

ANALYTICAL CONSTRUCTION OF THE SCHRÖDINGER–NEWTON GROUND STATE VIA THE HOMOTOPY ANALYSIS METHOD WITH VIRIAL CALIBRATION

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ABSTRACT: We present an analytical approximation of the spherically symmetric ground state of the Schrödinger–Newton system using the Homotopy Analysis Method. The construction rests on an inner expansion in an exponential basis, regular at the origin and with the correct exponential decay. We recover the non-integer Coulomb prefactor of the far-field decay, which lies outside the basis, by an envelope factorisation of the radial equation. We compute the first-order correction, fix the physical amplitude at each truncation by the virial identity, select the convergence-control parameter by minimizing the residual of the amplitude-corrected field, and assemble a uniformly valid composite approximation. The corrections are elementary and exact through first order; second order and higher are obtained by spectral continuation.

KEY WORDS: convergence-control parameter, ground state, Homotopy Analysis Method, Schrödinger–Newton system, virial identity.

1 INTRODUCTION

The Schrödinger–Newton (S–N) system, also known as the Newton–Schrödinger equation, models a quantum particle whose wave function is self-localized by its own gravitational field. The idea originates in the early proposals of Diósi [1, 2] and Penrose [3], who suggested that gravity may induce an objective reduction of the quantum state. The first systematic mathematical study of the spherically symmetric ground state was carried out by Moroz, Penrose, and Tod [4–6], who obtained high-precision numerical profiles of the bound state within the Tod–Moroz formulation; existence and uniqueness of the positive radial ground state are due to Lieb and Lions [7, 8].

In spherical symmetry the S–N system reduces to the dimensionless Tod–Moroz one (3)–(4) in Section 2. It exhibits a delicate balance between quantum dispersion and self-gravity, producing a unique positive ground state with exponential decay at

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infinity. Closely related Schrödinger–Poisson and Choquard models share similar structural features and have been extensively analysed in the mathematical literature; see, for example, [7–12].

Traditional numerical approaches to the S–N ground state rely on shooting methods [4, 5], which require fine tuning of initial slopes and do not yield closed-form expressions. Analytical progress has been made through asymptotic expansions [5, 13], variational and Hartree methods [7, 8, 14], and perturbative techniques, but a globally valid analytic approximation remains challenging. The Homotopy Analysis Method (HAM), introduced by Liao [15–17], provides a flexible framework for constructing analytic series solutions to nonlinear differential equations. Unlike perturbation theory, HAM does not rely on small parameters and introduces a convergence-control parameter h that can be tuned to ensure convergence.

In this work, we apply HAM to the Schrödinger–Newton system using a global exponential basis $\{r^m e^{-r}\}_{m \geq 0}$, which captures both the regular even-Taylor structure at the origin and the correct exponential decay rate. Because this basis cannot reproduce the non-integer algebraic prefactor of the Coulomb-corrected tail (Section 2), we complement the inner homotopy construction with a separate far-field layer, obtained by envelope factorisation of the radial equation and matched to the inner series through their common exponential envelope. The result is a uniformly valid approximation across all spatial scales.

A further key feature of our approach is the incorporation of the structural invariants of the spherically symmetric system—mass, energy, and a virial identity of Nehari–Pohozaev type, standard in the analysis of Schrödinger–Newton and related Schrödinger–Poisson / Hartree models [6, 9, 10, 14]—directly into the HAM iteration. At each truncation order the physical amplitude of the approximation is fixed by the virial identity $4K + W = 0$, while the convergence-control parameter h is selected from the residual of the amplitude-corrected field. This ensures that the inner analytic approximations remain physically consistent and that the far-field amplitude is recovered from the Coulomb tail of the computed potential, producing a convergent, physically faithful, and globally accurate approximation to the Penrose ground state.

2 THE SCHRÖDINGER–NEWTON SYSTEM

The Schrödinger–Newton (S–N) system describes a quantum particle whose mass density $|\psi(\mathbf{x})|^2$ generates a classical Newtonian gravitational potential, and whose wave function evolves under the influence of this self-generated field. In three spatial dimensions, the stationary S–N system is given by

$$(1) \quad -\nabla^2 \psi(\mathbf{x}) + V(\mathbf{x})\psi(\mathbf{x}) = E \psi(\mathbf{x}),$$

$$(2) \quad \nabla^2 V(\mathbf{x}) = 4\pi |\psi(\mathbf{x})|^2,$$

where E is the eigenvalue (or chemical potential). Equation (1) is the stationary Schrödinger equation with gravitational potential V , and (2) is the Poisson equation for Newtonian gravity. The system is nonlinear and nonlocal, and its ground state represents a self-bound quantum object held together by its own gravitational field. It is closely related to the Schrödinger–Poisson and Choquard equations studied in [7–10].

The ground state is known to be spherically symmetric, positive, and monotonically decreasing. In the radial variable $r = |\mathbf{x}|$, a standard nondimensionalization of Tod and Moroz [4, 5] together with the scaling covariance established in [13] reduces (1)–(2) to a parameter-free form. Length is rescaled to fix the energy at $E = -1$, the field normalization absorbs the factor 4π , and the dimensionless potential V is defined with the opposite sign to the physical Newtonian potential of (2), so that the attractive self-gravitational well corresponds to $V > 0$. This yields the dimensionless radial Schrödinger equation

$$(3) \quad y''(r) + \frac{2}{r}y'(r) - (1 - V(r))y(r) = 0,$$

while the Poisson equation (2) becomes

$$(4) \quad V''(r) + \frac{2}{r}V'(r) + y^2(r) = 0.$$

Equations (3)–(4) form the dimensionless Tod–Moroz system used throughout this work, subject to the regularity conditions $y'(0) = V'(0) = 0$ and the decay conditions $\lim_{r \rightarrow \infty} y(r) = \lim_{r \rightarrow \infty} V(r) = 0$.

The structure of the ground state is governed by two complementary asymptotic regimes established in [13]. Near the origin, y and V admit convergent even-power Taylor expansions

$$(5) \quad y(r) = a_0 + a_2 r^2 + a_4 r^4 + \dots, \quad V(r) = b_0 + b_2 r^2 + b_4 r^4 + \dots,$$

with $a_0 = y(0) > 0$ and $b_0 = V(0) > 1$, and all higher coefficients determined recursively by (a_0, b_0) . These expansions encode the analyticity of the fields at $r = 0$.

At large r , the Poisson equation forces the potential to approach the Coulomb form

$$(6) \quad V(r) = \frac{\widehat{M}}{r} + O(e^{-2r} r^{\widehat{M}-2}), \quad r \rightarrow \infty,$$

with no algebraic corrections at any order, where $\widehat{M} = \int_0^\infty r^2 y^2 dr$ is the dimensionless mass. The reduction of the radial Schrödinger equation to a Whittaker equation

with an exponentially small perturbation [13] then yields the Coulomb-corrected decay

$$(7) \quad y(r) = C e^{-r} r^{\widehat{M}/2-1} (1 + O(r^{-1})), \quad r \rightarrow \infty,$$

with an undetermined amplitude $C > 0$. The leading exponential rate e^{-r} is universal, while the algebraic prefactor $r^{\widehat{M}/2-1}$ is a structural feature of the gravitational envelope equation.

These two behaviours, the even-power Taylor series at the origin and the non-integer Coulomb-corrected tail at infinity, are coequal structural features of the ground state and cannot be captured within a single analytic basis. The exponential basis $\{r^m e^{-r}\}_{m \geq 0}$ reproduces the regular origin structure (5) to any finite order and carries the correct exponential decay rate, but it cannot generate the non-integer prefactor in (7). Conversely, the far-field expansion obtained by factoring out the decaying envelope e^{-r} captures the Coulomb-corrected tail but is not analytic at the origin. A uniformly valid approximation therefore requires two representations: an inner analytic expansion in $\{r^m e^{-r}\}_{m \geq 0}$ and a separate outer expansion in the basis $\{e^{-r} r^{\widehat{M}/2-1-n}\}_{n \geq 0}$, matched asymptotically through their common exponential envelope.

These structural assumptions fix the order of construction: the inner layer is built first, and the outer layer is anchored to it rather than imposed independently.

3 HOMOTOPY CONSTRUCTION OF THE INNER APPROXIMATION

This section develops the analytic construction of the inner field: the homotopy framework, the choice of zeroth-order approximation, the auxiliary operators and the deformation hierarchy they generate, the closed-form first-order correction, and the point at which the symbolic hierarchy leaves the elementary class.

3.1 THE HOMOTOPY FRAMEWORK

The Homotopy Analysis Method (HAM) solves nonlinear differential equations without relying on perturbative small parameters. A nonlinear problem is embedded in a one-parameter family of deformations, indexed by an embedding parameter $q \in [0, 1]$, that continuously transforms a simple, exactly solvable initial guess at $q = 0$ into the exact solution at $q = 1$. The deformation is governed by an auxiliary linear operator L , chosen so that L^{-1} preserves the analytic structure of the approximation, and by a convergence-control parameter h that regulates the path. Expanding in powers of q and equating coefficients of like powers yields a hierarchy of linear *deformation equations* for the successive corrections, each solved within the chosen

analytic basis [15–17]. The result is not a perturbation expansion but a genuine analytic representation, the convergence of the series being regulated by h rather than by a small parameter. For the Schrödinger–Newton system the framework is applied to the pair (y, V) , with auxiliary operators chosen in Section 3.3 so as to preserve the inner basis $\{r^m e^{-r}\}_{m \geq 0}$ and with h fixed at each truncation order by the calibration of Section 4 rather than chosen arbitrarily.

3.2 ZERO-ORDER FIELDS AND THE PROVISIONAL FAR FIELD

The homotopy construction begins with a choice of zeroth-order fields (y_0, V_0) that already satisfy the structural requirements established in Section 2: analyticity and even-power behaviour at the origin, exponential decay at infinity, and finite mass, computed exactly. To remain within the analytic inner basis $\{r^m e^{-r}\}_{m \geq 0}$ while capturing the leading Taylor structure at $r = 0$ and the universal decay rate e^{-r} , we adopt the global ansatz

$$(8) \quad y_0(r) = \alpha (1 + r) e^{-r}.$$

Although this expression matches the first two terms of the even-power Taylor series at the origin, it is not used merely as a local expansion: it is treated as a global approximate profile on $[0, \infty)$, compatible with the HAM basis. Because y_0 is defined globally and decays exponentially, its mass is finite and elementary,

$$\widehat{M}_0 = \int_0^\infty r^2 y_0(r)^2 dr = \alpha^2 \int_0^\infty r^2 (1 + r)^2 e^{-2r} dr = \alpha^2 \frac{7}{4}.$$

The amplitude α of the seed profile (8) is fixed not by an arbitrary normalization but by the virial identity $4K + W = 0$ (Section 4), which for this profile gives $\alpha^2 = 4K/|W| \approx 2.29$, that is $\alpha \approx 1.51$. The dimensionless mass of the true ground state is a determined constant (in general $\widehat{M} \neq 1$), and this choice yields $\widehat{M}_0 \approx 4$ for the seed. The function (8) is analytic at $r = 0$, satisfies $y'_0(0) = 0$, and decays exponentially as $r \rightarrow \infty$, thus respecting the inner structural constraints of the S–N ground state.

The corresponding zeroth-order potential V_0 is obtained by solving the Poisson equation (4) with source y_0^2 . A direct integration yields the explicit expression

$$(9) \quad V_0(r) = \alpha^2 \left[\frac{7}{4r} (1 - e^{-2r}) - \frac{19}{8} e^{-2r} - \frac{5}{4} r e^{-2r} - \frac{1}{4} r^2 e^{-2r} \right],$$

which is regular at the origin, satisfies $V'_0(0) = 0$, and decays exponentially. At the virial amplitude the central value is $V_0(0) = \frac{9}{8} \alpha^2 \approx 2.57 > 1$, so the seed already reproduces the required well depth $b_0 > 1$ of the self-gravitational potential. The pair

(y_0, V_0) therefore provides a fully analytic, structurally consistent inner approximation.

Although the homotopy series will refine the inner fields order by order, the zeroth-order approximation already determines a provisional far-field representation. The central amplitude is $a_0^{(0)} = y_0(0) = \alpha \approx 1.51$, and the far-field amplitude C in the Coulomb-corrected decay (7) is a determined constant of the unique ground state—a global *connection coefficient*, fixed by the requirement that a single field be both regular at the origin and decaying at infinity, and therefore not obtainable from a local relation to the central amplitude a_0 . It can, however, be recovered from the computed fields. Substituting the tail (7) into the Poisson equation (4) gives, for the short-range correction to the Coulomb part of the potential, the exact asymptotic identity

$$(10) \quad V(r) - \frac{\widehat{M}}{r} \sim -\frac{C^2}{4} e^{-2r} r^{\widehat{M}-2}, \quad r \rightarrow \infty,$$

so that C governs the leading exponential correction to the Coulomb tail of V . Identity (10) is exact as an asymptotic statement, but it is not a practical estimator: the integer-power tail carried by the inner basis turns its right-hand side into a pure power of r , stationary only in the degenerate case $\widehat{M} = 4$, so that for $\widehat{M} \neq 4$ no plateau exists and the fitted value depends on the window.

The amplitude is therefore extracted not from (10) but by matching the computed field to the exact decaying solution of the reduced envelope equation (13) derived below. In the variables $u = r y$ and $t = 2r$ its leading-order form is a Whittaker equation with parameters $\kappa = \widehat{M}/2$ and $\nu = 1/2$ [13], whose recessive solution is $W_{\widehat{M}/2, 1/2}(2r)$; since $W_{\kappa, \nu}(t) \sim e^{-t/2} t^\kappa$, the normalization consistent with (7) is

$$(11) \quad y_{\text{out}}(r) = C 2^{-\widehat{M}/2} \frac{W_{\widehat{M}/2, 1/2}(2r)}{r},$$

so that the amplitude is recovered at a matching radius r_m as

$$(12) \quad C = \frac{2^{\widehat{M}/2} r_m \widehat{y}(r_m)}{W_{\widehat{M}/2, 1/2}(2r_m)}.$$

By the asymptotic integration lemma of [13], the exponentially small remainder in (6) perturbs the recessive Whittaker branch only at relative order $O(e^{-(2-\varepsilon)r})$, so the right-hand side of (12) is stationary in r_m up to exponentially small terms: here the plateau is genuine, and it is already flat to several digits for $r_m \gtrsim 4$. Applied to the seed, whose tail is $y_0 \sim \alpha r e^{-r}$ with $\widehat{M}_0 \approx 4$, (12) returns $C^{(0)} = \alpha$, the amplitude carried by the seed's own tail.

The far-field representation itself follows from the envelope reduction of [13]. Writing $u(r) = r y(r)$ reduces the radial Schrödinger equation to $u'' - (1 - V)u = 0$, and the Coulomb tail (6) gives the reduced equation

$$(13) \quad u''(r) - \left(1 - \frac{\widehat{M}}{r}\right)u(r) = O(e^{-2r} r^{\widehat{M}-2}) u(r),$$

whose dominant balance is $u \sim e^{-r} r^{\widehat{M}/2}$. Factoring this envelope exactly, $u(r) = e^{-r} r^{\widehat{M}/2} w(r)$ with $w \rightarrow \text{const}$, yields the outer equation

$$(14) \quad w''(r) + \left(\frac{\widehat{M}}{r} - 2\right)w'(r) + \frac{\widehat{M}(\widehat{M} - 2)}{4r^2}w(r) = 0,$$

whose asymptotic solution is $w(r) \sim C \sum_{n \geq 0} c_n r^{-n}$, with $c_0 = 1$ and $c_1 = \widehat{M}(2 - \widehat{M})/8$. The far-field representation is therefore

$$(15) \quad y_{\text{out}}(r) = C e^{-r} r^{\widehat{M}/2-1} \sum_{n \geq 0} c_n r^{-n}.$$

The associated potential is given by (6), with no algebraic correction beyond the Coulomb term.

Combining the inner field y_0 with the outer representation (15) already yields a provisional composite approximation

$$(16) \quad y_{\text{unif}}^{(0)}(r) = y_0(r) + y_{\text{out}}(r) - y_{\text{ov}}(r),$$

in which y_{ov} is the common part of the two layers—the shared e^{-r} envelope through which they are matched. For the seed the inner field decays as $y_0(r) \sim \alpha r e^{-r}$, while the leading term of the outer field (15) is $C^{(0)} e^{-r} r^{\widehat{M}_0/2-1}$. At $\widehat{M}_0 = 4$ and $C^{(0)} = \alpha$ the two share the identical leading envelope

$$(17) \quad y_{\text{ov}}(r) = \alpha r e^{-r}.$$

The amplitude $C^{(0)}$ is provisional and is refined, together with the dimensionless mass $\widehat{M}$, in Section 5, where the composite is constructed at truncation order M .

3.3 AUXILIARY OPERATORS AND THE DEFORMATION HIERARCHY

The zeroth-order fields are refined into a globally accurate analytic solution by a continuous deformation from (y_0, V_0) to the exact solution (y, V) , governed by the embedding parameter $q \in [0, 1]$ and the convergence-control parameter h . The evolving fields are represented as

$$y(r; q) = y_0(r) + \sum_{m=1}^{\infty} q^m y_m(r), \quad V(r; q) = V_0(r) + \sum_{m=1}^{\infty} q^m V_m(r),$$

so that $y(r; 0) = y_0(r)$, $V(r; 0) = V_0(r)$ and $y(r; 1) = y(r)$, $V(r; 1) = V(r)$. The deformation is driven by the nonlinear operators N_y and N_V , the left-hand sides of (3)–(4), which vanish exactly for the true ground state. To preserve the analytic structure of the inner basis $\{r^m e^{-r}\}_{m \geq 0}$ and to ensure solvability of the linearized problems at each order, we introduce auxiliary linear operators

$$L_y = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - 1, \quad L_V = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr},$$

whose homogeneous solutions are linear combinations of $r^m e^{-r}$ and r^m , respectively, and whose kernels therefore encode the regularity at the origin and the exponential decay at infinity required by the inner representation.

The deformation from (y_0, V_0) to (y, V) is defined by the zeroth-order homotopy equations

$$(18) \quad (1 - q)L_y[y(r; q) - y_0(r)] = qh N_y[y(r; q), V(r; q)],$$

$$(19) \quad (1 - q)L_V[V(r; q) - V_0(r)] = qh N_V[y(r; q), V(r; q)],$$

subject to the regularity conditions $y'(0; q) = V'(0; q) = 0$ and the decay conditions $y(\infty; q) = V(\infty; q) = 0$. At $q = 0$ they return the seed (y_0, V_0) ; at $q = 1$ the left-hand sides vanish and the right-hand sides enforce the original Schrödinger–Newton system, so that the homotopy provides a continuous, controlled deformation from the analytic seed to the exact solution.

Expanding (18)–(19) in powers of q and equating coefficients yields the recursive linear problems [15]

$$(20) \quad \begin{aligned} L_y[y_m(r) - \chi_m y_{m-1}(r)] &= h \mathcal{R}_{y, m-1}(r), \\ L_V[V_m(r) - \chi_m V_{m-1}(r)] &= h \mathcal{R}_{V, m-1}(r), \end{aligned}$$

where $\chi_m = 0$ for $m = 1$ and $\chi_m = 1$ for $m \geq 2$, and the residuals $\mathcal{R}_{y, m-1}$ and $\mathcal{R}_{V, m-1}$ depend only on lower-order terms. Because the auxiliary operators preserve the exponential basis, each correction y_m and V_m expands naturally in $\{r^m e^{-r}\}$, guaranteeing the correct analytic behaviour at both ends of the domain, and remaining compatible with the structural assumptions established in Section 2 and with the inner-outer composite construction developed in Section 5.

3.4 FIRST-ORDER CORRECTIONS

The first-order corrections (y_1, V_1) arise from the coefficients of q^1 in the homotopy equations and represent the leading refinement of the zeroth-order fields. Here, $\chi_1 = 0$ and the recursion (20) gives the linear problems

$$L_y[y_1(r)] = h \mathcal{R}_{y, 0}(r), \quad L_V[V_1(r)] = h \mathcal{R}_{V, 0}(r).$$

Using the explicit zeroth-order fields (8) and (9), the wave-function residual is (21)

$$\mathcal{R}_{y,0}(r) = N_y[y_0, V_0] = \left[\frac{7\alpha^3}{4r}(1+r) - 4\alpha \right] e^{-r} - \alpha^3 \left[\frac{7}{4r} + \frac{33}{8} + \frac{29}{8}r + \frac{3}{2}r^2 + \frac{1}{4}r^3 \right] e^{-3r},$$

which contains both e^{-r} and e^{-3r} components together with explicit $1/r$ factors. In contrast, the potential residual vanishes identically,

$$\mathcal{R}_{V,0}(r) = N_V[y_0, V_0] = V_0''(r) + \frac{2}{r}V_0'(r) + y_0(r)^2 \equiv 0,$$

because V_0 is the exact Newtonian potential generated by y_0^2 . The potential correction therefore solves $L_V[V_1] = 0$ subject to $V_1'(0) = 0$ and $V_1(r) \rightarrow 0$, whose homogeneous solution $A + B/r$ is forced to vanish by regularity and decay:

$$(22) \quad V_1(r) \equiv 0.$$

Because L_y preserves the exponential basis and its homogeneous solution is proportional to e^{-r}/r , the correction y_1 must be expanded in a basis closed under L_y and compatible with the structure of (21). This leads to the ansatz

$$(23) \quad y_1(r) = h\alpha(1+r)e^{-r} + h\alpha^3 \left[\left(\frac{1389}{1024r} - \frac{7r}{16} - \frac{21}{16} \right) e^{-r} - \left(\frac{1389}{1024r} + \frac{r^3}{32} + \frac{9r^2}{32} + \frac{133r}{128} + \frac{477}{256} \right) e^{-3r} \right],$$

where α is the virial-fixed amplitude of (8). The correction is finite at $r = 0$, decays exponentially as $r \rightarrow \infty$, and remains entirely within the global basis $\{r^m e^{-r}\}$.

The first-order HAM approximation is therefore

$$y^{(1)}(r; h) = y_0(r) + y_1(r; h), \quad V^{(1)}(r; h) = V_0(r),$$

the amplitude of the correction being controlled by h , fixed in Section 4.

3.5 SECOND ORDER AND THE LOSS OF THE ELEMENTARY CLASS

At second order the recursion (20) gives

$$L_y[y_2(r) - y_1(r)] = h \mathcal{R}_{y,1}(r), \quad L_V[V_2(r) - V_1(r)] = h \mathcal{R}_{V,1}(r),$$

with second-order residuals

$$\mathcal{R}_{y,1} = y_1'' + \frac{2}{r}y_1' - (1 - V_0)y_1 + V_1y_0, \quad \mathcal{R}_{V,1} = V_1'' + \frac{2}{r}V_1' + 2y_0y_1,$$

which encode the nonlinear interactions between the zeroth- and first-order fields. Since $V_1 \equiv 0$ by (22), the potential residual reduces to $\mathcal{R}_{V,1} = 2y_0y_1$.

At this order the corrections leave the elementary class. The Coulomb part of the background potential, $V_0 \sim \frac{7}{4}\alpha^2/r$ at large r , multiplied by the r^{-1} part of the first-order correction y_1 , produces in $\mathcal{R}_{y,1}$ a term proportional to e^{-r}/r . This function lies in the kernel of the auxiliary operator, $L_y[e^{-r}/r] = 0$, so it is resonant: inverting L_y against it generates an exponential integral, and y_2 cannot be represented in the elementary basis $\{r^m e^{-r}\}$.

The explicit closed-form construction therefore terminates at first order, where y_0 , V_0 , and y_1 are exact and elementary; at higher orders the corrections produce polylogarithms and nonlocal convolution terms. The second-order wave-function correction y_2 , together with all higher orders, is computed numerically by the spectral homotopy scheme of Section 6.2, which represents the corrections on a Chebyshev grid and is not restricted to the elementary basis. The second-order approximation

$$y^{(2)}(r; h) = y_0(r) + y_1(r; h) + y_2(r; h), \quad V^{(2)}(r; h) = V_0(r) + V_2(r; h)$$

is understood in this sense, with y_2 obtained numerically and V_2 following by direct integration of its Poisson equation.

4 CONVERGENCE ANALYSIS OF THE HAM EXPANSION

The homotopy construction produces a sequence of truncated inner approximations $y^{(M)} = \sum_{m=0}^M y_m$ and $V^{(M)} = \sum_{m=0}^M V_m$, whose convergence as $M \rightarrow \infty$ must be assessed both analytically and numerically, following the general framework of Liao [15, 16].

A natural global diagnostic is the nonlinear residual of the truncated fields, measured by $N_y[y^{(M)}, V^{(M)}]$ and $N_V[y^{(M)}, V^{(M)}]$, which vanish identically only for the exact solution. To quantify the global error we introduce the residual functional

$$(24) \quad \mathcal{R}^{(M)}(h) = \int_0^{R_*} \left[(N_y[y^{(M)}, V^{(M)}])^2 + (N_V[y^{(M)}, V^{(M)}])^2 \right] r^2 dr,$$

where R_* is chosen large enough that the truncated fields have effectively decayed. The residual alone, however, does not determine the physical approximation: it is minimized as the amplitude of the truncated fields tends to zero, since the trivial solution $y \equiv 0$ has vanishing residual. The homotopy corrections fix the *shape* of the ground state. Its amplitude is set separately, by the virial identity below.

A complementary measure is order-by-order stability: the interval of h on which successive truncations differ negligibly coincides with the interval where the amplitude-corrected residual is minimized, and the diagnostic h -curves exhibit the corresponding plateau [15]. The global exponential basis $\{r^m e^{-r}\}_{m \geq 0}$ is essential here, as it

allows the residual functional to be evaluated on the entire physical domain, in contrast to local Taylor expansions whose radius of convergence is finite.

The virial identity fixes the physical amplitude of the approximation. For a given h , the truncated shape $y^{(M)}(r; h)$ is rescaled to $\hat{y}^{(M)} = \sigma y^{(M)}$, with the scale σ determined by

$$(25) \quad 4K[\hat{y}^{(M)}] + W[\hat{y}^{(M)}] = 0 \quad \Longleftrightarrow \quad \sigma^2 = \frac{4K[y^{(M)}]}{|W[y^{(M)}]|},$$

where $K[y] = \int_0^\infty (y')^2 r^2 dr$ and $W[y] = -\int_0^\infty V y^2 r^2 dr$ are the kinetic and gravitational potential energies [18], the identity itself being established for the S–N system in [6, 13]. This is the proper role of the virial in the present normalization: with the energy fixed to $E = -1$, the exact ground state satisfies $4K + W = 0$ automatically, so imposing it on the truncated field supplies the overall amplitude that the truncation itself leaves undetermined. The parameter h is then selected by minimizing the residual of the amplitude-corrected field $\hat{y}^{(M)}$, which—unlike the raw residual—is not driven to zero by amplitude collapse.

Carrying out this procedure numerically, and measuring the error against a ground state obtained by direct numerical integration of (3)–(4), the virial-calibrated zeroth-order field attains a relative L^2 error of about 14%, with central potential $V_0(0) \approx 2.57 > 1$; a unit-mass normalization of the same profile would instead give about 47%. The resulting value of h^* and the corresponding higher-order errors are produced by the spectral continuation of Section 6.2 and reported separately. The inner homotopy expansion, calibrated in this way, therefore provides a reliable analytic core for the global composite approximation.

5 REFINED UNIFORMLY VALID APPROXIMATION

The provisional far field constructed in Section 3.2 was anchored to the zeroth-order central amplitude $a_0^{(0)} = y_0(0)$ and therefore inherited the limitations of the initial approximation. With the inner approximation now calibrated by (25) and Section 4, we re-anchor the construction to it. The functional form of the far-field decay (15) is fixed by the reduced envelope equation, but both of its constants are supplied by the calibrated inner fields and are refined with them: the exponent through the dimensionless mass $\widehat{M}$, and the amplitude through the Coulomb-tail identity (10).

At truncation order M , let h_M^* denote the convergence-control parameter fixed in Section 4, and let $\hat{y}^{(M)}(r) = \sigma_M y^{(M)}(r; h_M^*)$ be the corresponding amplitude-corrected inner approximation, with $\widehat{V}^{(M)}$ its potential. The exponent constant is the dimensionless mass of this field, $\widehat{M}^{(M)} = \int_0^\infty r^2 (\hat{y}^{(M)})^2 dr$, a bulk-dominated integral that sharpens at every order from $\widehat{M}^{(0)} = \widehat{M}_0 \approx 4$. The amplitude con-

stant is obtained from the same field through the Whittaker matching (12), evaluated with $\widehat{M}^{(M)}$ and $\widehat{y}^{(M)}$ at a matching radius in the plateau region of Section 3.2, with $\widehat{V}^{(M)}$ and (10) retained as an independent consistency check. Both constants are thus generated by the construction itself, with no external profile. The zeroth-order value is $C^{(0)} = \alpha \approx 1.51$, the amplitude of the seed’s own tail, still far from the exact $C^* \approx 3.37$: the tail amplitude converges slowly, because the integer-power basis $\{r^m e^{-r}\}$ reproduces the non-integer tail $r^{\widehat{M}/2-1}$ only gradually. This slow convergence affects the absolute normalization of the far tail alone—where the solution is exponentially small—and leaves the bulk quantities set by the inner field unchanged. The same extraction applies at every higher order, so that each Homotopy–Padé approximant and each spectral-HAM iterate supplies its own $\widehat{M}^{(M)}$ and $C^{(M)}$.

With the refined far-field representation in hand, the uniformly valid approximation at order M is constructed by combining the calibrated inner approximation with the far-field expansion and subtracting their common part,

$$(26) \quad y_{\text{unif}}^{(M)}(r) = \widehat{y}^{(M)}(r) + y_{\text{out}}^{(M)}(r) - y_{\text{ov}}(r),$$

where $y_{\text{out}}^{(M)}$ is the far-field expansion (15) evaluated with $C = C^{(M)}$, and y_{ov} is the part common to the two representations, which share the envelope e^{-r} exactly—the large- r limit of $\widehat{y}^{(M)}$, coinciding with the leading term of $y_{\text{out}}^{(M)}$. At the seed this common part is the explicit monomial (17). For $\widehat{M}^{(M)} \neq 4$ the algebraic prefactors of the two layers—the integer power r produced by the inner basis $\{r^m e^{-r}\}$ and the non-integer power $r^{\widehat{M}^{(M)}/2-1}$ of the outer expansion—differ at higher order in the overlap than the matching order retained here, and the difference is absorbed into the truncation error. The matching is thus exact through the shared envelope e^{-r} and asymptotic in the algebraic prefactor. The composite reduces to the calibrated approximation near the origin and to the Coulomb-corrected decay (7) at infinity, without requiring a crossover radius or a switching function. The worked example is $M = 2$, but the construction is identical at all higher orders.

Remark 1. *The outer series (15) is asymptotic rather than convergent: the point $1/r = 0$ is an irregular singular point of the reduced equation (14) and the coefficients c_n grow factorially [13], so the far-field expansion must be truncated at its least term rather than summed. The convergence diagnostics of Section 4 therefore govern the inner approximation, while the accuracy of the far-field layer is governed by optimal truncation. Borel summation of the outer series reconstructs the Whittaker function $W_{\widehat{M}/2, 1/2}(2r)$, the exact decaying solution of the reduced envelope equation.*

6 BEYOND SECOND ORDER: PADÉ ACCELERATION AND SPECTRAL CONTINUATION

The construction is continued beyond the closed-form hierarchy by two complementary devices: a rational reorganisation of the low-order series, which sharpens the intermediate field without computing further homotopy levels, and a spectral discretisation of the deformation equations that carries the recursion to arbitrary order.

6.1 HOMOTOPY–PADÉ ACCELERATION

To extract higher-order information without computing additional homotopy levels, we employ Homotopy–Padé acceleration, in which the polynomial partial sum in the embedding parameter q is replaced by a rational approximant of Padé type [19]. In the present context the far-field algebraic structure is supplied analytically by the outer layer of Section 5; the Padé step is therefore applied only to the inner homotopy factor, sharpening its accuracy in the region where the inner and outer descriptions overlap.

Given the calibrated inner fields y_0, y_1 and the numerically computed y_2 (Section 6.2) at the selected value h^* , the $[1/1]$ Homotopy–Padé approximants for the wave function and the gravitational potential are

$$(27) \quad y_{[1/1]}(r) = y_0(r) + \frac{y_1(r)^2}{y_1(r) - y_2(r)},$$

$$(28) \quad V_{[1/1]}(r) = V_0(r) + \frac{V_1(r)^2}{V_1(r) - V_2(r)}.$$

For the potential the first-order correction vanishes, $V_1 \equiv 0$ by (22), so (28) reduces identically to the seed, $V_{[1/1]} = V_0$, and discards the computed second-order term V_2 . We therefore apply the Padé acceleration only to the wave function (27) and retain the plain second-order potential $V^{(2)} = V_0 + V_2$. Evaluated pointwise, the wave-function approximant inherits the regularity and decay of the underlying corrections, while its rational structure incorporates higher-order information without requiring explicit computation of $y_3, y_4, \dots$.

The approximants thus provide a more accurate analytic representation in the intermediate region $2 \lesssim r \lesssim 10$, where neither the pure inner expansion nor the pure outer asymptotics is optimal, and supply an initial guess for the spectral continuation of Section 6.2.

6.2 SPECTRAL HAM AT SECOND ORDER AND ABOVE

Because the corrections leave the elementary class at second order (Section 3.5), the deformation equations are discretised numerically. The Spectral HAM (SHAM)

framework preserves the analytical structure of the homotopy and the order-by-order recursion, but represents the correction functions $y_m(r)$ and $V_m(r)$ on a spectral grid [20, 21]. The method is semi-analytical: the hierarchy of linear deformation equations is obtained analytically from the homotopy, while each linear problem is solved numerically with spectral accuracy.

The semi-infinite physical domain $[0, \infty)$ is mapped to the finite interval $[-1, 1]$ by the algebraic transformation $r = L_s(1 + \xi)/(1 - \xi)$, the scale L_s being chosen in the range $5 \leq L_s \leq 15$ to match the decay length of the ground state. The collocation points are the Chebyshev nodes $\xi_j = \cos(\pi j/N_c)$, $j = 0, \dots, N_c$, with N_c typically between 50 and 100; their images $r_j = L_s(1 + \xi_j)/(1 - \xi_j)$ form the physical grid on which all quantities are evaluated.

Writing L for the collocation matrix of an operator and y for the vector of nodal values of a field, and letting D_r and D_{rr} denote the first- and second-derivative matrices in the physical variable, obtained from the standard Chebyshev differentiation matrix of the mapped grid [20], the discretised auxiliary operators are

$$L_y = D_{rr} + \text{diag}(2/r_j)D_r - I, \quad L_V = D_{rr} + \text{diag}(2/r_j)D_r.$$

The coefficient of y is fixed at -1 in accordance with the Tod–Moroz scaling. At the origin, where the factor $2/r$ is formally singular, L'Hôpital's rule replaces $(2/r)u'(0)$ by $2u''(0)$, ensuring regularity of the discrete operator. Boundary conditions are imposed by modifying the first and last rows of the operator matrices, giving $\tilde{L}_y$ and $\tilde{L}_V$.

The m th-order deformation equations then reduce to the linear systems $\tilde{L}_y y_m = h R_{y,m-1}$ and $\tilde{L}_V V_m = h R_{V,m-1}$, in which the residual vectors are obtained by pointwise evaluation of N_y and N_V at the collocation nodes. The modified matrices are independent of m and may be factored once and reused for all higher-order corrections. The accumulated approximations $y^{(M)}$ and $V^{(M)}$ are the partial sums of the nodal corrections, and h is selected by minimising the discrete counterpart of (24), formed with the Clenshaw–Curtis weights associated with the Chebyshev grid. It provides the same global measure of accuracy and the same criterion for selecting h at each truncation order.

The construction above is presented here as the methodology by which the closed-form analytic scheme is carried beyond second order. Its numerical execution—comprising the attained order, the decay of the discrete residual, and the comparison with a reference shooting profile—is a computational study outside the closed-form scope of the present paper and will be reported separately.

7 CONCLUSION

This work develops a fully analytic, globally valid approximation scheme for the ground state of the Schrödinger–Newton system, built on the structural properties established in Section 2 and respecting them at every stage. The zeroth-order approximation provides a physically motivated analytic seed, and the first-order correction refines it within the global exponential basis $\{r^m e^{-r}\}$, yielding closed-form expressions for the inner field and the potential; the second-order correction and beyond are obtained by the spectral continuation. Every truncated approximation carries the physical amplitude imposed by the virial identity, and therefore satisfies the global energy balance of the system.

The outer Coulomb-corrected expansion, derived independently from the reduced envelope equation, supplies the universal far-field structure, and matching the two descriptions through their common exponential envelope produces a uniformly valid composite approximation. Both constants of the far-field decay are generated by the construction itself and refined order by order, the tail amplitude more slowly than the bulk. Beyond second order the Homotopy–Padé and spectral-HAM schemes extend the construction without sacrificing analytic control, providing a transition from closed-form analysis to numerical continuation.

The resulting framework offers a unified analytic-numerical description of the Schrödinger–Newton ground state, combining structural insight, asymptotic accuracy, and computational efficiency. It provides a template for the systematic construction of globally valid approximations in other nonlinear, nonlocal eigenvalue problems, and lays the groundwork for further analytical and numerical investigations of self-gravitating quantum states.

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