_0(t, x_1) - \xi_0\,, \qquad \varepsilon_i = a_i - \Phi_i(t, x_1) - \xi_i\,, \quad i = \overline{1,2n}\,. \tag{5}$$

If the equalities $\varepsilon_0(t) = \varepsilon_1(t) = \cdots = \varepsilon_{2n}(t) \equiv 0$ holds on some trajectories of the extended system of differential equations (2), (3) for certain functions $U(t, \xi, x_1)$, $\Phi_i(t, x_1)$, $i = \overline{0,2n}$, then the relations (4) become invariant and determine $x_2(t)$ and coefficients a_i, $i = \overline{1,2n}$ on these trajectories. Non-zero deviations ε_i, $i = \overline{0,2n}$ appear on other trajectories.

Consider formulas (5) as a substitution of variables in the original system. Then the differential equations for $\dot{x}_2$ and $\dot{a}_i = 0$ are transformed into differential relations (here $(\bullet)'_q$ means the partial derivative $\partial(\bullet)/\partial q$)

$$\begin{aligned} &\dot{\xi}_0 + \dot{\varepsilon}_0 = -g(x_1) + \hat{u} - \Phi'_{0t} - \Phi'_{0x_1}\hat{v}\,,\\ &\dot{\xi}_i + \dot{\varepsilon}_i = -\Phi'_{it} - \Phi'_{ix_1}\hat{v}\,, \qquad i = \overline{1,2n}\,, \end{aligned} \tag{6}$$

where

$$\hat{u} = \sum_{k=1}^{n}[(\Phi_{2k-1} + \xi_{2k-1} + \varepsilon_{2k-1})\cos k\omega t + (\Phi_{2k} + \xi_{2k} + +\varepsilon_{2k})\sin k\omega t]\,,$$
$$\hat{v} = \Phi_0 - F + \xi_0 + \varepsilon_0\,.$$

Let us determine the structure of the additional system of differential equations (3). At the first step of the proposed approach aimed at determining the function $U\,(t, y(t), x_1(t))$, we select from the previous equalities all terms that do not depend on deviations. As a result we get

$$\begin{aligned} &\dot{\xi}_0 = -\Phi'_{0x_1}\xi_0 - g(x_1) - \Phi'_{0t} - \Phi'_{0x_1}(\Phi_0 - F) + \tilde{u}\,,\\ &\dot{\xi}_i = -\Phi'_{ix_1}\xi_0 - \Phi'_{it} - \Phi'_{ix_1}(\Phi_0 - F)\,, \quad i = \overline{1,2n}\,, \end{aligned} \tag{7}$$

where $\tilde{u} = \sum\limits_{k=1}^{n} (\Phi_{2k-1} + \xi_{2k-1}) \cos k\omega t + (\Phi_{2k} + \xi_{2k}) \sin k\omega t$.

According to this choice, we have that if ξ_i satisfy (7), then the terms remaining in relationships (6) form the following linear homogeneous differential equations for the deviations:

$$\begin{aligned} \dot{\varepsilon}_0 &= -\Phi'_{0x_1}\varepsilon_0 + \sum\nolimits_{k=1}^{n} (\varepsilon_{2k-1} \cos k\omega t + \varepsilon_{2k} \sin k\omega t)\,, \\ \dot{\varepsilon}_i &= -\Phi'_{ix_1}\varepsilon_0\,, \quad i = \overline{1, 2n}\,. \end{aligned} \tag{8}$$

System (8) has the trivial solution $\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{2n} = 0$. So, we have obtained a family of additional systems (7), depending on arbitrary functions $\Phi_i(t, x_1)$, $i = \overline{0, 2n}$ and their derivatives. Each of them, together with the original system (2), transforms equalities (4) on some trajectories into invariant relations, from which the $x_2(t)$, a_i, $i = \overline{1, 2n}$ can be found.

4 Stabilization of Deviations

We do not know on which trajectories of the system of differential equations (2), (3) the relations (4) become invariant. Therefore, it is natural to consider the problem of choosing functions $\Phi_i(t, x_1)$, $i = \overline{0, 2n}$ in such a way that deviations in formulas (5) tend to zero. In other words, the trivial solution of equations (8) should become globally asymptotically stable. The second stage of the proposed method is related to solving this problem. There are several approaches to stabilize non-autonomous systems. At this stage, we assume that the appropriate method is selected depending on the analytical form of the original dynamic system.

To stabilize deviations, we use partial derivatives $\Phi'_{ix_1}(t, x_1)$ of uncertain functions as a control. The purpose of this control is to fulfill the conditions

$$\lim_{t\to\infty} \varepsilon_i(t) = 0\,, \qquad i = \overline{0, 2n}\,. \tag{9}$$

Let us consider the positive definite function $V = \frac{1}{2}\sum_{i=0}^{2n} \varepsilon_i^2$ and write its derivative along the solutions of system (8):

$$\dot{V} = -\Phi'_{0x_1}\varepsilon_0^2 + \sum\nolimits_{k=1}^{n} [(\cos k\omega t - \Phi'_{2k-1x_1})\varepsilon_0\varepsilon_{2k-1} + (\sin k\omega t - \Phi'_{2kx_1})\varepsilon_0\varepsilon_{2k}].$$

Let us choose the remaining undefined functions by setting

$$\Phi_0(t, x_1) = \lambda x_1\,, \quad \Phi_{2k-1}(t, x_1) = x_1 \cos k\omega t\,, \quad \Phi_{2k}(t, x_1) = x_1 \sin k\omega t\,, \tag{10}$$

where $\lambda > 0$. Under such functions, the derivative $\dot{V} = -\lambda\varepsilon_0^2 \le 0$. That is, the function V becomes a Lyapunov function with a negative semidefinite derivative. In this case, the zero solution of the equations in deviations is uniformly stable.

Let us show that it is asymptotically stable. To do this, we will use the LaSalle's invariance principle [14] for autonomous systems. In our case, equations (8) depend on $x_1(t)$ and time t. Therefore, we will investigate the stability of the extended autonomous system with respect to the part of variables $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{2n}$ composed by the initial system (2), equations in deviations and the system

$$\dot{z}_{2k-1} = -k\omega z_{2k}\,, \qquad \dot{z}_{2k} = k\omega z_{2k-1}\,, \quad k = \overline{1,n}\,,$$

where $z_{2k-1}(t) = \cos k\omega t$, $z_{2k}(t) = \sin k\omega t$. Taking into account (10) equations of the autonomous system under study takes the form

$$\begin{aligned}
&\dot{x}_1(t) = x_2(t) - F\left(x_1(t)\right), \quad \dot{x}_2(t) = -g\left(x_1(t)\right) + u(t)\,,\\
&\dot{\varepsilon}_0 = -\lambda\varepsilon_0 + \sum\nolimits_{k=1}^{n}\left(\varepsilon_{2k-1}z_{2k-1} + \varepsilon_{2k}z_{2k}\right),\\
&\dot{\varepsilon}_{2k-1} = -\varepsilon_0 z_{2k-1}\,, \quad \dot{\varepsilon}_{2k} = -\varepsilon_0 z_{2k}\,,\\
&\dot{z}_{2k-1} = -k\omega z_{2k}\,, \quad \dot{z}_{2k} = k\omega z_{2k-1}\,, \quad k = \overline{1,n}\,.
\end{aligned} \tag{11}$$

The phase vector of system (11) is $(x_1, x_2, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{2n}, z_1, \ldots z_{2n}) \in \bar{S}_x \times S_\varepsilon \times S_z = \Omega \subset R^{4n+3}$, where $\bar{S}_x$ is a closure of the bounded domain S_x, $S_\varepsilon = \{(\varepsilon_0, \ldots, \varepsilon_{2n}) : \sum\limits_{i=0}^{2n}\varepsilon_i^2(t) \le \sum\limits_{i=0}^{2n}\varepsilon_i^2(0)\}$ is a ball in R^{2n+1}, and $S_z = \{(z_1, \ldots, z_{2n}) : \sum\limits_{i=1}^{2n} z_i^2(t) = n$ is a sphere in R^{2n}. Thus, the set $S_x \times S_\varepsilon \times S_z = \Omega$ is compact and positive invariant to (11).

Let E be the set of all points $(x_1, x_2, \varepsilon_0, \ldots, \varepsilon_{2n}, z_1, \ldots, z_{2n})$, where $\dot{V} = -\varepsilon_0^2 \equiv 0$. It is clear that the set $M = \{(x_1(t), x_2(t), \varepsilon_0 = 0, \ldots, \varepsilon_{2n} = 0, z_1(t), \ldots, z_{2n}(t))$ is invariant in E. If M is the largest invariant set, then according to LaSalle's invariant principle all solutions with initial values in E tend to M, i.e. condition (9) is satisfied.

Let us prove that M is the largest invariant set in E. To this end, let us assume the opposite, i.e., the set E contains trajectories with non-zero deviations $\varepsilon_i(t)$, $i = \overline{0,2n}$. From the differential equations (11) it follows that these non-zero deviations are constant $\varepsilon_0(t) = \varepsilon_0^*, \ldots, \varepsilon_{2n}(t) = \varepsilon_{2n}^*$. Hence, our assumption leads to the existence of the relation

$$\sum_{k=1}^{n}\left(\varepsilon_{2k-1}^* z_{2k-1}(t) + \varepsilon_{2k}^* z_{2k}(t)\right) \equiv 0\,.$$

Since $z_{2k-1}(t) = \cos k\omega t$, $z_{2k}(t) = \sin k\omega t$, the last equality means that the next harmonic functions $\cos\omega t, \sin\omega t, \ldots, \cos 2n\omega t, \sin 2n\omega t$ are linearly dependent, which is impossible. The resulting contradiction proves the statement that the set $M$is the largest invariant set in E, and therefore, according to LaSalle's invariant principle, conditions (9) are satisfied.

5 Final Form of the Relationships

In the proposed algorithm, all degrees of freedom are fixed. Specifically, the structure of additional differential equations is defined by equation (7). The choice of functions $\Phi_i(t, x_1)$, $i = \overline{0, 2n}$ according to formulas (10) determines the final form of these equations

$$
\begin{aligned}
&\dot{\xi}_0 = -\lambda\xi_0 - g(x_1) - \lambda(\lambda x_1 - F(x_1)) + \tilde{u}\,,\\
&\dot{\xi}_{2k-1} = x_1 k\omega \sin k\omega t - (\lambda x_1 - F(x_1) + \xi_0)\cos k\omega t\,,\\
&\dot{\xi}_{2k} = -x_1 k\omega \cos k\omega t - (\lambda x_1 - F(x_1) + \xi_0)\sin k\omega t, \quad k = \overline{1, n}\,,
\end{aligned}
\tag{12}
$$

where $\tilde{u} = \sum_{k=1}^{n} [(x_1 \cos k\omega t + \xi_{2k-1})\cos k\omega t + (x_1 \sin k\omega t + \xi_{2k})\sin k\omega t]$.

According to this scheme, the terms $\Phi_i(t, x_1)$, $\xi_i(t)$ in formulas (5) become known and the deviations $\varepsilon_i(t)$ asymptotically tend to zero, which allows us to directly obtain asymptotic estimates of the $x_2(t)$ and parameters a_i, $i = \overline{0, 2n}$. Thus, the statement formulated below has been proved:

Statement. *The formulas*

$$
\begin{aligned}
&\hat{x}_2(t) = \lambda x_1(t) + \xi_0(t)\,,\\
&\hat{a}_{2k-1}(t) = x_1(t)\cos k\omega t + \xi_{2k-1}(t)\,,\\
&\hat{a}_{2k}(t) = x_1(t)\sin k\omega t + \xi_{2k}(t)\,, \quad k = \overline{1, n}\,,
\end{aligned}
$$

where $\xi_i(t)$, $i = \overline{0, 2n}$ are any solutions of the Cauchy problem for the system of differential equations (12) give asymptotic estimates of the unknowns $x_2(t)$, a_i, $i = \overline{1, 2n}$.

6 Simulation Results. Duffing Oscillator

Below are the results of some tests carried out to identify the possibility of implementing the proposed methodology for estimating the parameters of periodic signals. The Duffing oscillator is a physical model of an oscillating system that consists of a material point attached to a nonlinear spring and a linear shock absorber. We will discuss the results of numerical determination of the parameters of the external harmonic force that forces the Duffing oscillator to oscillate. The equation of forced oscillations can be written as follows:

$$
\ddot{x}(t) + \delta\dot{x}(t) + \alpha x + \beta x^3 = \sum_{k=1}^{5} (a_{2k-1}\cos k\omega t + a_{2k}\sin k\omega t)\,.
\tag{13}
$$

Here, $x(t)$ is the displacement of the material point; δ characterizes dissipation in the system; α and β are the linear and nonlinear stiffness coefficients, respectively; ω is

the angular frequency of the external force; a_i, $i = \overline{1,10}$ are unknown parameters. It should be noted that if β is negative, the dynamics of the trajectories are trivial, as all solutions of the differential equation (14) exponentially tend to infinity. Hence, we consider the case where $\beta \geq 0$. As in the general case, we assume that with such parameters, solutions (13) are correctly defined on the interval $[0, \infty)$.

Let us rewrite equation (13) in the form of a system in the space of states by substituting the Liénard variables. We obtain

$$\begin{aligned} &\dot{x}_1 = x_2 - \delta x_1 \,, \\ &\dot{x}_2 = -\alpha x_1 - \beta x_1^3 + \sum_{k=1}^{5} (a_{2k-1} \cos k\omega t + a_{2k} \sin k\omega t) \,. \end{aligned} \tag{14}$$

The scheme proposed in this work was numerically tested within a wide range of initial conditions and parameters of the dynamic system (14). This system was used to obtain the output $x_1(t)$ and to find the asymptotic estimates of $x_2(t)$ and a_i, $i = \overline{1,10}$. The parameters of the simulated system were taken equal to

$$\begin{gathered} \delta = 0.5; \quad \alpha = 3.5; \quad \beta = 0.25; \quad \omega = 0.5; \quad \lambda = 2; \\ a_i = a_{i+5} = -3 + i, \qquad i = \overline{1,5.} \,. \end{gathered}$$

The initial conditions imposed on the system (14) for numerical simulation, here are $x(0) = (-1.0; 1.0)$. The initial conditions for the variables of the auxiliary system of differential equations are chosen arbitrarily; in the illustrated case $\xi_i(0) = -3$, $i = \overline{1,10}$. With such parameters, the auxiliary system looked like

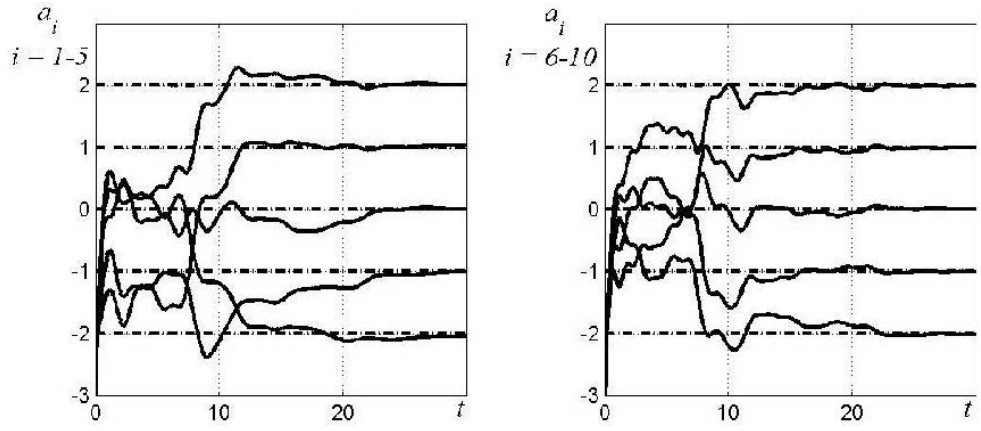
$$\begin{aligned} &\dot{\xi}_0 = -\lambda\xi_0 - (\lambda^2 - \lambda\delta + \alpha)x_1 - \beta x_1^3 + \tilde{u} \,, \\ &\dot{\xi}_{2k-1} = x_1 k\omega \sin k\omega t - [(\lambda - \delta)x_1 + \xi_0)] \cos k\omega t \,, \\ &\dot{\xi}_{2k} = -x_1 k\omega \cos k\omega t - [(\lambda - \delta)x_1 + \xi_0] \sin k\omega t \,, \quad k = \overline{1,5} \,, \end{aligned} \tag{15}$$

where

$$\tilde{u} = \sum\nolimits_{k=1}^{5} [(x_1 \cos k\omega t + \xi_{2k-1}) \cos k\omega t + (x_1 \sin k\omega t + \xi_{2k}) \sin k\omega t] \,.$$

The simulation results for estimating Fourier series coefficients a_i, $i = \overline{1,10}$ are presented in Fig. 1. Solid lines show the graphs of the following functions:

$$\begin{aligned} &\hat{a}_{2k-1}(t) = x_1(t) \cos k\omega t + \xi_{2k-1}(t) \,, \\ &\hat{a}_{2k} = x_1(t) \sin k\omega t + \xi_{2k}(t) \,, \quad k = \overline{1,5} \,. \end{aligned}$$



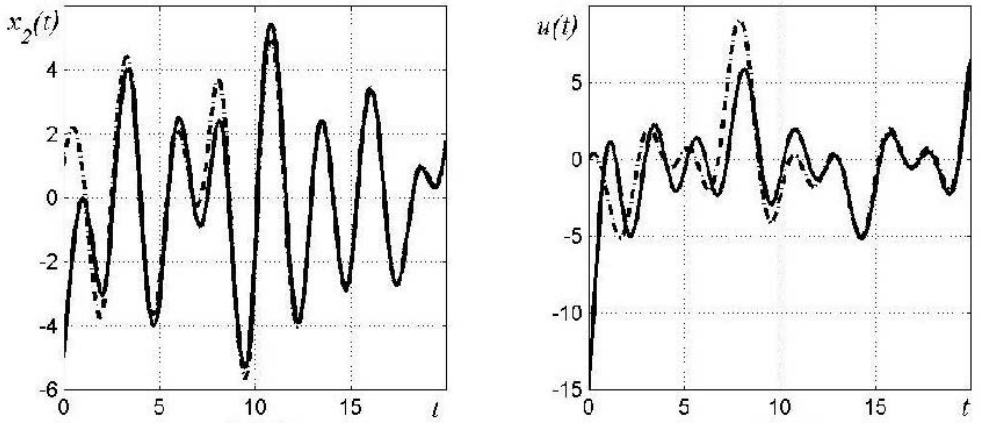
(a) (b)

Fig. 1. Convergence of estimates $\hat{a}_i$ to exact values of coefficients a_i: (a) $i = \overline{1,5}$; (b) $i = \overline{6,10}$.

These estimates, in complete accordance with the statement in Section 5, asymptotically tend to the true values of the unknown parameters (represented by horizontal dotted lines). Similarly, the oscillation speed is estimated using the formula

$$\hat{x}_2(t) = \lambda x_1(t) + \xi_0(t)\,.$$

The graph of this function is shown in Fig. 2a with a solid line, and the dashed line represents the $x_2(t)$ graph.



(a) (b)

Fig. 2. Estimation of phase vector component $x_2(t)$ (a); and tracking of the external signal $u(t)$ (b).

Figure 2b shows the tracking of an external signal, where the dotted line represents the external force acting on the oscillator

$$u(t) = \sum_{k=1}^{5} (a_{2k-1} \cos k\omega t + a_{2k} \sin k\omega t)$$

and the solid line corresponds to its approximation based on the simulation results

$$\hat{u}(t) = \sum_{k=1}^{5} (\hat{a}_{2k-1} \cos k\omega t + \hat{a}_{2k} \sin k\omega t) .$$

As can be seen from these graphs, the simulation results confirm the effectiveness of the proposed method of asymptotic estimation of $x_2(t)$ and parameters of external periodic force.

7 Conclusions

The oscillations of a Liéard oscillator subjected to an external load specified as a partial sum of a Fourier series, are considered. Assuming the availability of the signal concerning the oscillator's position as a function of time from measurements, we solve the problem of online asymptotic reconstruction of unknown quantities: the oscillation speed and the Fourier coefficients of the external force. A method of asymptotic estimation has been proposed based on the synthesis of invariant relations for the trajectories of an auxiliary dynamic system with invariant relations reflecting the relationship between the unknown and known quantities in the family of solutions under study. The convergence of the estimates of unknown variables to their exact values has been proved based on LaSalle's principle of invariance. An example of numerical simulation of the proposed approach using the Duffing oscillator as an example has been given.

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