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On the secondary stationary flows of a fluid between concentric rotating cylinders*

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Introduction

Let a viscous incompressible homogeneous fluid fills in the space between two concentric cylinders rotating in the same direction with radii respectively $r=R_1$ and $r=R_2$ when a cylindric coordinate system $Or\varphi z$ is introduced. It is supposed that the cylinders are hard and rotate with angle velocities ω_1 and ω_2 respectively, and the vector of the applied vortical mass forces $\mathbf{F}$ has the form $(0, \nu F(r), 0)$, (ν — viscosity coefficient). We shall study the stationary flows for which the following condition is fulfilled: the flux of velocity $\mathbf{v}=(u, v, w)$ through the cross-section of the space between the two rotating cylinders to be zero, i. e.

$$\int_{-\pi}^{\pi}\int_{R_1}^{R_2} w(r, \varphi, z)\, r dr d\varphi = 0.$$

The solution of the stationary Navier-Stokes boundary value system corresponding to this problem is unique at small Reynolds numbers, small mass forces, etc., and it has the following form

$$\begin{matrix} u_0=w_0=0, \\ v_0=v_0(r), \end{matrix} \quad P_0=\int_{R_1}^{R_2} \frac{v_0^2(r)}{r}\, dr+\text{const}, \tag{1}$$

where $v_0(r)$ function is defined as a solution of the boundary problem

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right) v_0(r) = -F(r), \tag{2}$$

with boundary conditions

$$v_0(R_1)=\omega_1 R_1$$
$$v_0(R_2)=\omega_2 R_2.$$

Under definite conditions, however, at sufficiently big Reynold's numbers, the laminar flow $v_0(r)$ is possible to lose stability and secondary steady motions to

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arise. This means that the corresponding stationary problem has more than one solution. In [1] it is shown that for the investigation of this problem the bifurcation theory of solution of the non-linear operator equations developed in [3] is applicable. In [2] the problem of the arising axisymmetric stationary flows is investigated, i. e. such ones that the components of velocity **v** to depend only on r and z. It is supposed that u, v, w are periodical in z with a period $2\pi/\alpha_0$. A solution of the problem which should be different from (1) and has the form that follows is looked for:

$$\mathbf{v}=\mathbf{v}_0+\mathbf{v}_1,$$

$$P=P_0+\nu p_1,$$

where $\mathbf{v}_0$ and P_0 is the solution given above, and for u_1, v_1, w_1 it is supposed to be $2\pi/\alpha_0$ periodical in z.

Taking into account the relations (1) and (2) from the equations of Navier - Stokes, the following system of equations for v_1 and p_1 is obtained

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_1)+\frac{\partial w_1}{\partial z}=0, \tag{3}$$

$$\Delta u_1-\frac{u_1}{r^2}-\frac{\partial p_1}{\partial r}=\lambda\left[u_1\frac{\partial u_1}{\partial r}+w_1\frac{\partial u_1}{\partial z}-\frac{v_1^2}{r}-\frac{2v_0}{r}v_1\right],$$

$$\Delta v_1-\frac{v_1}{r^2}=\lambda\left[u_1\frac{\partial v_1}{\partial r}+w_1\frac{\partial v_1}{\partial z}+\frac{u_1v_1}{r}+\left(\frac{dv_0}{dr}+\frac{v_0}{r}\right)u_1\right),$$

$$\Delta w_1-\frac{\partial p_1}{\partial z}=\lambda\left[u_1\frac{\partial w_1}{\partial r}+w_1\frac{\partial w_1}{\partial r}\right],$$

$$\left(\lambda=\frac{1}{\nu},\ \Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)$$

with boundary conditions

$$u_1(R_1)=u_1(R_2)=0,\quad v_1(R_1)=v_1(R_2)=0$$

$$w_1(R_1)=w_1(R_2)=0,\quad \int_{R_1}^{R_2} w_1(r,z)r\,dr=0,$$

for which the conditions of existence of bifurcation points of the corresponding operator equation are established and therefore, the conditions of the arising of secondary stationary flows periodical in z with a period $2\pi/\alpha_0$ of the form

$$u_1=u_1(r)e^{i\alpha z},\quad v_1=v_1(r)e^{i\alpha z},$$

$$w_1=w_1(r)e^{i\alpha z},\quad p_1=p_1(r)e^{i\alpha z}.$$

Formulation of the problem

At the investigations made so far and stated above the existence of secondary stationary flows periodic in z is established. The experiments, however, show that the arising of flows periodical in z as well as in φ is possible. To this end let us consider solutions of the form

$$\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_0,\quad P=P_0+\nu p_1+\nu p_2,$$

where $\mathbf{v}_0$, $\mathbf{v}_1$, P_0, p_1 are the solutions discussed above and for $\mathbf{v}_2$ with components u_2, v_2, w_2 and p_2 it is supposed that they depend on r and φ and at that are periodical in φ with a period $2\pi/\beta_0$.

Substituting in the equations of Navier-Stokes we obtain the following system of equations

$$\frac{\partial}{\partial r}(ru_2)+\frac{\partial v_2}{\partial \varphi}=0,$$

$$\Delta u_2-\frac{u_2}{r^2}-\frac{2}{r^2}\frac{\partial v_2}{\partial \varphi}-\frac{\partial p_2}{\partial r}=\lambda\left[u_2\frac{\partial u_2}{\partial r}+u_2\frac{\partial u_1}{\partial r}+u_1\frac{\partial u_2}{\partial r}\right.$$

$$\left.+\frac{v_0}{r}\frac{\partial u_2}{\partial \varphi}+\frac{v_1}{r}\frac{\partial u_2}{\partial \varphi}+\frac{v_2}{r}\frac{\partial u_2}{\partial \varphi}+w_2\frac{\partial u_1}{\partial z}-\frac{2v_0}{r}v_2-\frac{2v_1}{r}v_2-\frac{v_1^2}{r}\right],$$

$$\Delta v_2-\frac{v_2}{r^2}+\frac{2}{r^2}\frac{\partial u_2}{\partial \varphi}-\frac{1}{r}\frac{\partial p_2}{\partial \varphi}=\lambda\left[u_2\frac{\partial v_2}{\partial r}+u_1\frac{\partial v_2}{\partial r}\right.$$

(4) $$\left.+u_2\frac{\partial}{\partial r}(v_0+v_1)+\frac{v_0+v_1+v_2}{r}\frac{\partial v_2}{\partial \varphi}+w_2\frac{\partial v_1}{\partial z}+\frac{u_1v_2+u_2(v_0+v_1+v_2)}{r}\right],$$

$$\Delta w_2=\lambda\left[u_2\frac{\partial w_2}{\partial r}+u_1\frac{\partial w_2}{\partial r}+u_2\frac{\partial w_1}{\partial r}+w_2\frac{\partial w_1}{\partial z}\right.$$

$$\left.+\frac{v_0+v_1+v_2}{r}\frac{\partial w_2}{\partial \varphi}\right]$$

$$\left(\lambda=\frac{1}{\nu};\ \Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial_2}{\partial \varphi_2}+\frac{\partial^2}{\partial z^2}\right),$$

with boundary conditions

$$u_2(R_1)=u_2(R_2)=0,\quad v_2(R_1)=v_2(R_2)=0,$$

$$w_2(R_1)=w_2(R_2)=0,\ \int_{-\pi}^{\pi}\int_{R_1}^{R_2}w_2(r,\varphi)r\,dr\,d\varphi=0.$$

The linearized problem corresponding to problem (4) has the form

$$\frac{\partial}{\partial r}(ru_2')+\frac{\partial v_2'}{\partial \varphi}=0,$$

$$\Delta u_2'-\frac{u_2'}{r^2}-\frac{2}{r^2}\frac{\partial v_2'}{\partial \varphi}-\frac{\partial p_2'}{\partial r}=\lambda\left[u_1\frac{\partial u_2'}{\partial r}+u_2'\frac{\partial u_1}{\partial r}\right.$$

$$\left.+\frac{v_0}{r}\frac{\partial u_2'}{\partial \varphi}+\frac{v_1}{r}\frac{\partial u_2'}{\partial \varphi}+w_2'\frac{\partial u_1}{\partial z}-\frac{2v_0}{r}v_2'-\frac{2v_1}{r}v_2'\right],$$

(5) $$\Delta v_2'-\frac{v_2'}{r^2}+\frac{2}{r^2}\frac{\partial u_2'}{\partial \varphi}-\frac{1}{r}\frac{\partial p_2'}{\partial \varphi}=\lambda\left[u_1\frac{\partial v_2'}{\partial r}+u_2'\frac{\partial v_0}{\partial r}\right.$$

$$\left.+u_2'\frac{\partial v_1}{\partial r}+\frac{v_0+v_1}{r}\frac{\partial v_2'}{\partial \varphi}+w_2'\frac{\partial v_1}{\partial z}+\frac{u_1v_2'+u_2'(v_0+v_1)}{r}\right],$$

$$\Delta w_2'=\lambda\left[u_1\frac{\partial w_2'}{\partial r}+u_2'\frac{\partial w_1}{\partial r}+\frac{v_0+v_1}{r}\frac{\partial w_2'}{\partial \varphi}+w_2'\frac{\partial w_1}{\partial z}\right]$$

with the same boundary conditions.

Let consider the set M of the solenoidal vectors $\{\mathbf{v}_2\}$ possessing continuous second derivatives, determined in the closed region $\{R_1 \leq r \leq R_2;\ \pi \leq \varphi \leq \pi\}$, depending on r and φ, annulling at $r=R_1, R_2$ and having a flux through the cross-section of the space between the two cylinders equal to zero. We denote by H_1 the Hilbert space obtained by the completing of the set M by a norm arisen from the scalar product

$$(\mathbf{v}_2, \mathbf{v}_2')_{H_1} = -\int_{-\pi/\beta_0}^{\pi/\beta_0} \int_{R_1}^{R_2} \Delta \mathbf{v}_2 \mathbf{v}_2' r dr = -\int_{-\pi/\beta_0}^{\pi/\beta_0} \int_{R_1}^{R_2} \left[\left(\Delta u_2 - \frac{u_2}{r} - \frac{2}{r^2} \frac{\partial v_2}{\partial \varphi} \right) u_2' \right.$$

$$\left. + \left(\Delta v_2 - \frac{v_2}{r^2} - \frac{2}{r^2} \frac{\partial u_2}{\partial \varphi} \right) v_2' + \Delta w_2 w_2' \right] r dr d\varphi.$$

At $\lambda=0$ we obtain the inverse operator of the operator determined by relation (4). Thus the problem is reduced to the operator equation

$$\mathbf{v}_2 = \lambda A \mathbf{v}_2. \tag{4, a}$$

Analogously, problem (15) is reduced to the operator equation

$$\mathbf{v}_2' = \lambda \Phi \mathbf{v}_2'. \tag{5, a}$$

From [1] immediately follows that:

1. The operators A and Φ are completely continuous in the space H_1.
2. The operator Φ appears as a differential of Freche in the point $\mathbf{v}_2=\theta$ of the operator A in H_1.

This permits us to apply the theorem of M. A. Krasnoselskii which states:

Let A is a completely continuous operator having in the point θ a differential of Freche Φ and satisfying the condition $A\theta=\theta$. Then each real characteristic number λ_0 with uneven multiplicity of the linear operator Φ appears as a point of bifurcation of the operator A at which a continuous branch of proper vectors of the operator, A corresponds to this point of bifurcation.

Investigation of the problem

Representing in a series of Fourier we get convinced that the general solution of the spectral problem (5) represents a linear combination of solutions of the form

$$u_2' = u_2(r) e^{i\beta\varphi}, \quad v_2' = v_2(r) e^{i\beta\varphi},$$
$$w_2' = w_2(r) e^{i\beta\varphi}, \quad p_2' = p_2(r) e^{i\beta\varphi} \tag{6}$$

where $\beta = n\beta_0$(n—a whole number).

Substituting in (5) we obtain the system

$$\frac{1}{r}\frac{d}{dr}(ru_2) + \frac{i\beta}{r} v_2(r) = 0,$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1+\beta^2}{r^2} \right) u_2 - i\beta \frac{2}{r^2} v_2 - \frac{dp_2}{dr} = \lambda \left[u_1 \frac{du_2}{dr} \right.$$

$$\left. + u_2 \frac{\partial u_1}{\partial r} + i\beta \frac{v_0}{r} u_2 + i\beta \frac{v_1}{r} u_2 - \frac{2v_0}{r} v_2 - \frac{2v_1}{r} v_2 + w_2 \frac{\partial u_1}{\partial z} \right],$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1+\beta^2}{r^2} \right) v_2 + \frac{2i\beta}{r^2} u_2 - \frac{i\beta}{r} p_2 = \lambda \left[u_1 \frac{dv_2}{dr} \right. \tag{7}$$

$$+u_2\frac{dv_0}{dr}+u_2\frac{\partial v_1}{\partial r}+\frac{v_0+v_1}{r}\,i\beta v_2+\frac{\partial v_1}{\partial z}$$

$$+\frac{(v_0+v_1)u_2+u_1v_2}{r}\Big],$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{\beta^2}{r^2}\right)w_2=\lambda\left[u_2\frac{\partial w_1}{\partial r}+u_1\frac{dw_2}{dr}\right.$$

$$\left.+\frac{v_0+v_1}{r}\,i\beta w_2+\frac{\partial w_1}{\partial z}w_2\right].$$

At that only those solutions which correspond to one and the same λ enter the linear combination.

Here the following two cases are possible:

A. When the conditions

$$\frac{v_0(r)}{r}>0,\quad -\left(\frac{dv_0}{dr}+\frac{v_0}{r}\right)<0\ (R_1<r<R_2) \tag{8}$$

are fulfilled, i. e. the laminar flow $v_0(r)$ is stable to axisymmetric perturbations [2].

B. When the conditions

$$\frac{v_0(r)}{r}>0,\quad -\left(\frac{dv_0}{dr}+\frac{v_0}{r}\right)>0\ (R_1<r<R_2) \tag{9}$$

are fulfilled, i. e. the laminar flow is unstable in z and the system (3) has a solution different from the zero one.

Here we shall consider the case when conditions (8) are fulfilled, i. e. when the system (3) has the only solution $u_1=v_1=w_1=0$. In this case system (7) achieves the following form

$$\frac{d}{dr}[ru_2(r)]+i\beta v_2(r)=0,$$

$$Lu_2(r)-\frac{2}{r^2}i\beta v_2(r)-\frac{dp_2(r)}{dr}=\lambda\left[i\beta\frac{v_0(r)}{r}u_2(r)-\frac{2v_0(r)}{r}v_2(r)\right],$$

$$Lv_2(r)+\frac{2}{r^2}i\beta u_2(r)-\frac{i\beta}{r}p_2(r)=\lambda\left[\left(\frac{dv_0(r)}{dr}+\frac{v_0(r)}{r}\right)u_2(r)+\frac{i\beta}{r}v_0(r)v_2(r)\right], \tag{10}$$

$$L_1w_2(r)=\lambda i\beta\frac{v_0(r)}{r}w_2(r),$$

$$\left(L=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1+\beta^2}{r^2};\ L_1=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{\beta^2}{r^2}\right).$$

Determining $v_2(r)$ and $p_2(r)$ from (10) by means of $u_2(r)$ through the expressions

$$v_2(r)=-\frac{1}{i\beta}\frac{d}{dr}[ru_2(r)],$$

$$p_2(r)=\frac{r}{i\beta}\left\{Lv_2(r)+\frac{2}{r^2}i\beta u_2(r)-\lambda\left[\left(\frac{dv_0(r)}{dr}+\frac{v_0(r)}{r}\right)u_2(r)\right.\right.$$

$$\left.\left.+\frac{i\beta}{r}v_0(r)v_0(r)\right]\right\}$$

for $u_2(r)$, $w_2(r)$ and λ we obtain the following spectral problem

$$\left(\frac{d^2}{d_r^2}+\frac{1}{r}\frac{d}{dr}-\frac{\beta^2}{r^2}\right)w_2(r)=\lambda imw_2(r),$$

(11)
$$\frac{4}{r^2}u_2(r)+\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1+\beta^2}{r^2}\right)u_2(r)$$

$$-\frac{1}{\beta^2}\frac{d}{dr}\left\{r\left|\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1+\beta^2}{r^2}\right|\frac{d}{dr}[ru_2(r)]\right\}=\lambda iMu_2(r),$$

where

$$M=A+B\frac{d}{dr}+C\frac{d^2}{dr^2},$$

$$A=\frac{\beta}{r}v_0(r)-\frac{2}{\beta r}v_0(r)+\frac{1}{\beta}\frac{dv_0(r)}{dr}+\frac{r}{\beta}\frac{d^2v_0(r)}{dr^2},$$

$$B=-\frac{3}{\beta}v_0(r),$$

$$C=-\frac{r}{\beta}v_0(r),$$

$$m=\frac{\beta}{r}v_0(r).$$

Let us transform the above two equations to the integro-differential equations. To this end let denote by $G_1(r,\varrho)$ Greene's function of the operator rL_1 and by $G_2(r,\varrho)$ Greene's function of the operator $\beta^2 rL_2$:

$$L_2=\frac{4}{r^2}u_2(r)+Lu_2(r)-\frac{1}{\beta^2}\frac{d}{dr}\left\{r\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1+\beta^2}{r^2}\right]\frac{d}{dr}[ru_2(r)]\right\}.$$

Green's functions $G_1(r,\varrho)$ and $G_2(r,\varrho)$ are continuous by r and symmetrical. This, as known, follows from the symmetry of their corresponding differential operators.

Let denote by $G_k\,(k=1,2)$ the integral operators determined by the formulas

$$G_kf=\int_{R_1}^{R_2}G_k(r,\varrho)f(\varrho)\varrho d\varrho.$$

If we denote by H_2 Hilbert's space L_2 with a weight r on the interval (R_1,R_2) and a scalar product

$$(\varphi,\psi)_{H_2}=\int_{R_1}^{R_2}\varphi(r)\psi(r)rdr$$

then the operators G_k are symmetric and completely continuous in H_2.

Then equations (11) are reduced to the following system independent integro-differential equations

(12)
$$u_2=\mu G_2Mu_2=\mu G_2^0u_2,$$

$$w_2=\mu G_1mw_2=\mu G_1^0w_2$$

$$(\mu=\lambda i,\ G_1^0=G_1m,\ G_2^0=G_2M).$$

The differential operator rL_1 may be represented in the following form

$$rL_1 u = \varrho_0 \frac{d}{dr} \varrho_1 \frac{d}{dr} (\varrho_2 u)$$

$$\left(\varrho_0 = \varrho_2 = r^\beta,\ \varrho_1 = \frac{r}{\varrho_0^2}\right).$$

Then from the investigations of M. G. Kreyn on some functions of Greene of ordinary differential operators [4], [5], it follows that the operator G_1^0 is an oscillatory one and therefore its spectrum is real, positive and simple.

Hence, it immediately follows that λ are purely imaginary numbers and according to the theorem of M. A. Krasnoselskii the nonlinear equation has not points of bifurcation.

Conclusion

The laminar flow $\mathbf{v}_0$ is stable in comparison to φ, i. e. the arising of secondary stationary flows periodical in φ at any numbers of Reynolds is not possible before the arising of secondary stationary flows periodical in z. Moreover, the independence of equations (12) shows that the arising of flows with a third component $w_2 = 0$ is possible, which will be the object of further investigations.

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