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Free Convection Effects on MHD Stokes Problem for a Vertical Plate

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Nomenclature

B_0 — Magnetic induction
c_p — Specific heat at constant pressure
g — Acceleration due to gravity
G — Grashof number $(\nu g\beta(T'_w - T'_\infty)/U_0^3)$
K — Thermal conductivity
M — Magnetic field parameter
P — Prandtl number $(\mu c_p/K)$
T'_∞ — Temperature of fluid
T'_∞ — Temperature of free stream
T'_w — Temperature of the plate
u' — Velocity component
U_0 — Velocity of the plate
x', y' — Co-ordinate axes
β — Coefficient of volume expansion
ν — Kinematic viscosity
σ — Electrical conductivity
ϱ' — Density
θ — Non-dimensional temperature
erfc — Complementary error function
τ' — Skin-friction

1. Introduction

One of the simple solutions of the Navier-Stokes equation was first given by Stokes [1] in case of the flow of an incompressible viscous fluid past an infinite horizontal plate moving impulsively in its own plate. Hence it is known as Stokes first problem. It is also known as Rayleigh's problem in the literature. This study has been extended to the flow past bodies of different shapes, moving impulsively in their own plane. Notable amongst them are by Stewartson [2], Hall [3], Dennis [4] etc. In [2], [3], the analytical study of Rayleigh's problem for a semi-infinite plate was made, whereas in [3] the problem was studied by finite difference method. But the study of the flow past an infinite vertical plate, moving impulsively in its own plane, has not been presented for a long time.

Now in this type of flow two types of physical situations can be considered: i) the temperature of the plate and the fluid extending to infinity is equal, ii) the plate temperature differs from the temperature of the fluid at infinity. From physical and technological point of view, the second case is more important. In this case, if the difference between the plate temperature T'_w and that of the fluid at infinity T'_∞, viz. $T'_w - T'_\infty$ is appreciably large, there is a flow of free convection currents in the neighbourhood of the plate and the free convection currents do affect the flow past an impulsively started vertical plate. This study of an important physical phenomenon was recently presented by Soundalgekar [5]. In this paper Soundalgekar studied the effects of the free convection currents on the velocity field in a quantitative manner. Illingworth [6] also studied Stokes problem for a vertical plate, by the method of successive approximation, for a compressible fluid and this was extended to the compressible flow past an infinite plate with time-dependent velocity and temperature by Eliott [7]. In both [6, 7] the problem was solved mathematically without any discussion of the physical situation.

Now in recent years, the effects of the transverse magnetic field on the flow of an incompressible, viscous, electrically conducting fluids, have also been studied extensively by many research workers. The magnetohydrodynamic aspect of Stokes or Rayleigh's problem, on neglecting the induced magnetic field, was first presented by Rossow [8] in case of a horizontal plate. How a transversely applied magnetic field affects the flow of an electrically conducting, viscous incompressible fluid past an impulsively started vertical plate? This indeed is the motivation of the present study.

In section 2, the mathematical analysis is presented under proper assumptions followed by a discussion. In section 3 the conclusions are set out.

2. Mathematical Analysis

Here x'-axis is taken along the plate in the vertical direction and the y'-axis is taken normal to it. An uniformly applied transverse magnetic field is in the direction of y'-axis. It is also assumed that the induced magnetic field is negligible. All the fluid properties are assumed constant except that the influence of the density variation with temperature is considered only in the body force term. This is the usual Boussinesq's approximation. Then under these assumptions the problem is governed by the following equations:

$$\frac{\partial u'}{\partial t'}=g\beta(T'-T'_{\infty})+\nu\frac{\partial^2 u'}{\partial y'^2}-\frac{\nu B_0^2}{\varrho'}u', \tag{1}$$

$$\varrho' c_p\frac{\partial T'}{\partial t'}=k\frac{\partial^2 T'}{\partial y'^2}. \tag{2}$$

Here u' is the velocity of the fluid, T' — the temperature of the fluid near the plate, T'_{∞} — the temperature of the fluid far away from the plate, g — the acceleration due to gravity, β — the coefficient of volume expansion, ν — the kinematic viscocity, σ — the scalar electrical conductivity, B_0 — the applied uniform magnetic field and ϱ' is the density of the fluid, k — the thermal conductivity and t' — the time. In equation (2), the heat due to viscous dissipation is assumed to be negligible. This is possible when the velocity is small. The boundary conditions are

$$\begin{aligned} &u'=U_0, \quad T'=T'_w \quad \text{as} \quad y'=0,\\ &u'\to 0, \quad T'\to T'_{\infty} \quad \text{as} \quad y'\to\infty. \end{aligned} \tag{3}$$

On introducing the following non-dimensional quantities

$$y=y'U_0/\nu,\ t=t'U_0^2/\nu,\ u=u'/U_0,$$

$$\theta=\frac{T'-T'_\infty}{T'_w-T'_\infty},\quad P=\frac{\mu C_P}{k}\ \text{the Prandtl number,}$$

(4)

$$G=\frac{\nu g\beta(T'_w-T'_\infty)}{U_0^3},\ \text{the modified Grashof number,}$$

$M^2=\sigma B_0^2\nu/\varrho' U_0^2$, the magnetic field parameter, in (1), (2) and (3), we have

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial y^2}+G\theta-M^2u, \tag{5}$$

$$P\frac{\partial\theta}{\partial t}=\frac{\partial^2\theta}{\partial y^2} \tag{7}$$

and

$$u=1,\ \theta=1\quad\text{at}\quad y=0,$$

(8)

$$u=0,\ \theta=0\quad\text{at}\quad y\to\infty.$$

These are the coupled equations whose solutions can be obtained by the usual Laplace-Transform technique and are given as follows:

$$\theta=\operatorname{erfc}(\eta\sqrt{P}) \tag{9}$$

for $P<1$,

$$u=\frac{1}{2}\left(1-\frac{G}{M^2}\right)[e^{-2\eta M\sqrt{t}}\operatorname{erfc}(\eta-M\sqrt{t})+e^{2\eta M\sqrt{t}}\operatorname{erfc}(\eta+M\sqrt{t})]$$

$$-\frac{G}{2M^2}e^{bt}[e^{-2\eta\sqrt{Pbt}}\operatorname{erfc}(\eta\sqrt{P}-\sqrt{bt})+e^{2\eta\sqrt{Pbt}}\operatorname{erfc}(\eta\sqrt{P}+\sqrt{bt})]$$

$$+\frac{G}{M^2}\operatorname{erfc}(\eta\sqrt{P})+\frac{G}{2M^2}e^{bt}[e^{-2\eta\sqrt{Pbt}}\operatorname{erfc}(\eta-\sqrt{Pbt})+e^{2\eta\sqrt{Pbt}}\operatorname{erfc}(\eta+\sqrt{Pbt})], \tag{10}$$

where

$$b=M^2/(P-1)\ \text{and}\ \eta=y/2\sqrt{t}$$

and for $P=1$

$$u=\frac{1}{2}\left(1-\frac{G}{M^2}\right)[e^{-2\eta M\sqrt{t}}\operatorname{erfc}(\eta-M\sqrt{t})+e^{2\eta M\sqrt{t}}\operatorname{erfc}(\eta+M\sqrt{t})]+\frac{G}{M^2}\operatorname{erfc}(\eta). \tag{11}$$

Now, for all the electrical conducting fluids, P is either less than unity or $P\sim 1$. For $P<1$, the argument of the complementary error function in (10) is imaginary and hence for calculation purpose this has to be separated into real and imaginary parts. This can be done, following Strand [9], as follows:

For $x>0$, $y\geq 0$,

$$\operatorname{erfc}(x+iy)=e^{-2ixy}\sum_{n=0}^{\infty}(xy)^n[Y_n(x)-ixy(n+1)Y_{n+1}(x)]=e^{-2ixy}\Phi(x,y), \tag{12}$$

where

$$Y_{n+1}(x)=\frac{2}{(2n+1)\sqrt{\pi}}\left[\frac{e^{-x^2}}{(n+1)!\,x^{2n+1}}-\frac{\sqrt{\pi}}{n+1}Y_n(x)\right]\quad(n=0,1,2,\ldots),$$

and

$$Y_0(x)=\operatorname{erfc}(x).$$

Hence, with the help of (12), the velocity profiles for any value of P can be calculated. Here, to show the effects of G and t, we have calculated the velocity and temperature for $P=1$, 0.5, 0.7 and $t=0.5$, 1, 1.5, 2 and for realistic purpose, all the real values of G are chosen as they are interesting from the physical point of view. Now, the free convection currents are in existence due to the temperature difference $T'_w-T'_\infty$ and hence $T'_w-T'_\infty$ may be positive, zero or negative. Then the value of the Grashof number $G(=\nu g\beta\,(T'_w-T'_\infty)/U_0^3)$ will assume positive, zero or negative values. From the physical point of view, $G<0$ corresponds to an externally heated plate as the free convection currents are carried towards the plate. Then $G>0$ corresponds to an externally cooled plate and $G=0$ corresponds to the absence of the free convection currents.

On Figs. 1 and 2 the velocity profiles are shown for $P<1$. The curves on Fig. 1 correspond to the velocity profiles in the presence of the plate being heated by the free convection currents. We conclude from the nature of these curves that they are of reverse type. An increase in P leads to an increase in the velocity when the values of G, t and M are constant. An increase in M also leads to an increase in the velocity. This leads us to conclude that the reverse type of flow may be avoided, in the presence of the plate being heated by the free convection currents, by increasing the strength of the magnetic field. As time is increased, the velocity still decreases when the values of G, M and P are constant. Greater heating of the plate by the free convection currents also causes a decrease in the velocity. Quantitatively, for $M=2$, $P=0.5$, $t=0.5$, there is 100% fall in the velocity at $\eta=1$ when the plate is heated such that G is increased from 4 to 6. On Fig. 2, the velocity profiles are shown in case of the plate being cooled by the free convection currents. They being all positive, we conclude that there is a non-reverse type of flow. Here, an increase in P

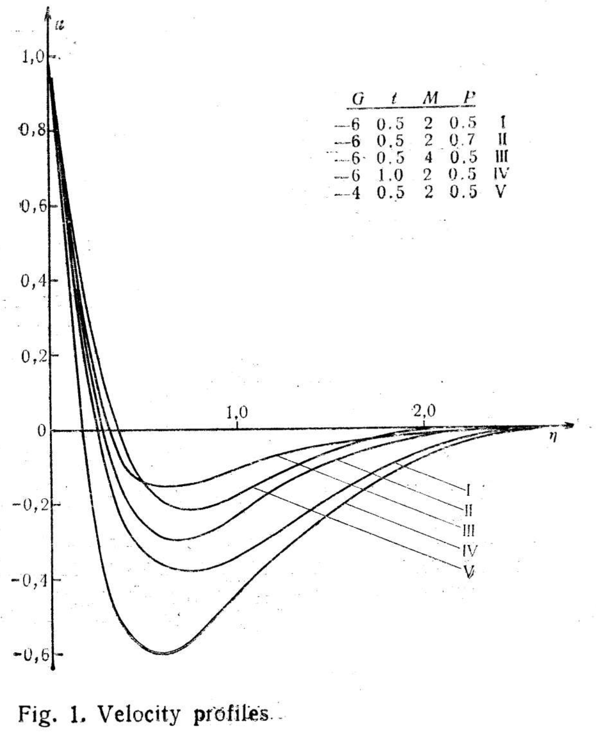

Fig. 1. Velocity profiles.

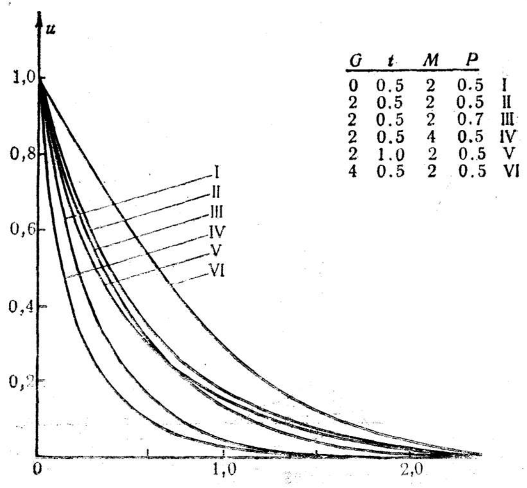

Fig. 2. Velocity profiles

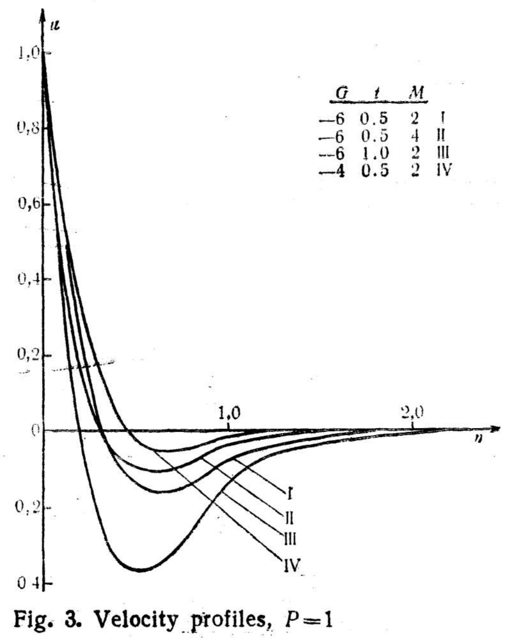

Fig. 3. Velocity profiles, $P=1$

leads to a decrease in the velocity when the values of G, t, M are constant. An increase in M or t leads to a decrease in the velocity. But greater cooling of the plate by the free convection currents causes a rise in the velocity. Quantitatively, for $t=0{,}5$, $M=2$, $P=0.5$, there is 76 % rise in the velocity at $\eta=1$, when G is increased from 2 to 4 owing to greates cooling of the plate.

On Figs. 3 and 4 the velocity profiles are shown for fluids whose Prandtl number is unity. The trend is the same as described above.

Knowing the velocity field, we can now calculate the skin friction. It is defined as

$$\tau' = -\mu\left(\frac{\partial u'}{\partial y'}\right)_{y'=0} \tag{13}$$

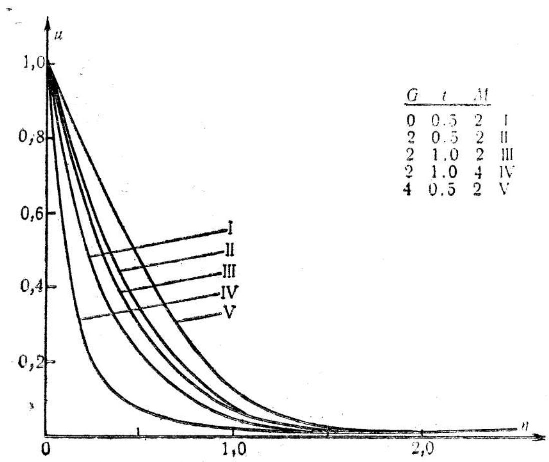

Fig. 4. Velocity profiles, $P=1$

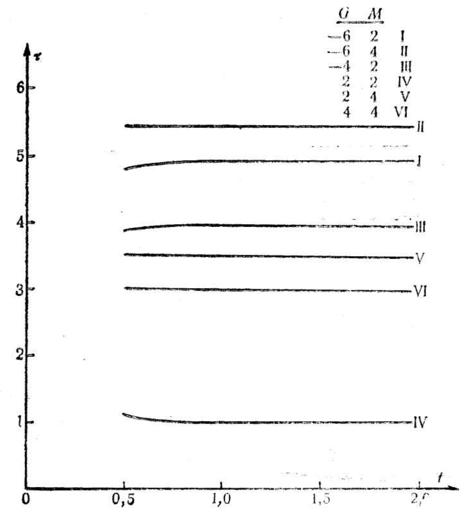

Fig. 5. Skin-Friction, $P=0.7$

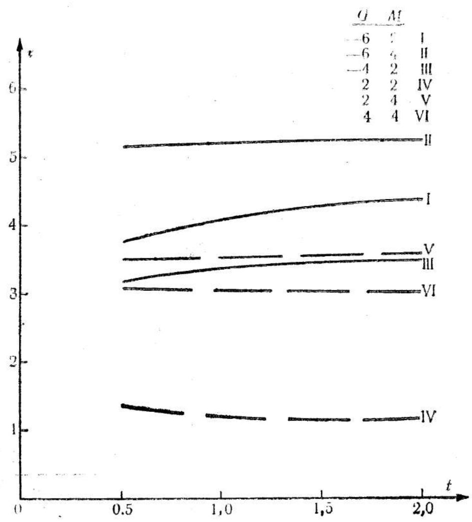

Fig. 6. Skin-Friction, $P=1$

and in virtue of (4), (13) reduces to

$$\tau=\tau'/\varrho U_0^2=-\left(\frac{\partial u}{\partial \eta}\right)_{\eta=0}.$$

Substituting (10) and (11) in (14), we have for $P<1$

$$(15)\quad \tau=\frac{G}{M^2\sqrt{\pi t}}+\left(1-\frac{G}{M^2}\right)\left(\frac{1}{\sqrt{\pi t}}\,\mathrm{e}^{M^2 t}+M\operatorname{erf}(M\sqrt{t})\right)-\frac{G}{M^2\sqrt{\pi t}}\,\mathrm{e}^{bt}\,(\sqrt{P}\,\mathrm{e}^{bt}-\mathrm{e}^{bPt},$$

and for $P=1$,

$$(16)\quad \tau=\left(1-\frac{G}{M^2}\right)\left(\frac{1}{\sqrt{\pi t}}\,\mathrm{e}^{-M^2 t}+M\operatorname{erf}(M\sqrt{t})\right)+\frac{G}{M^2\sqrt{\pi t}}.$$

τ is plotted on Figs. 5 and 6 for $P=0.7$ and 1 respectively. We conclude from Fig. 5 that the value of the skin-friction is more, when the plate is being heated by the free convection currents. For $G<0$ (heating of the plate), an increase in M leads to an increase in the skin-friction whereas owing to greater heating of the plate by the free convection currents, there is a rise in the skin-friction. There is also a rise in the skin-friction when M is increased for $G>0$, but owing to greater cooling of the plate there is a fall in the value of the skin-friction. The same is the trend of the skin-friction for $P=1$.

3. Conclusions

1. In the presence of the plate being heated by free convection currents, the velocity profiles are of reverse type.
2. Owing to greater heating of the plate, the velocity decreases whereas due to greater cooling of the plate the velocity increases.
3. Owing to greater heating of the plate, the skin-friction increases and due to greater cooling of the plate it decreases.

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