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SPECIAL ISSUE

LOCAL SOLVABILITY OF THE PARABOLIC POINCARÉ PROBLEM WITH EMERGENT TYPE VECTOR FIELD

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To Academician Petar Popivanov, with our very best wishes on the occasion of his 80th anniversary, and with sincere thanks for all that we have learned from him as a mathematician and as an extraordinary person

ABSTRACT: We prove local strong solvability of the oblique derivative problem for linear second-order parabolic equations in the case when the vector field, defining the boundary operator, can be tangent to the boundary and is of emergent type. *A priori* estimates in suitable Sobolev spaces are derived for the solution as well.

KEY WORDS: *A priori* estimates, Emergent vector field, Oblique derivative problem, Poincaré problem, Second order PDE, Sobolev space, Strong solution, Uniformly parabolic.

1 INTRODUCTION

The tangential oblique derivative problem is a boundary value problem for a given second-order partial differential equation (PDE) with a boundary condition expressed in terms of a directional derivative with respect to a unit vector field ℓ that becomes *tangent* to the boundary of the underlying domain. It was Poincaré who first arrived at a problem of that type in his studies on tides [1] and, in the mathematical literature the term *Poincaré problem* is commonly used for problems with tangential vector field ℓ.

It is well known from the general theory of PDEs that problems with boundary condition given as directional derivative along a field ℓ which is *strictly transversal* to the boundary are *regular* because the Shapiro–Lopatinskij complementary condition

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is satisfied. As a consequence, such problems are of finite index and their solutions gain *full regularity* from the corresponding data. The situation changes drastically when ℓ is tangent to the boundary at the points of some non-empty set because the Shapiro–Lopatinskij condition is violated there and the corresponding Poincaré problem is *no longer regular* (see [2–5]). Depending on whether the normal component of ℓ changes its orientation with respect to the boundary along the ℓ-trajectories when these cross the set of tangency, the Poincaré problem may have *kernel* or *co-kernel* (cf. [6]) of *infinite dimension* (cf. [5]) and the solution always *loses regularity* with respect to the data. Moreover, the exact loss of regularity depends on the order of contact between ℓ and the boundary of the underlying domain as shown in [7].

Setting γ to be the scalar product of ℓ with the outward normal to the boundary, the simplest case arises when γ, although vanishing on the set of tangency, preserves its sign. According to the physical interpretation of the Poincaré problems in the theory of Brownian motion, this corresponds to the case of *neutral* vector field ℓ, for which the Poincaré problem has finite dimensional index, although the solution still loses regularity with respect to the data (see [3, 4]). The field ℓ is classified as *emergent* when γ changes sign from negative to positive along the ℓ-integral curves as they cross the set of tangency where $\gamma = 0$, and the Poincaré problem, besides the usual loss of regularity, has a *kernel of infinite dimension*, as shown in [6]. In order to obtain a well-posed problem, the values of the solution must be prescribed on a codimension-one submanifold of the tangency set.

In the remaining case, when the sign of γ changes from positive to negative along the trajectories of ℓ, the field is called *submergent*. Then the *co-kernel* of the corresponding Poincaré problem is of *infinite dimension* (cf. [5]), and a well-posed modification is possible by violating the directional derivative condition on the set of tangency which, however, produces finite jump of the solution across the set $\gamma = 0$.

Cocluding this brief overview, it is worth to note the interesting paper [8] where existence of viscosity solutions has been proved for tangential oblique derivative problems for fully nonlinear elliptic operators, and also [9, 10] where boundary value problems violating the Shapiro–Lopatinskij condition have been studied for the bi- and polyharmonic operators.

The general goal of the present paper is to study the strong solvability of the *emergent Poincaré problem* for second-order linear uniformly parabolic equations. More precisely, we deal with the following problem

$$(1) \qquad \begin{cases} \mathcal{P}u \equiv D_t u - a^{ij}(x,t) D_{x_i x_j} u = f(x,t) & \text{in } Q, \\ \dfrac{\partial u}{\partial \ell} = \psi(x,t) \quad \text{on } S, \qquad u(x,0) = \varphi(x) & \text{in } \Omega. \end{cases}$$

The problem is posed in $(n+1)$-dimensional cylinder $Q = \Omega \times (0,T)$, with an

emergent vector field $\boldsymbol{\ell}$ that becomes tangent to the later surface S at the points of an $(n-1)$-dimensional manifold $S_0 \subset S$. According to what has already been mentioned above, in order to obtain a well-posed problem one has to modify (1) by prescribing the values $u = \mu$ on S_0.

We are interested in *strong solutions* of (1), that is, functions belonging to suitable parabolic Sobolev spaces and satisfying the second-order equation for almost all (a.a.) $(x,t) \in Q$, while the boundary and the initial conditions are verified almost everywhere (a.e.) in the sense of the corresponding traces. The coefficients a^{ij} will be assumed to be at least continuous in $\overline{Q}$, while the data f, ψ, φ, and μ are taken in suitable Lebesgue or Besov spaces. In order to compensate the loss of regularity caused by the tangency, we require additional smoothness of the data near the set S_0.

The strong solvability theory for parabolic Poincaré problems has already been developed in [11] in the case of *neutral* field $\boldsymbol{\ell}$, while the corresponding elliptic problems have been studied in [12] for *neutral* fields $\boldsymbol{\ell}$ and in [13,14] when $\boldsymbol{\ell}$ is *emergent.*

Actually, we deal only with the *local* strong solvability of the Poincaré problem near the tangency set S_0. Bearing in mind that (1) is a regular problem away from S_0, this constitutes a crucial step toward the *global* solvability result, which will be presented in a forthcoming paper [15]. To prove the local solvability of (1) supplemented with the condition $u = \mu$ on S_0, we follow the approach already used in [13] in the study of the *emergent* Poincaré problem for uniformly *elliptic* equations.

Namely, we first construct an n-dimensional manifold $\Sigma \subset Q$ by extending S_0 inside Q in the interior normal direction, which automatically defines μ on Σ. Considering a suitable restriction $\mathcal{P}'$ of the operator $\mathcal{P}$ to Σ, there exists a unique strong solution u' of the Cauchy–Dirichlet problem on Σ for the operator $\mathcal{P}'$ with boundary data μ and initial condition $\varphi|_{\Sigma \cap \{t=0\}}$.

Further, taking into account the higher regularity of the data near S_0, we "differentiate" the equation in (1) along a suitable extension of $\boldsymbol{\ell}$ near S_0, and solve another Cauchy–Dirichlet problem around Σ for a parabolic equation with right-hand side $\partial f/\partial \boldsymbol{\ell}$, boundary condition $w = \psi$, and initial condition $w = \partial \varphi/\partial \boldsymbol{\ell}$.

At this point, integrating the unique solution w of that problem along the $\boldsymbol{\ell}$-trajectories, and adjusting it by means of u', yields an "approximate" local solution to (1) which allows us to establish a Fredholm alternative and thereby obtain the desired local solvability result.

2 Notation, assumptions and auxiliary results

In the sequel, we shall use the following notation:

- $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with $C^{1,1}$-smooth boundary, and $\boldsymbol{\nu}(x)$ denotes the unit outward normal to $\partial\Omega$;
- $Q = \Omega \times (0,T)$, $S = \partial\Omega \times (0,T)$, $T > 0$;

• $\boldsymbol{\ell}(x,t) = \big(\ell^1(x,t),\dots,\ell^n(x,t),0\big)$ is a unit vector field defined on S, decomposable as $\boldsymbol{\ell}(x,t) = \boldsymbol{\tau}(x,t) + \gamma(x,t)\boldsymbol{\nu}(x)$ for all $(x,t) \in S$, where $\gamma(x,t) = \boldsymbol{\ell}(x,t)\cdot\boldsymbol{\nu}(x)$;

• $S^{\pm} = \big\{(x,t)\in S\colon\ \gamma(x,t) \gtrless 0\big\}$, $S_0 = \big\{(x,t)\in S\colon\ \gamma(x,t) = 0\big\}$;

• for a given vector field $\boldsymbol{L}$, $s \mapsto \mathrm{e}^{s\boldsymbol{L}}(x,t)$ denotes the natural parametrization of the integral curve of $\boldsymbol{L}$ passing through (x,t), that is,

$$\partial_s \mathrm{e}^{s\boldsymbol{L}}(x,t) = \boldsymbol{L}(\mathrm{e}^{s\boldsymbol{L}}(x,t)), \quad \mathrm{e}^{0\boldsymbol{L}}(x,t) = (x,t);$$

• $L^p(Q)$ with $1 \le p \le \infty$, is the collection of all measurable functions $v\colon\ Q \to \mathbb{R}$ with finite norm $\|v\|_{L^p(Q)} = \Big(\int_Q |v(x,t)|^p \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/p}$;

• $W^{2,1}_p(Q)$, $1 \le p \le \infty$, is the Sobolev space equipped with the usual norm

$$\|v\|_{W^{2,1}_p(Q)} = \sum_{j=0}^{2}\sum_{2r+s=j} \|D_t^r D_x^s v\|_{L^p(Q)};$$

• $W^{\theta}_p(\Omega)$ and $W^{\theta,\theta/2}_p(S)$, with non-integer $\theta > 0$, are the Besov spaces on Ω and S, respectively (see [11, 16] for the corresponding norms);

• $V^1_p(Q) = \big\{v\colon\ v \in L^p(Q),\ D_x v \in L^p(Q\big\}$;

• $\mathrm{Lip}(Q)$ is the space of the Lipschitz continuous functions $v\colon\ Q \to \mathbb{R}$;

• $W^{3,1}_{\infty}(Q) = \big\{v \in W^{2,1}_{\infty}(Q)\colon\ D_x^2 v \in V^1_{\infty}(Q)\big\}$;

• the usual summation convention on the repeated indices is adopted throughout.

2.1 Assumptions

In what follows, $K \subset \overline{Q}$ will denote a closed neighbourhood of the set of tangency S_0.

Our first hypothesis concerns the *emergent* behaviour of $\boldsymbol{\ell}$ on S and the structure of the *tangency* set S_0 :

$$\text{(2)}\qquad \begin{cases} \boldsymbol{\ell} \text{ is tangent to } S \text{ only at the points of } S_0,\ \operatorname{codim}_S S_0 = 1,\\ \boldsymbol{\ell} \text{ is } \textit{strictly transversal} \text{ to } S_0 \text{ and points from } S^- \text{ to } S^+ \text{ there},\\ \ell^i(x,t) \in \mathrm{Lip}(\overline{S}) \cap W^{2,1}_{\infty}(\overline{S}\cap K), \quad S_0 \text{ and } S\cap K \text{ are } W^{3,1}_{\infty}\text{-smooth}. \end{cases}$$

The operator $\mathcal{P}$ in (1) is *uniformly* parabolic, that is, there exists a constant $\lambda > 0$ such that

$$\text{(3)}\qquad a^{ij}(x,t)\xi_i\xi_j \ge \lambda|\xi|^2, \qquad a^{ij} = a^{ji}, \qquad a^{ij} \in C^0(\overline{Q}) \cap V^1_{\infty}(K).$$

2.2 Geometric constructions near the set S_0

Since $\partial\Omega \in C^{1,1}$, setting $R = \{(x,t) \in \overline{Q}\colon\ \operatorname{dist}(x,\partial\Omega) \ll 1\}$, to each $(x,t) \in R$ there corresponds a unique closest point $(y(x),t) \in S$ and the mapping $(x,t) \mapsto (y(x),t)$ is $\mathrm{Lip}(R) \cap W^{2,1}_\infty(R \cap K)$-smooth.

We now extend the field $\boldsymbol{\ell}$ to R by the formula $\boldsymbol{L}(x,t) = \boldsymbol{\ell}(y(x,t))\ \forall (x,t) \in R$ and, without loss of generality, assume that $\boldsymbol{L}$ is defined in the whole $\overline{Q}$ in such a way that $\boldsymbol{L} \in \mathrm{Lip}(\overline{Q}) \cap W^{2,1}_\infty(\overline{Q} \cap K)$. Let

$$\mathcal{H} = \{(x', x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times (0,T)\colon\ x_n = \Phi(x',t),\ (x',t) \in \mathcal{O} \subset \mathbb{R}^{n-1} \times (0,T)\}$$

be an n-dimensional hyperplane with $\Phi \in W^{2,1}_\infty(\mathcal{O})$, and let $v, D_{x_n} v \in L^p_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$. Then $v(x', x_n, t)$ is absolutely continuous with respect to x_n and $D_{x_n} v$ is a.e. a classical derivative. Define now the trace $\tilde{v}|_{\mathcal{H}}$ on $\mathcal{H}$ by the formula

$$\tilde{v}(x', \Phi(x',t), t) = v(x', x_n, t) - \int_{\Phi(x',t)}^{x_n} \frac{\partial v}{\partial x_n}(x', \theta, t)\, \mathrm{d}\theta \tag{4}$$

for a.a. $(x', \Phi(x',t), t) \in \mathcal{H}$, and note that the Fubini theorem implies $\tilde{v} \in L^p_{\mathrm{loc}}(\mathcal{H})$. Moreover, if $v, D_{x_n} v \in W^{2,1}_{p,\mathrm{loc}}(\mathbb{R}^n \times (0,T))$ then $\tilde{v} \in W^{2,1}_{p,\mathrm{loc}}(\mathcal{H})$ and the trace operator $v \mapsto \tilde{v}$ is compact when considered as a mapping from $W^{2,1}_p$ into V^1_p.

This procedure is also applicable in the more general case when $\partial/\partial x_n$ is replaced by $\partial/\partial \boldsymbol{L}$, where $\boldsymbol{L} = (L^1, \dots, L^n, 0)$ is a unit vector field strictly transversal to $\mathcal{H}$. Indeed, it suffices first to straighten $\boldsymbol{L}$ in a neighbourhood of a point of $\mathcal{H}$ so that $\partial/\partial \boldsymbol{L} \equiv \partial/\partial x_n$ and then apply (4).

Define now $\Sigma = \{(x,t) \in R\colon\ (y(x),t) \in S_0\}$ and set $\boldsymbol{\tau}(x,t) = \boldsymbol{\tau}(y(x),t)$ for all $(x,t) \in R$. It should be noted that Σ is an n-dimensional hyperplane lying in $\overline{Q}$, with $\Sigma \in W^{2,1}_\infty$, and whose normal vector is nowhere parallel to the axis Ot, because of the strict transversality of $\boldsymbol{\ell}$ to S_0. Moreover, $\boldsymbol{\tau} \in W^{2,1}_\infty(R \cap K)$ and $\boldsymbol{\tau}|_\Sigma \equiv \boldsymbol{L}|_\Sigma \equiv \boldsymbol{\ell}|_{S_0}$.

Define the closed neighbourhood U of Σ by

$$U = \{(x,t) \in R \cap K\colon\ |\boldsymbol{\tau}(x,t)| \geq 1/2\}$$

and set $\boldsymbol{\tau'}(x,t)$ for the normalized field $\boldsymbol{\tau}(x,t)$, $(x,t) \in U$. Because of the strict transversality of $\boldsymbol{\tau'}$ to Σ, each point $(\bar{x},t) \in U$ can be reached from a unique point $(\bar{x}',t) \in \Sigma$ along a trajectory of $\boldsymbol{\tau'}$, that is, $(\bar{x},t) = \mathrm{e}^{\xi \boldsymbol{\tau'}(\bar{x}',t)}$ for some ξ satisfying $\operatorname{sign}\xi = \operatorname{sign}(\gamma(y(\bar{x}),t))$.

We now introduce new coordinates $(\xi, \eta, \zeta, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times [0,T]$ in U as follows. For each $(\bar{x},t) \in U$, $\xi(\bar{x},t)$ is the length with sign of the $\boldsymbol{\tau'}$-trajectory joining $(\bar{x},t)$ with $(\bar{x}',t) \in \Sigma$. Furthermore,

$$\eta(\bar{x},t) = \operatorname{dist}\big((\bar{x}',t), S\big) = \operatorname{dist}\big(\mathrm{e}^{-\xi(\bar{x},t)\boldsymbol{\tau'}}(\bar{x},t), S\big)$$

and $\zeta(\bar{x},t)=y\big(\mathrm{e}^{-\xi(\bar{x},t)\boldsymbol{\tau}'}(\bar{x},t),t\big)$, while the time variable t remains unchanged.

In the $O\eta\xi$-plane consider the $C^{1,1}$-smooth convex domain D, symmetric with respect to the $O\eta$-axis, whose boundary ∂D for $\xi>0$ consists of the segment $[0,\xi]$ for $\xi\in[0,1]$, the graph of $1+\sqrt{1-(\eta-1)^2}$ for $\eta\in[0,1]$, the segment $[\eta,2]$ for $\eta\in[1,2]$, and the graph of $\sqrt{4-(\eta-2)^2}$ for $\eta\in[2,4]$.

Setting δD for the dilation of D by a small factor $\delta>0$, we define the set

$$D_{\delta,T}=\big\{(x,t)\in U\colon\quad \zeta(x,t)\in S_0,\ (\eta(x,t),\xi(x,t))\in\delta D,\ t\in(0,T)\big\}.$$

It is clear that $D_{\delta,T}\subset\overline{Q}$ is non-cylindrical domain with $W^{2,1}_\infty$-smooth lateral surface $\partial D_{\delta,T}$, while $S\cap\partial D_{\delta,T}$ is $W^{3,1}_\infty$-smooth, and the normal vector to $\partial D_{\delta,T}$ is nowhere parallel to Ot. Furthermore, if δ is sufficiently small, then the vector field $\boldsymbol{L}$ is tangent to $\partial D_{\delta,T}$ *only* at the points of $\Sigma\cap\partial D_{\delta,T}=S_0\cup S_{0\delta}$, where $S_{0\delta}=\{(x,t)\in U\colon\ \eta(x,t)=4\delta\}$, and it points inward/outward to $D_{\delta,T}$ at $(x,t)\in\partial D_{\delta,T}\setminus(S_0\cup S_{0\delta})$ with $(y(x),t)\in S^-/S^+$, that is, $\boldsymbol{L}$ is emergent field on $\partial D_{\delta,T}$.

3 Local solvability of the Poincaré problem

As already mentioned, in order (1) to be a well-posed problem, the values of u must be prescribed on the set of tangency S_0, namely $u(x,t)=\mu(x,t)$ on S_0. Furthermore, due to the degeneracy of (1) near S_0, in order to solve the problem we require additional regularity of the data f,ψ,φ and μ near S_0. In particular, $\psi\in W^{2-1/p,1-1/(2p)}(S\cap K)$ and $\mu\in W^{2-1/p,1-1/(2p)}(S_0)$. Without loss of generality, we may assume that ψ and μ are extended inside Q to functions $\psi\in W^{2,1}_p(K)$ and $\mu\in W^{2,1}_p(\Sigma)$.

Our main result is the following

Theorem 1: *Under the hypotheses* (2) *and* (3), *let* $f\in V^1_p(K)$, $\psi\in W^{2,1}_p(K)$, $\mu\in W^{2,1}_p(\Sigma)$, *and* $\varphi, D_x\varphi\in W^{2-2/p}_p(K\cap\Omega)$ *for a given* $p>1$. *When* $p>3/2$, *assume moreover the following compatibility conditions*

$$\frac{\partial\varphi}{\partial\boldsymbol{\ell}}(x,0)=\psi(x,0),\quad \mu(x,0)=\varphi(x)\quad \textit{for a.a. } x\in\partial\Omega\cap K.$$

Then for each sufficiently small $\delta>0$, *the problem*

$$\text{(5)}\qquad \begin{cases} \mathcal{P}u\equiv D_tu-a^{ij}(x,t)D_{x_ix_j}u=f(x,t) & \textit{a.e. in } D_{\delta,T},\\ \dfrac{\partial u}{\partial\boldsymbol{L}}=\psi(x,t)\quad \textit{on } \partial D_{\delta,T}, \qquad u=\mu(x,t) & \textit{on } S_0\cup S_{0\delta},\\ u(x,0)=\varphi(x) & \textit{on } \Omega\cap\overline{D_{\delta,T}} \end{cases}$$

possesses a unique solution $u\in W^{2,1}_p(D_{\delta,T})$ *such that* $\partial u/\partial\boldsymbol{L}\in W^{2,1}_p(D_{\delta,T})$.

Moreover, there exists a constant depending on the data of (5) such that

$$
\begin{aligned}
(6)\quad \|u\|_{W^{2,1}_p(D_{\delta,T})},\ \|\partial u/\partial \boldsymbol{L}\|_{W^{2,1}_p(D_{\delta,T})} &\le C\Big(\|f\|_{L^p(D_{\delta,T})} + \|\partial f/\partial \boldsymbol{L}\|_{L^p(D_{\delta,T})} \\
&+ \|\psi\|_{W^{2,1}_p(D_{\delta,T})} + \|\mu\|_{W^{2,1}_p(\Sigma\cap D_{\delta,T})} \\
&+ \|\varphi\|_{W^{2-2/p}_p(\Omega\cap\overline{D_{\delta,T}})} + \|\partial\varphi/\partial \boldsymbol{L}\|_{W^{2-2/p}_p(\Omega\cap\overline{D_{\delta,T}})}\Big).
\end{aligned}
$$

PROOF. The estimate (6) implies that the homogeneous problem (5) admits only the trivial solution, whence the unicity does follow.

Turning to the existence assertion, we first note that, by subtracting a suitable function $\bar u \in W^{2,1}_p(D_{\delta,T})$ with $D_x\bar u \in W^{2,1}_p(D_{\delta,T})$ and satisfying $\bar u(x,0) = \varphi(x)$ for $x \in \Omega \cap \overline{D_{\delta,T}}$, problem (5) reduces to a problem with zero initial condition. Indeed, such a function $\bar u$ exists thanks to the higher regularity assumptions imposed on the data of (5), the compatibility conditions satisfied by φ when $p > 3/2$, and the regularity theory for linear parabolic equations (cf. [16, 17]. For instance, $\bar u$ may be chosen as the unique strong solution of the Cauchy–Dirichlet problem for the heat equation with initial datum φ. Thus, without loss of generality, we study (5) under the assumption $\varphi \equiv 0$, namely

$$
(7)\qquad \begin{cases} \mathcal{P}u \equiv D_t u - a^{ij}(x,t)D_{x_ix_j}u = f(x,t) & \text{a.e. in } D_{\delta,T}, \\ \dfrac{\partial u}{\partial \boldsymbol{L}} = \psi(x,t) \quad \text{on } \partial D_{\delta,T}, \qquad u = \mu(x,t) & \text{on } S_0 \cup S_{0\delta}, \\ u(x,0) = 0 & \text{on } \Omega \cap \overline{D_{\delta,T}}. \end{cases}
$$

Consider the following Cauchy–Dirichlet problem in $D_{\delta,T}$:

$$
(8)\qquad \begin{cases} \mathcal{P}w = \partial f/\partial \boldsymbol{L}(x,t) & \text{a.e. in } D_{\delta,T}, \\ w = \psi(x,t) \quad \text{on } \partial D_{\delta,T}, \qquad w(x,0) = 0 & \text{on } \Omega \cap \overline{D_{\delta,T}}. \end{cases}
$$

The coefficients a^{ij} of the uniformly parabolic operator $\mathcal{P}$ are continuous, the right-hand side $\partial f/\partial \boldsymbol{L} \in L^p(D_{\delta,T})$, $\psi \in W^{2,1}_p(D_{\delta,T})$, and the normal vector to the $W^{2,1}_\infty$-smooth lateral surface $\partial D_{\delta,T}$ is nowhere parallel to the axis Ot. Therefore, according to the regularity theory of linear parabolic equations with continuous coefficients (see [17, 18] and [16], there exists a unique strong solution $w \in W^{2,1}_p(D_{\delta,T})$ of (8).

The action of the operator $\mathcal{P} \equiv D_t - a^{ij}(x,t)D_{x_ix_j}$ on the functions defined in U and constant along the trajectories of the vector field $\boldsymbol{L}$ defines a second-order linear differential operator $\mathcal{P}'$ on the manifold Σ. The coefficients of $\mathcal{P}'$ are Lipschitz continuous on Σ, thanks to (3) and the regularity of $\boldsymbol{L}$, and $\mathcal{P}'$ is *strictly*

parabolic on Σ because of the strong transversality of $\boldsymbol{L}$ to Σ. Since each point $(x,t) \in U$ can be reached from $(x',t) \in \Sigma$ along an $\boldsymbol{L}$-trajectory in either the positive or negative direction, there exists a unique parameter $s(x,t) \in W^{2,1}_{\infty}(U)$ such that $\mathrm{e}^{-s(x,t)\boldsymbol{L}}(x,t) = (x',t) \in \Sigma$. Without loss of generality, we may assume that $|s(x,t)| \le \delta$ for each $(x,t) \in D_{\delta,T}$.

We now define, for a.a. $(x',t) \in \Sigma$, the trace of $f(x,t)$ on Σ along the $\boldsymbol{L}$-trajectories by

$$\tilde{f}(x',t) = f(x,t) - \int_0^{s(x,t)} \frac{\partial f}{\partial \boldsymbol{L}}\big(\mathrm{e}^{\theta \boldsymbol{L}}(x',t)\big)\,\mathrm{d}\theta \tag{9}$$

and observe that $\tilde{f} \in L^p(\Sigma)$ (cf. (4)).

Furthermore, the manifold $\Sigma_{\delta,T} = \Sigma \cap D_{\delta,T}$ is $W^{2,1}_{\infty}$-smooth with normal nowhere parallel to the axis Ot, and we consider the Cauchy–Dirichlet problem

$$\begin{cases} \mathcal{P}'v = \tilde{f} & \text{a.e. in } \Sigma_{\delta,T}, \\ v = \mu & \text{on } S_0 \cup S_{0\delta}, \qquad v(x,0) = 0 \text{ on } \Omega \cap \Sigma_{\delta,T}. \end{cases} \tag{10}$$

Thanks to the uniform parabolicity of $\mathcal{P}'$, the continuity of its coefficients, $\tilde{f} \in L^p(\Sigma_{\delta,T})$, and $\mu \in W^{2,1}_p(\Sigma_{\delta,T})$, there exists a unique solution $v \in W^{2,1}_p(\Sigma_{\delta,T})$ to (10) (see [17, 18] once again).

With the functions w and v at hand, we define

$$\begin{aligned} u(x,t) &= v\big(\mathrm{e}^{-s(x,t)\boldsymbol{L}}(x,t)\big) + \int_0^{s(x,t)} w\big(\mathrm{e}^{(\theta - s(x,t)\boldsymbol{L})}(x,t)\big)\,\mathrm{d}\theta \\ &= v(x',t) + \int_0^{s(x,t)} w\big(\mathrm{e}^{\theta \boldsymbol{L}}(x',t)\big)\,\mathrm{d}\theta \end{aligned} \tag{11}$$

for a.a. $(x,t) \in D_{\delta,T}$. Bearing in mind (8) and (10), it is clear that the function $u(x,t)$ satisfies the boundary and initial conditions of (7). Moreover, employing the *a priori* estimates for (8) and (10), we obtain

$$\begin{aligned} \|\mathcal{P}u\|_{L^p(D_{\delta,T})},\, \|\partial \mathcal{P}u/\partial \boldsymbol{L}\|_{L^p(D_{\delta,T})} \le C\Big(&\|\tilde{f}\|_{L^p(\Sigma_{\delta,T})} \\ &+ \|\partial f/\partial \boldsymbol{L}\|_{L^p(D_{\delta,T})} + \|\psi\|_{W^{2,1}_p(D_{\delta,T})} + \|\mu\|_{W^{2,1}_p(\Sigma_{\delta,T})}\Big). \end{aligned}$$

In particular, in order to prove existence for (7) (and therefore also for (5)) and derive estimate (6), we may first subtract a function of the type (11). This reduces (7) to the equivalent problem with $\psi \equiv 0$ and $\mu \equiv 0$:

$$\begin{cases} \mathcal{P}u \equiv D_t u - a^{ij}(x,t) D_{x_i x_j} u = f(x,t) & \text{a.e. in } D_{\delta,T}, \\ \dfrac{\partial u}{\partial \boldsymbol{L}} = 0 \quad \text{on } \partial D_{\delta,T}, \qquad u = 0 & \text{on } S_0 \cup S_{0\delta}, \\ u(x,0) = 0 & \text{on } \Omega \cap \overline{D_{\delta,T}}. \end{cases} \tag{12}$$

Consider now the linear space

$$\mathcal{H}_p(D_{\delta,T}) = \big\{ f \in L^p(D_{\delta,T}) \colon\ \partial f/\partial \boldsymbol{L} \in L^p(D_{\delta,T}) \big\}$$

and equip it with the norm

$$\|f\|_{\mathcal{H}_p(D_{\delta,T})} = \|\tilde{f}\|_{L^p(\Sigma_{\delta,T})} + \delta^{1-1/p} \|\partial f/\partial \boldsymbol{L}\|_{L^p(D_{\delta,T})}.$$

Obviously, $\mathcal{H}_p(D_{\delta,T})$ is a Banach space since the trace $\tilde{f}$ is uniquely defined by (9).

Define the linear operator $\mathcal{A}\colon \mathcal{H}_p(D_{\delta,T}) \mapsto \mathcal{H}_p(D_{\delta,T})$ by the formula $\mathcal{A}f = \mathcal{P}u$ with u given by (11), where w and v are the solutions of (8) and (10), respectively, with $\psi \equiv 0$ and $\mu \equiv 0$.

It is clear that the invertibility of $\mathcal{A}$ would imply solvability of (12). In fact, given $f \in \mathcal{H}_p(D_{\delta,T})$, one computes $\partial(\mathcal{A}^{-1}f)/\partial \boldsymbol{L}$ and the trace $\mathcal{A}^{-1}f|_{\Sigma_{\delta,T}}$ and with these right-hand sides solves (8) and (10) with $\psi \equiv 0$ and $\mu \equiv 0$. This gives the corresponding solutions $w \in W_p^{2,1}(D_{\delta,T})$ and $v \in W_p^{2,1}(\Sigma_{\delta,T})$ which define the unique solution u of (12) through the formula (11).

We will employ the Fredholm alternative in order to prove invertibility of the operator $\mathcal{A}$ and for this goal we are going to show that

$$\mathcal{A} = \mathrm{Id}_{\mathcal{H}_p(D_{\delta,T})} + \mathcal{B} + \mathcal{C} \tag{13}$$

where the operator norm of $\mathcal{B}$ is small enough when δ is small, while $\mathcal{C}$ is a compact operator.

To realize this, we compute first the commutator of $\mathcal{P}$ and $\partial/\partial \boldsymbol{L}$ in order to get

$$\mathcal{P}\frac{\partial}{\partial \boldsymbol{L}} - \frac{\partial}{\partial \boldsymbol{L}}\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$$

where

$$\begin{aligned} \mathcal{P}_1 &= \big(D_t L^i(x,t) - a^{kj}(x,t) D_{kj} L^i(x,t)\big) D_i, \\ \mathcal{P}_2 &= \big(D_k a^{ij}(x,t) L^k(x,t) - 2a^{kj}(x,t) D_k L^i(x,t)\big) D_{ij} \end{aligned}$$

are differential operators with *essentially bounded* coefficients involving *only* derivatives with respect to the spatial variables x.

Applying the commutator identity to the function u given by (11), we obtain

$$\mathcal{P}\frac{\partial u}{\partial \boldsymbol{L}} - \frac{\partial(\mathcal{P}u)}{\partial \boldsymbol{L}} = \mathcal{P}_1 u + \mathcal{P}_2 u \quad \text{for a.a. } (x,t) \in D_{\delta,T}.$$

Since $\partial u/\partial \boldsymbol{L} = w$ a.e. $D_{\delta,T}$, this becomes

$$\frac{\partial(\mathcal{P}u)}{\partial \boldsymbol{L}} = \mathcal{P}w - \mathcal{P}_1 u - \mathcal{P}_2 u \quad \text{for a.a. } (x,t) \in D_{\delta,T}.$$

At this point, we integrate the last identity along the $\boldsymbol{L}$-integral curve through $(x,t) \in D_{\delta,T}$, obtaining

$$\begin{aligned} \mathcal{P}u(x,t) &= \mathcal{P}u(x',t) \ + \int_0^{s(x,t)} \big(\mathcal{P}w - \mathcal{P}_1 u - \mathcal{P}_2 u\big)\big(\mathrm{e}^{\theta \boldsymbol{L}}(x',t)\big)\,\mathrm{d}\theta \\ &= \mathcal{P}u(x',t) \ + \int_0^{s(x,t)} \frac{\partial f}{\partial \boldsymbol{L}}\big(\mathrm{e}^{\theta \boldsymbol{L}}(x',t)\big)\,\mathrm{d}\theta \\ &\quad - \int_0^{s(x,t)} \big(\mathcal{P}_1 u + \mathcal{P}_2 u\big)\big(\mathrm{e}^{\theta \boldsymbol{L}}(x',t)\big)\,\mathrm{d}\theta \end{aligned} \tag{14}$$

by (8), where $(x',t) = \mathrm{e}^{-s(x,t)\boldsymbol{L}}(x,t) \in \Sigma_{\delta,T}$.

To compute $\mathcal{P}u(x',t)$, we apply the operator $\mathcal{P}$ to (11), obtaining for a.a. $(x,t) \in D_{\delta,T}$ that

$$\begin{aligned} \mathcal{P}u(x,t) = \mathcal{P}'v(x',t) \ &+ \overline{\mathcal{P}}_1 w(x,t) + \int_0^{s(x,t)} D_t w\big(\mathrm{e}^{\theta \boldsymbol{L}}(x',t)\big)\,\mathrm{d}\theta \\ &- a^{ij}(x,t) \int_0^{s(x,t)} \overline{\mathcal{P}}_2^{ij}\big(\mathrm{e}^{\theta \boldsymbol{L}}(x',t)\big)\,\mathrm{d}\theta, \end{aligned}$$

where $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2^{ij}$ are first- and second-order differential operators with L^∞-coefficients, acting *only* in the spatial variables x. In particular, $\overline{\mathcal{P}}_1 w \in V_p^1(D_{\delta,T})$ and, setting $\mathcal{T}\colon V_p^1(D_{\delta,T}) \mapsto L^p(\Sigma_{\delta,T})$ for the trace operator and noting that taking the trace on $\Sigma_{\delta,T}$ makes vanish $s(x,t)$, we obtain

$$\mathcal{P}u(x',t) = \mathcal{P}'v(x',t) + \big(\mathcal{T} \circ \overline{\mathcal{P}}_1 w\big)(x',t).$$

This way,

$$\begin{aligned} \mathcal{P}u(x',t) &= \tilde{f}(x',t) + \big(\mathcal{T} \circ \overline{\mathcal{P}}_1 w\big)(x',t) \\ &= f(x,t) - \int_0^{s(x,t)} \frac{\partial f}{\partial \boldsymbol{L}}\big(\mathrm{e}^{\theta \boldsymbol{L}}(x',t)\big)\,\mathrm{d}\theta + \big(\mathcal{T} \circ \overline{\mathcal{P}}_1 w\big)(x',t) \end{aligned}$$

by virtue of (10) and (9). Finally, setting

$$\begin{aligned} \mathcal{B}f(x,t) &= \ -\int_0^{s(x,t)} \big(\mathcal{P}_2 u\big)\big(\mathrm{e}^{\theta \boldsymbol{L}}(x',t)\big)\,\mathrm{d}\theta, \\ \mathcal{C}f(x,t) &= \big(\mathcal{T} \circ \overline{\mathcal{P}}_1 w\big)(x',t) - \int_0^{s(x,t)} \big(\mathcal{P}_1 u\big)\big(\mathrm{e}^{\theta \boldsymbol{L}}(x',t)\big)\,\mathrm{d}\theta, \end{aligned}$$

we obtain $\mathcal{P}u(x,t) = f(x,t) + \mathcal{B}f(x,t) + \mathcal{C}f(x,t)$ a.e. $D_{\delta,T}$, that is, (13).

In what follows, we restrict our considerations to the set D_{δ,δ^2}.

STEP 1. *The operator norm of* $\mathcal{B}\colon\ \mathcal{H}_p(D_{\delta,\delta^2}) \mapsto \mathcal{H}_p(D_{\delta,\delta^2})$ *is small for sufficiently small* δ.

To start with, note that for any $(x',t) \in \Sigma_{\delta,\delta^2}$, the integral curve of $\boldsymbol{L}$ passing through (x',t) reaches $\partial D_{\delta,\delta^2}$ at exactly two points

$$(x^{\pm},t) = \mathrm{e}^{s(x^{\pm},t)\boldsymbol{L}}(x',t) \ \text{ with } \ s(x^{\pm},t) \gtrless 0,\ 0 < s(x^{+},t) - s(x^{-},t) \le 2\delta.$$

It follows by Fubini's theorem that

$$\int_{\Sigma_{\delta,\delta^2}} \mathrm{d}x' \int_{s(x^{-},t)}^{s(x^{+},t)} g\big(\mathrm{e}^{\theta\boldsymbol{L}}(x',t)\big)\,\mathrm{d}\theta = \int_{D_{\delta,\delta^2}} g(x,t)\,\mathrm{d}x\,\mathrm{d}t \quad \forall g \in L^1(D_{\delta,\delta^2}).$$

Hence, using Jensen's integral inequality and the fact that $s(x,t) \le \delta$, we obtain

$$\left\| \int_0^{s(x,t)} g\big(\mathrm{e}^{\theta\boldsymbol{L}}(x',t)\big)\,\mathrm{d}\theta \right\|_{L^p(D_{\delta,\delta^2})} \le C\delta \|g\|_{L^p(D_{\delta,\delta^2})} \quad \forall g \in L^p(D_{\delta,\delta^2}). \tag{15}$$

In particular, assuming $\tilde{f}(x',t)$ in (9) to be extended from Σ_{δ,δ^2} as a constant along the $\boldsymbol{L}$-trajectory through $(x',t) \in \Sigma_{\delta,\delta^2}$, we have

$$\begin{aligned} \|\tilde{f}\|_{L^p(D_{\delta,\delta^2})} &= \Big\| \big(s(x^{+},t) - s(x^{-},t)\big)^{1/p} \tilde{f}(x',t) \Big\|_{L^p(\Sigma_{\delta,\delta^2})} \\ &\le C\Big(\|f\|_{L^p(D_{\delta,\delta^2})} + \delta \|\partial f/\partial \boldsymbol{L}\|_{L^p(D_{\delta,\delta^2})} \Big). \end{aligned} \tag{16}$$

Turning to the operator $\mathcal{B}$, we note that $\widetilde{\mathcal{B}f} \equiv 0$ and

$$\|\mathcal{B}f\|_{\mathcal{H}_p(D_{\delta,\delta^2})} = \delta^{1-1/p} \|\partial(\mathcal{B}f)/\partial\boldsymbol{L}\|_{L^p(D_{\delta,\delta^2})} = \delta^{1-1/p} \|\mathcal{P}_2 u\|_{L^p(D_{\delta,\delta^2})},$$

as follows from $s(x,t) \le \delta$, (15), and (16). On the other hand,

$$\begin{aligned} \|\mathcal{P}_2 u\|_{L^p(D_{\delta,\delta^2})} &\le C\Bigg(\|\mathcal{P}_2 v\|_{L^p(D_{\delta,\delta^2})} + \|w\|_{L^p(D_{\delta,\delta^2})} + \|Dw\|_{L^p(D_{\delta,\delta^2})} \\ &\qquad + \left\| \int_0^{s(x,t)} \mathcal{P}_2 w\big(\mathrm{e}^{\theta\boldsymbol{L}}(x',t)\big)\,\mathrm{d}\theta \right\|_{L^p(D_{\delta,\delta^2})} \Bigg) \\ &\le C\Big(\|\mathcal{P}_2 v\|_{L^p(\Sigma_{\delta,\delta^2})} + \|w\|_{L^p(D_{\delta,\delta^2})} + \|Dw\|_{L^p(D_{\delta,\delta^2})} \\ &\qquad + \delta \|\mathcal{P}_2 w\|_{L^p(D_{\delta,\delta^2})} \Big). \end{aligned}$$

Hence, using the *a priori* estimates for $v \in \Sigma_{\delta,\delta^2}$ (cf. (10)) and the fact that $\Sigma_{\delta,\delta^2} \in W^{2,1}_\infty$ uniformly in δ, we deduce

$$
\begin{aligned}
(17) \qquad \|\mathcal{B}f\|_{\mathcal{H}_p(D_{\delta,\delta^2})} &\le C\Big(\delta\|v\|_{W^{2,1}_p(\Sigma_{\delta,\delta^2})} + \delta^{1-1/p}\Big(\|w\|_{L^p(D_{\delta,\delta^2})} \\
&\qquad + \|Dw\|_{L^p(D_{\delta,\delta^2})} + \delta\|D^2w\|_{L^p(D_{\delta,\delta^2})}\Big)\Big) \\
&\le C\delta\Big(\|\tilde f\|_{L^p(\Sigma_{\delta,\delta^2})} + \delta^{-1/p}\Big(\|w\|_{L^p(D_{\delta,\delta^2})} \\
&\qquad + \|Dw\|_{L^p(D_{\delta,\delta^2})} + \delta\|D^2w\|_{L^p(D_{\delta,\delta^2})}\Big)\Big).
\end{aligned}
$$

In order to estimate the expression containing w above, we use the *a priori* estimates for the solution w of (8), which depend on δ because $\partial D_{\delta,\delta^2}$ is not uniformly $W^{2,1}_\infty$-smooth in δ. To overcome this, we make use of suitable scaling, considering

$$
\delta^{-1}D_{\delta,\delta^2} = \big\{(y,s) \in \mathbb{R}^n \times \mathbb{R}^+ \colon\ (\delta y, \delta^2 s) \in D_{\delta,\delta^2}\big\}.
$$

The definition of D_{δ,δ^2} ensures that $\partial(\delta^{-1}D_{\delta,\delta^2})$ is $W^{2,1}_\infty$-smooth independently of δ as it follows from the definition of D_{δ,δ^2}. Moreover, the function $W(y,s) = w(\delta y, \delta^2 s)$ is a $W^{2,1}_p(\delta^{-1}D_{\delta,\delta^2})$-strong solution to the Cauchy–Dirichlet problem

$$
(18) \qquad \begin{cases} D_s W - A^{ij}(y,s)D_{ij}W = \delta^2 F(y,s) & \text{a.e. in } \delta^{-1}D_{\delta,\delta^2}, \\ W = 0 \quad \text{on } \partial(\delta^{-1}D_{\delta,\delta^2}), \qquad W = 0 & \text{on } \overline{\delta^{-1}D_{\delta,\delta^2}} \cap \{s=0\} \end{cases}
$$

(cf. (8)), where $A^{ij}(y,s) = a^{ij}(\delta y, \delta^2 s)$ and $F(y,s) = \partial f/\partial \boldsymbol{L}(\delta y, \delta^2 s)$.

By the *a priori* estimates for solutions of (18), we have

$$
\|W\|_{W^{2,1}_p(\delta^{-1}D_{\delta,\delta^2})} \le C\|\delta^2 F\|_{L^p(\delta^{-1}D_{\delta,\delta^2})}.
$$

Passing back to the unscaled variables (x,t) yields

$$
\begin{aligned}
\|w\|_{L^p(D_{\delta,\delta^2})} &+ \delta\|Dw\|_{L^p(D_{\delta,\delta^2})} + \delta^2\|D^2w\|_{L^p(D_{\delta,\delta^2})} + \delta^2\|D_t w\|_{L^p(D_{\delta,\delta^2})} \\
&\le C\delta^2\|\partial f/\partial \boldsymbol{L}\|_{L^p(D_{\delta,\delta^2})},
\end{aligned}
$$

which transforms (17) into

$$
\begin{aligned}
\|\mathcal{B}f\|_{\mathcal{H}_p(D_{\delta,\delta^2})} &\le C\delta\Big(\|\tilde f\|_{L^p(\Sigma_{\delta,\delta^2})} + \delta^{-1/p}(\delta^2 + 2\delta)\|\partial f/\partial \boldsymbol{L}\|_{L^p(D_{\delta,\delta^2})}\Big) \\
&\le C\delta\|f\|_{\mathcal{H}_p(D_{\delta,\delta^2})}
\end{aligned}
$$

provided additionally that $\delta < 1$. Therefore

$$\|\mathcal{B}\|_{\mathcal{H}_p(D_{\delta,\delta^2})\mapsto\mathcal{H}_p(D_{\delta,\delta^2})} < 1$$

when $\delta < 1$ is small enough.

STEP 2. *Compactness of the operator* $\mathcal{C}$.

We first verify that $\mathcal{C}f \in \mathcal{H}_p(D_{\delta,\delta^2})$ whenever $f \in \mathcal{H}_p(D_{\delta,\delta^2})$. For, note that $\widetilde{\mathcal{C}f}(x,t) = (\mathcal{T}\circ\overline{\mathcal{P}}_1 w)(x',t)$ and, the continuity of the trace operator $\mathcal{T}$, together with the L^p-estimates for the solution of (8) yields

$$\|\widetilde{\mathcal{C}f}\|_{L^p(\Sigma_{\delta,\delta^2})} \le C\|\overline{\mathcal{P}}_1 w\|_{V_p^1(D_{\delta,\delta^2})} \le C\|w\|_{W_p^{2,1}(D_{\delta,\delta^2})} \le C\|\partial f/\partial \boldsymbol{L}\|_{L^p(D_{\delta,\delta^2})}.$$

Later on, $\partial(\mathcal{C}f)/\partial\boldsymbol{L}(x,t) = -\mathcal{P}_1 u(x,t)$ and, recalling (11), (15), (16), and that $s(x,t) \le \delta < 1$, we have

$$\begin{aligned}\left\|\frac{\partial(\mathcal{C}f)}{\partial \boldsymbol{L}}\right\|_{L^p(D_{\delta,\delta^2})} &\le C\Bigg(\|\mathcal{P}_1 v\|_{L^p(D_{\delta,\delta^2})} + \|w\|_{L^p(D_{\delta,\delta^2})} \\ &\qquad + \left\|\int_0^{s(x,t)} \mathcal{P}_1 w\big(\mathrm{e}^{\theta\boldsymbol{L}}(x',t)\big)\,\mathrm{d}\theta\right\|_{L^p(D_{\delta,\delta^2})}\Bigg) \\ &\le C\Big(\delta^{1/p}\|v\|_{V_p^1(D_{\delta,\delta^2})} + \|w\|_{V_p^1(D_{\delta,\delta^2})}\Big) \\ &\le C\Big(\delta^{1/p}\|\tilde{f}\|_{L^p(\Sigma_{\delta,\delta^2})} + \|\partial f/\partial\boldsymbol{L}\|_{L^p(D_{\delta,\delta^2})}\Big).\end{aligned}$$

Therefore,

$$\|\mathcal{C}f\|_{\mathcal{H}_p(D_{\delta,\delta^2})} = \|\widetilde{\mathcal{C}f}\|_{L^p(\Sigma_{\delta,\delta^2})} + \delta^{1-1/p}\left\|\frac{\partial(\mathcal{C}f)}{\partial\boldsymbol{L}}\right\|_{L^p(D_{\delta,\delta^2})} \le C\delta^{-1+1/p}\|f\|_{\mathcal{H}_p(D_{\delta,\delta^2})}$$

which shows that $\mathcal{C}f \in \mathcal{H}_p(D_{\delta,\delta^2})$ for any fixed $\delta \in (0,1)$.

To prove the compactness of $\mathcal{C}$ we pick a bounded in $\mathcal{H}_p(D_{\delta,\delta^2})$ sequence $\{f_m\}$ and set w_m and v_m for the solutions to the corresponding problems (8) and (10). Since w_m and v_m are bounded in $W_p^{2,1}(D_{\delta,\delta^2})$ and $W_p^{2,1}(\Sigma_{\delta,\delta^2})$, respectively, and since $W_p^{2,1}(D_{\delta,\delta^2})$ is compactly embedded into $V_p^1(D_{\delta,\delta^2})$, convergent in $V_p^1(D_{\delta,\delta^2})$ subsequences w_m' and v_m' do exist, whence the above estimates and the compactness of the trace operator (4) imply

$$\begin{aligned}\|\mathcal{C}f_m' - \mathcal{C}f_k'\|_{\mathcal{H}_p(D_{\delta,\delta^2})} \le C\Big(&\|(\mathcal{T}\circ\overline{\mathcal{P}}_1)w_m' - (\mathcal{T}\circ\overline{\mathcal{P}}_1)w_k'\|_{L^p(\Sigma_{\delta,\delta^2})} \\ &+ \|v_m' - v_k'\|_{V_p^1(\Sigma_{\delta,\delta^2})} + \|w_m' - w_k'\|_{V_p^1(D_{\delta,\delta^2})}\Big).\end{aligned}$$

Since the right-hand side is infinitesimal as $m, k \to \infty$, the sequence $\{\mathcal{C} f'_m\}$ is fundamental in $\mathcal{H}_p(D_{\delta,\delta^2})$ and therefore convergent there.

Turning back to (13), since the norm of $\mathcal{B}$ is small when δ is sufficiently small, the operator $\mathrm{Id}_{\mathcal{H}_p(D_{\delta,\delta^2})} + \mathcal{B}$ is invertible, and the compactness of $\mathcal{C}$ implies that $\mathcal{A}$ is a Fredholm operator. Therefore, the invertibility of $\mathcal{A}$ is equivalent to the triviality of $\operatorname{Ker}\mathcal{A}$ which is immediate. Indeed, assume that $f \in \operatorname{Ker}\mathcal{A}$ with $f \not\equiv 0$. Then $\mathcal{P}u = 0$ a.e. in D_{δ,δ^2} which implies $u \equiv 0$ there by the uniqueness for (12). Consequently, $w = \partial u/\partial \boldsymbol{L} \equiv 0$ in D_{δ,δ^2}, and $v = u|_{\Sigma_{\delta,\delta^2}} \equiv 0$. Hence,

$$\begin{cases} \partial f/\partial \boldsymbol{L} = \mathcal{P}w = 0 & \text{a.e. in } D_{\delta,\delta^2} \\ f|_{\Sigma_{\delta,\delta^2}} = \mathcal{P}'v = 0 & \text{in } \Sigma_{\delta,\delta^2} \end{cases}$$

which gives the desired contradiction $f \equiv 0$ by virtue of (9).

Once having invertibility of $\mathcal{A}\colon\ \mathcal{H}_p(D_{\delta,\delta^2}) \mapsto \mathcal{H}_p(D_{\delta,\delta^2})$, we note that the unique solution of (5) in D_{δ,δ^2} is defined by (11), where v and w are the solutions of (10) and (8), respectively, with right-hand sides defined by $\mathcal{A}^{-1}f$.

The procedure described above can be iterated finitely many times on the portions

$$D_{\delta,T} \cap \{k\delta^2 \le t \le (k+1)\delta^2\}, \qquad k \ge 1,$$

in order to get solvability in the whole $D_{\delta,T}$ and this completes the proof. □

4 Concluding Remarks

1. Since the solution of (5) is given by (11) and the Cahchy–Dirichlet problems (8) and (10) support the *improving-of-integrability* property, we obtain the following

Corollary 2: *Under the hypotheses of* Theorem 1, *let* $u \in W_q^{2,1}(D_{\delta,T})$ *be a solution of (5) with* $q \in (1,p)$. *Then* $u \in W_p^{2,1}(D_{\delta,T})$ *and the estimate (6) holds.*

2. A careful look at the proof of Theorem 1 shows that the requirement (cf. (2)) $\boldsymbol{\ell}$ to be tangent to S *only at the points of* S_0 is *not essential* for the approach adopted. The crucial assumptions, besides $\operatorname{codim}_S S_0 = 1$, are that $\boldsymbol{\ell}$ is *strictly transversal* to S_0 and points from S^- to S^+ there. Therefore, the conclusions of Theorem 1 and Corollary 2 remain valid if $\boldsymbol{\ell}$ is tangent to S on a larger set $S' \supset S_0$, provided that every point of S' can be reached from S_0 along an $\boldsymbol{\ell}$-trajectory of sufficiently small length. In this case, the set K becomes a closed neighbourhood of S', while the choice of the codimension-one submanifold $S_0 \subset S'$ remains flexible, subject only to the assumptions in (2).

3. The additional integrability assumptions on $D_x a^{ij}$ (cf. (3), $D_x f$, and $D_x \varphi$ may be replaced by the integrability in K of the corresponding directional derivatives with respect to the extended vector field $\boldsymbol{L}$.

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REFERENCES

[1] H. POINCARÉ (1910) "Lecons de Méchanique Céleste, Tome III, Théorie de Marées". Gauthiers–Villars, Paris.

[2] P.R. POPIVANOV (2004) On the tangential oblique derivative problem – methods, results, open problems. In: J. Gil et al. (eds) "Aspects of Boundary Problems in Analysis and Geometry". *Operator Theory: Advances and Applications* vol. **151** Birkhäuser, Basel, pp. 430-471.

[3] P.R. POPIVANOV, D.K. PALAGACHEV (1997) "The Degenerate Oblique Derivative Problem for Elliptic and Parabolic Equations". Akademie–Verlag, Berlin.

[4] B.P. PANEAH (2000) "The Oblique Derivative Problem. The Poincaré Problem". Wiley–VCH, Berlin.

[5] V.G. MAZ'YA, B.P. PANEAH (1974) Degenerate elliptic pseudodifferential operators and oblique derivative problem. *Transactions of the Moscow Mathematical Society* **31** 247-305.

[6] Y.V. EGOROV, V.A. KONDRAT'EV (1969) The oblique derivative problem. *Mathematics of the USSR-Sbornik* **7** 139-169.

[7] P. GUAN, E. SAWYER (1993) Regularity estimates for the oblique derivative problem. *Annals of Mathematics* **137**(1) 1-70.

[8] P. POPIVANOV, N. KUTEV (2005) Viscosity solutions to the degenerate oblique derivative problem for fully nonlinear elliptic equations. *Mathematische Nachrichten* **278** 888-903.

[9] P. POPIVANOV (2019) Boundary value problems for the biharmonic operator in the unit ball. In: A. Slavova (ed) "Proceedings of the 6th International Conference NTADES 2019, Varna". *AIP Conference Proceedings* **2159** 030028.

[10] P. POPIVANOV, A. SLAVOVA (2023) Boundary value problems for the polyharmonic operators. In: A. Slavova (ed) "Proceedings of the 9th International Conference NTADES 2022, Sozopol". *Springer Proceedings in Mathematics & Statistics* **412**, Springer, Cham, pp. 35-50.

[11] L.G. SOFTOVA (2004) $W_p^{2,1}$-solvability for the parabolic Poincaré problem. *Communications in Partial Differential Equations* **29** 1783-1789.

[12] D.K. PALAGACHEV (2006) Neutral Poincaré problem in L^p-Sobolev spaces: Regularity and Fredholmness. *International Mathematics Research Notices* **2006** Art. ID 87540.

[13] D.K. PALAGACHEV (2005) The Poincaré problem in L^p-Sobolev spaces I: Codimension one degeneracy. *Journal of Functional Analysis* **229** 121-142.

[14] D.K. PALAGACHEV (2008) The Poincaré problem in L^p-Sobolev spaces II: Full dimension degeneracy. *Communications in Partial Differential Equations* **33** 209-234.

[15] D.K. PALAGACHEV, L.G. SOFTOVA, S. TRAMONTANO (2026) Global strong solvability of the parabolic Poincaré problem with emergent type vector field. preprint.

[16] O.A. LADYZHENSKAYA, V.A. SOLONNIKOV, N.N. URAL'TSEVA (1968) "Linear and Quasilinear Equations of Parabolic Type". American Mathematical Society, Providence, RI.

[17] G.M. LIEBERMAN (1996) "Second Order Parabolic Differential Equations". World Scientific Publishing Co., Inc., River Edge, NJ.

[18] H. DONG, D. KIM (2015) Boundary value problems for parabolic operators in a time-varying domain. *Communications in Partial Differential Equations* **40** 1282-1313.