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# Wave Propagation in Viscoelastic Rods Composed by Two Materials

G. Brankov, H. Boncheva, N. Petrov

The problem of wave propagation in a viscoelastic rod is investigated by many authors supposing that initially plane cross-section remain plane, the rod is thin (without lateral motion and transverse shear stress) and considering that the stress, strain, displacement and velocity are continuous functions of the coordinate $x$ (along the axis of the rod) and the time $t$. There are some articles concerning the solution of this problem for thin semi-infinite rod under the action of an impact with constant velocity for Maxwell material [1], Voigt material [2], three-parameter model [3], four-parameter model [4,5] and five-parameter model [6]. The solution of constant stress impact and impact with infinite rigid mass are considered in [2,7]. The wave propagaton in finite rod for Maxwell material [1], three-parameter model [8] and five-parameter model [9] is a solved problem too. The treated of wave propagation in finite rod for viscoelastic material under the action of some impact is given in [10]. In these mentioned articles the viscoelastic rod is composed by one material. The problem of wave propagation in thin rod composed by two materials becomes very complicated. It is known from the paper of Kolsky and Lee [11], who considered the reflection of a pulse travelling from a viscoelastic rod into an elastic rod. Assuming that the value of acoustic impedance $\varrho c$ for the viscoelastic rod is less than that of the elastic rod at higher frequencies, they found the reflected pulse.

In this paper the authors offer the solution of wave propagation in a rod composed by two viscoelastic materials (with different stress-strain laws). The boundary between two materials is $x=0$. The rod is impacted with a pulse $f(t)$ at $x=-l$ and there is a mass at $x=L$. The waves propagate from first viscoelastic material with length $l$ into second viscoelastic material with length $L$.

The equation of motion in uniaxial strain state is

$$\frac{\partial \sigma(x,t)}{\partial x}=\varrho\frac{\partial^2 u(x,t)}{\partial t^2}, \tag{1}$$

where $\sigma(x,t)$ is the strain in the rod; $\varrho$ is the mass density in the initial undeformed state; $u(x,t)$ is the displacement along $x$.

Denoting the stress by $\varepsilon$

$$\varepsilon(x,t)=\frac{\partial u(x,t)}{\partial x} \tag{2}$$

the stress-strain relation of a linear viscoelastic material can be expressed by

(3) $$L_1(t)\sigma = L_2(t)\varepsilon,$$

where $L_1(t)$ and $L_2(t)$ are two operators

$$L_1(t)=\sum_{j=0}^{m} a_j \frac{\partial^j}{\partial t^j}; \quad L_2(t)=\sum_{j=0}^{n} b_j \frac{\partial^j}{\partial t^j},$$

where $a_j$ and $b_j$ are combinations of elastic modulus and viscous coefficients of the material and they are independent of $t$.

Applying the operator $L_1$ to equation (1), differentiating equation (3) with respect to $x$ and taking into account (2) the equation of motion becomes

(4) $$L_2 \frac{\partial^2 u}{\partial x^2} = \varrho L_1 \frac{\partial^2 u}{\partial t^2}.$$

Let us assume that a harmonic wave propagates in the rod, i. e.,

(5) $$u = e^{i(\omega t + k^* x)},$$

where $\omega$ is the frequency, and $k^*$ is a complex wave vector, i. e.,

(6) $$k^* = k - i\alpha,$$

where $k$ is the real wave vector and $\alpha$ is the attenuation constant.

By putting (5) into (4) it is obtained the following expression [10]:

(7) $$(ik^*)^2 \sum_{j=0}^{n} b_j (i\omega)^j - \varrho (i\omega)^2 \sum_{j=0}^{m} a_j (i\omega)^j = 0$$

from which

(8) $$k^{*2} = \varrho \omega^2 \frac{\sum_{j=0}^{m} a_j (i\omega)^j}{\sum_{j=0}^{n} b_j (i\omega)^j}.$$

The expression (8) is the dispersion correlation and it is equivalent to two real equations

(9) $$k = \pm \sqrt{\frac{|k^{*2}| + R_e(k^{*2})}{2}}, \quad \alpha = \pm \sqrt{\frac{|k^{*2}| - R_e(k^{*2})}{2}}.$$

These expressions are valid for two materials, but for the rod with length $l$ the symbols will be with index 1 and for the other material $(0<x<L)$ — index 2.

The particular solution of the equation (4) for first media is

(10) $$u_1 = (C_1 e^{-ik_1^* x} + C_2 e^{ik_1^* x}) e^{i\omega t},$$

where $C_1$ and $C_2$ are to be determined by boundary conditions.

The general solution is

(10') $$u_1 = \sum_{\omega} [C_1(\omega) e^{-ik_1^* x} + C_2(\omega) e^{ik_1^* x}] e^{i\omega t}.$$

For second media it is obtained respectively

(11) $$u_2=(C_3e^{-i\overset{*}{k}_2x}+C_4e^{i\overset{*}{k}_2x})e^{i\omega t};$$

(11′) $$u_2=\sum_{\omega}[C_3(\omega)e^{-i\overset{*}{k}_2x}+C_4(\omega)e^{i\overset{*}{k}_2x}]e^{i\omega t}.$$

The unknown constants $C_1(i=4)$ are been determined by four boundary conditions. Two of them can be given by the condition of stress and strain continuous at $x=0$, i. e.,

$$u_1=u_2 \text{ or}$$

$$\sigma_1=\sigma_2;$$

(12) $$C_1+C_2-C_3-C_4=0;$$

(13) $$\frac{\varrho_1}{\overset{*}{k}_1}(C_2-C_1)-\frac{\varrho}{\overset{*}{k}_2}(C_4-C_3)=0.$$

The other two equations for determining of $C_i(i=4)$ are given by the boundary conditions at the ends $x=-l$ and $x=L$.

If the end $x=-l$ is impacted with a pulse aroused a displacement $f(t)$, i. e.,

$$u_1(-l,t)=f(t),$$

(14) $$C_1e^{i\overset{*}{k}_1l}+C_2e^{-i\overset{*}{k}_1l}=f(\omega),$$

where

$$f(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)e^{i\omega t}dt.$$

In the case when there is a mass $M$ at the end $x=L$, and this mass is uniformly distributed upon the cross-section of the rod, the boundary condition is given by the equation of motion of the mass

$$\sigma_2(L,t)=M\frac{\partial^2u_2(L)}{\partial t^2}$$

and

(15) $$\frac{i\varrho_2}{\overset{*}{k}_2}(C_4e^{i\overset{*}{k}_2L}-C_3e^{-i\overset{*}{k}_2L})=-M(C_3e^{-i\overset{*}{k}_2L}+C_4e^{i\overset{*}{k}_2L}).$$

If the end $x=L$ is a free one, i. e.,

$$\sigma_2(L,t)=0,$$

(16) $$C_3e^{-i\overset{*}{k}_2L}-C_4e^{i\overset{*}{k}_2L}=0.$$

If the end $x=L$ is fixed, i. e.,

$$u_2(L,t)=0 \quad \text{or} \quad M=\infty,$$

(17) $$C_3e^{-i\overset{*}{k}_2L}+C_4e^{i\overset{*}{k}_2L}=0.$$

When $L\to\infty$, i. e., the rod is semi-infinite $C_4=0$, i. e., there is not a reflection pulse.

If one of the materials is elastic, for example the first one, $\overset{*}{k}_1=k_1$, i. e., the attenuation is zero.

The constants are determined by equations (12), (13), (14) and (15) in the most general case when the rod is composed by two viscoelastic materials, impacted by a pulse $f(t)$ and at the end $x=L$ a mass $M$ exists.

After some transformations the expressions for the constants obtain the following complex forms

(18) $$\frac{C_2}{C_1}=\frac{PQ+Q+1}{PQ+Q-1};$$

(19) $$\frac{C_3}{C_1}=\frac{2Q}{PQ+Q-1};$$

(20) $$\frac{C_4}{C_1}=\frac{2PQ}{PQ+Q-1};$$

(21) $$C_1=f(\omega)\frac{PQ+Q-1}{(PQ+Q-1)e^{ik_1^*l}+(PQ+Q+1)e^{-ik_1^*l}},$$

where

$$P=\frac{i\varrho_2-k_2^*M}{i\varrho_2+k_2^*M}e^{-2ik_2^*L},$$

$$Q=\frac{\varrho_1 k_2^*}{\varrho_2 k_1^*}\frac{i\varrho_2+k_2^*M}{i\varrho_2\gamma-k_2^*M\beta},$$

$$\gamma=e^{-2ik_2^*L}-1,\quad \beta=e^{-2k_2^*iL}+1.$$

The displacements $u_1$ and $u_2$ become respectively

(22) $$u_1=f(\omega)\frac{(PQ+Q-1)e^{-ik_1^*x}+(PQ+Q+1)e^{ik_1^*x}}{(PQ+Q-1)e^{ik_1^*l}+(PQ+Q+1)e^{-ik_1^*l}}e^{i\omega t};$$

(23) $$u_2=2f(\omega)\frac{PQe^{ik_2^*x}+Qe^{-ik_2^*x}}{(PQ+Q-1)e^{ik_1^*l}+(PQ+Q+1)e^{-ik_1^*l}}e^{i\omega t}.$$

In the particular case when the length of first material $l$ is very big, because of attenuation in the material the reflection wave can not take part in the boundary condition at $x=-l$ and

$$|C_1e^{ik_1^*l}|\gg|C_2e^{-ik_2^*l}|.$$

In this case the constants are

$$C_1=f(\omega)e^{-ik_1^*l},$$

(24) $$C_3=f(\omega)\frac{e^{-ik_1^*l}}{1+P},$$

$$C_4=f(\omega)\frac{Pe^{-ik_1^*l}}{1+P}.$$

The expressions for $u_1$ and $u_2$ have the form

(25) $$u_1=f(\omega)e^{-ik_1^*(x+l)}e^{i\omega t};$$

(26) $$u_2=f(\omega)\frac{e^{-ik_1^*l}}{1+P}(e^{-ik_2^*x}+Pe^{ik_2^*x})e^{i\omega t}$$

or

(25′) $$Reu_1 = f(\omega)e^{-\alpha_1(x+l)}[\cos k_1(x+l)\cos\omega t+\sin k_1(x+l)\sin\omega t];$$

(26′) $$Reu_2=\frac{f(\omega)}{2\varrho_2(\varrho_2-M\alpha_2)}\{Be^{-\alpha_1 l-\alpha_2(x-2L)}[\cos\eta\cos\omega t-\sin\eta\sin\omega t]$$
$$+[\varrho_2^2-M^2(k_2^2+\alpha_2^2)]e^{-\alpha_1 l+\alpha_2 x}[\cos\psi\cos\omega t-\sin\psi\sin\omega t]\},$$

where

$$B=\varrho_2^2+M^2(k_2^2+\alpha_2^2)2\varrho_2 M\alpha_2,$$
$$\eta=k_1 l-2k_2 L-k_2 x,\quad \psi=k_1 l+k_2 x.$$

Another specific case is when $L\to\infty$ and

$$P=0,\quad \beta=1,\quad \gamma=-1,$$
$$Q=-\frac{\varrho_1 k_2^*}{\varrho_2 k_1^*}.$$

The expressions for the constants become

(27) $$C_1=f(\omega)\frac{\varrho_1 k_2^*+\varrho_2 k_1^*}{A},$$
$$C_2=f(\omega)\frac{\varrho_1 k_2^*-\varrho_2 k_1^*}{A},$$
$$C_3=f(\omega)\frac{2\varrho_1 k_2^*}{A},$$
$$C_4=0,$$

where

$$A=(\varrho_1 k_2^*+\varrho_2 k_1^*)e^{ik_1^*l}+(\varrho_1 k_2^*-\varrho_2 k_1^*)e^{-ik_1^*l}.$$

The expressions for the displacements take the following form

(28) $$u_1=f(\omega)\frac{(\varrho_1 k_2^*+\varrho_2 k_1^*)e^{-ik_1^*x}+(\varrho_1 k_2^*-\varrho_2 k_1^*)e^{ik_1^*x}}{A}e^{i\omega t};$$

(29) $$u_2=f(\omega)\frac{2\varrho_1 k_2^* e^{-ik_2^*x}}{A}e^{i\omega t};$$

(28′) $$Reu_1=f(\omega)\frac{F\cos\omega t+G\sin\omega t}{H};$$

(29′) $$Reu_2=2f(\omega)e^{-\alpha_2 x}\frac{K\cos\omega t+N\sin\omega t}{H},$$

where

(30) $$F=\cos k_1(x+l)[(a_1^2+a_2^2)e^{-\alpha_1(x-l)}+(b_1^2+b_2^2)e^{\alpha_1(x-l)}]$$
$$+[e^{\alpha_1(x+l)}+e^{-\alpha_1(x+l)}][\sin k_1(x-l)(a_1b_2-a_2b_1)+\cos k_1(x-l)(a_1b_1+a_2b_2)];$$

$$(31)\qquad G=\sin k_1(x+l)[(a_1^2+a_2^2)e^{-\alpha_1(x-l)}-(b_1^2+b_2^2)e^{\alpha_1(x-l)}]$$
$$+[e^{\alpha_1(x+l)}-e^{-\alpha_1(x+l)}][\cos k_1(x-l)(a_1b_2-a_2b_1)-\sin k_1(x-l)(a_1b_1+a_2b_2)];$$

$$(32)\qquad H=e^{2\alpha_1 l}(a_1^2+a_2^2)+e^{-2\alpha_1 l}(b_1^2+b_2^2);$$

$$(33)\qquad K=\cos k_2x\{\cos k_1l\,[k_2(a_1e^{\alpha_1 l}+b_1e^{-\alpha_1 l})+\alpha_2(a_2e^{\alpha_1 l}+b_2e^{-\alpha_1 l})]$$
$$+\sin k_1l[k_2(a_2e^{\alpha_1 l}-b_2e^{-\alpha_1 l})-\alpha_2(a_1e^{\alpha_1 l}-b_1e^{-\alpha_1 l})]\}$$
$$-\sin k_2x\{\cos k_1l\,[-k_2(a_2e^{\alpha_1 l}+b_2e^{-\alpha_1 l})+\alpha_2(a_1e^{\alpha_1 l}+b_1e^{-\alpha_1 l})]$$
$$+\sin k_1l[k_2(a_1e^{\alpha_1 l}-b_1e^{-\alpha_1 l})+\alpha_2(a_2e^{\alpha_1 l}-b_2e^{-\alpha_1 l})]\};$$

$$(34)\qquad N=\cos k_2x\{\cos k_1l[\alpha_2(a_1e^{\alpha_1 l}+b_1e^{-\alpha_1 l})-k_2(a_2e^{\alpha_1 l}+b_2e^{-\alpha_1 l})]$$
$$+\sin k_1l[\alpha_2(a_2e^{\alpha_1 l}-b_2e^{-\alpha_1 l})+k_2(a_1e^{\alpha_1 l}-b_1e^{-\alpha_1 l})]\}$$
$$+\sin k_2x\{\cos k_1l[\alpha_2(a_2e^{\alpha_1 l}+b_2e^{-\alpha_1 l})+k_2(a_1e^{\alpha_1 l}+b_1e^{-\alpha_1 l})]$$
$$+\sin k_1l[-\alpha_2(a_1e^{\alpha_1 l}-b_1e^{-\alpha_1 l})+k_2(a_2e^{\alpha_1 l}-b_2e^{-\alpha_1 l})]\};$$
$$a_1=k_2+\frac{\varrho_2}{\varrho_1}k_1;\quad a_2=\alpha_2+\frac{\varrho_2}{\varrho_1}\alpha_1;\quad b_1=k_2-\frac{\varrho_2}{\varrho_1}k_1;$$
$$b_2=\alpha_2-\frac{\varrho_2}{\varrho_1}\alpha_1.$$

The expressions for the reflection and transmission coefficients are the following

$$(35)\qquad \left|\frac{C_2}{C_1}\right|^2=\frac{\left(\frac{\varrho_2^2}{\varrho_1^2}k_1^2+\alpha_2^2\right)^2+\left(k_2^2+\frac{\varrho_2^2}{\varrho_1^2}\alpha_1^2\right)^2+2\left(\alpha_2k_2-\frac{\varrho_2^2}{\varrho_1^2}\alpha_1k_1\right)^2-2\left(\frac{\varrho_2}{\varrho_1}k_1k_2+\frac{\varrho_2}{\varrho_1}\alpha_1\alpha_2\right)^2}{\left(k_2+\frac{\varrho_2}{\varrho_1}k_1\right)^4+\left(\alpha_2+\frac{\varrho_2}{\varrho_1}\alpha_1\right)^4+2\left(k_2+\frac{\varrho_2}{\varrho_1}k_1\right)^2\left(\alpha_2+\frac{\varrho_2}{\varrho_1}\alpha_1\right)^2};$$

$$(36)\qquad \left|\frac{C_3}{C_1}\right|^2=\frac{4(k_2^2+\alpha_2^2)\left[k_2^2+\alpha_2^2+\frac{\varrho_2^2}{\varrho_1^2}(k_1^2+\alpha_1^2)\right]}{\left(k_2+\frac{\varrho_2}{\varrho_1}k_1\right)^4+\left(\alpha_2+\frac{\varrho_2}{\varrho_1}\alpha_1\right)^4+2\left(k_2+\frac{\varrho_2}{\varrho_1}k_1\right)^2\left(\alpha_2+\frac{\varrho_2}{\varrho_1}\alpha_1\right)^2}.$$

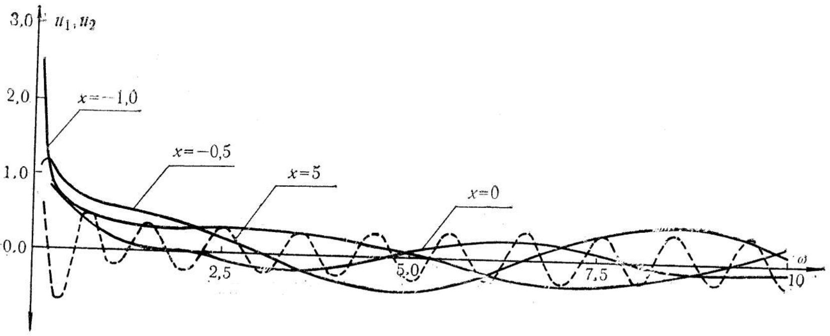
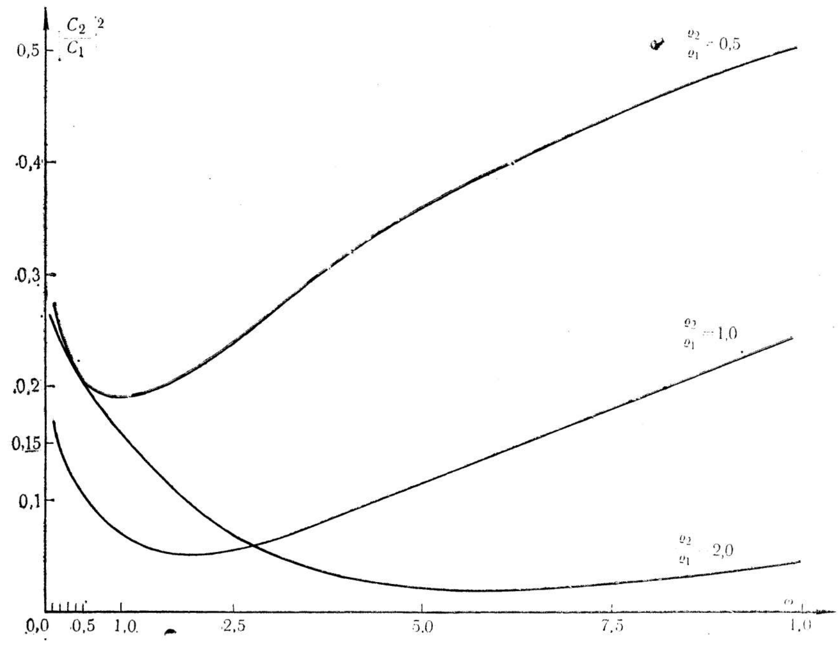


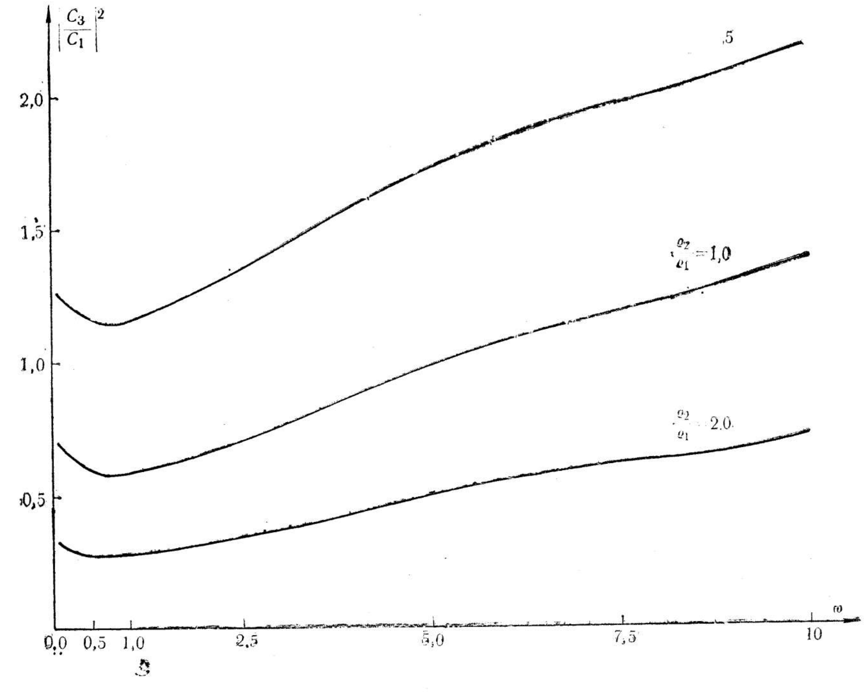

Fig. 1

$t_1=1;\ \frac{\varrho_2}{\varrho_1}=1;\ —\ u_1;\ ----\ u_2$



$\left|\frac{C_2}{C_1}\right|^2$
0,5
0,4
0,3
0,2
0,15
0,1
$\frac{\varrho_2}{\varrho_1}=0,5$
$\frac{\varrho_2}{\varrho_1}=1,0$
$\frac{\varrho_2}{\varrho_1}=2,0$
0,0
0,5
1,0
2,5
5,0
7,5


Fig. 2



$\left|\frac{C_3}{C_1}\right|^2$
2,0
1,5
1,0
0,5
,5
$\frac{\varrho_2}{\varrho_1}=1,0$
$\frac{\varrho_2}{\varrho_1}=2,0$
$\omega$
0,0
0,5
1,0
2,5
5,0
7,5
10


Fig. 3

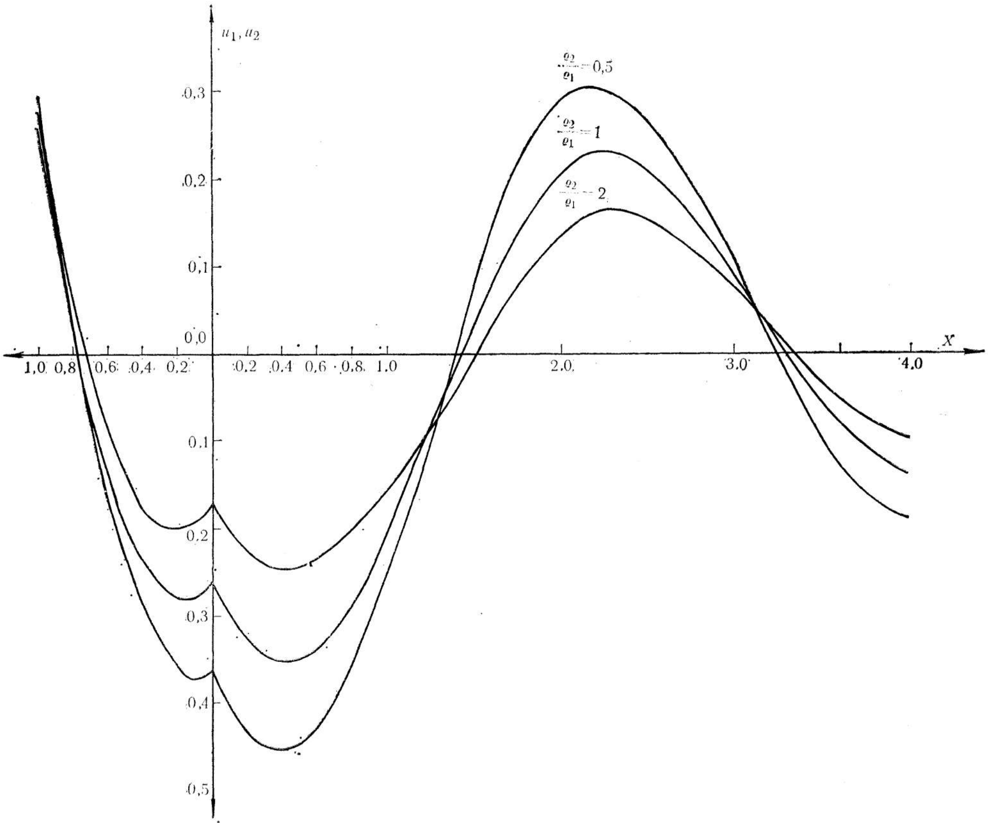



Fig. 4
$t=5$; $\omega=1$

Let us consider, like a numerical example, the following more simple case $f(t)=\delta(t)$ (Dirak's function) and
$f(\omega)=1$, $l=1$, $L\to\infty$.

Let us assume that the first material is two-parameter material of Voigt and the second one — five-parameter model of Brankov. The values of $k_1$, $\alpha_1$, $k_2$ and $\alpha_2$ for each $\omega$ are calculated in [10].

The graphs of $u_1$ and $u_2$ in function of $\omega$ at $t=1$ and $\frac{\varrho_2}{\varrho_1}=1$ for various values of $x$ are shown in Fig. 1. The dash line is for $u_2$, and the thick one — for $u_1$.

In Figs. 2 and 3 are shown the reflection $\left|\frac{C_2}{C_1}\right|^2$ and transmission $\left|\frac{C_3}{C_1}\right|^2$ coefficients in function of $\omega$ for various ratios of the density of two materials.

The influence of the ratio $\frac{\varrho_2}{\varrho_1}$ on the displacements $u_1$ and $u_2$ in function of $x$ is shown in Fig. 4. At $x=0$ $u_1=u_2$ (boundary condition) and in this point the bend is due to the fact that the deformations are different because of the different constitutive equations for two materials and the boundary condition (equation of strains). i. e., at $x=0$ the derivatives of $u_1$ and $u_2$ with respect to $x$ are different.

Fig. 5 shows the change of displacements $u_1$ and $u_2$ in function of $x$ at $\omega=1$ and various time $t(t=0;\ 5;\ 10)$.

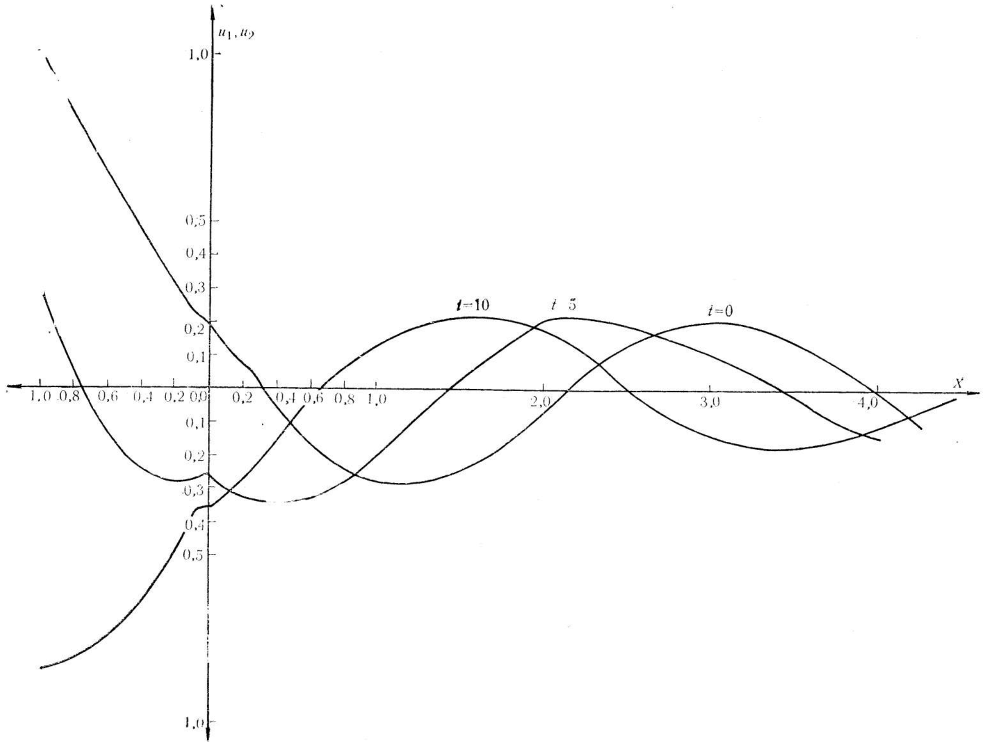



Fig. 5

$\omega=1$ ; $\frac{\varrho_2}{\varrho_1}=1$

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