БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ
ТЕОРЕТИЧНА И ПРИЛОЖНА МЕХАНИКА
София · 1977, год. VIII, № 1

BULGARIAN ACADEMY OF SCIENCES
THEORETICAL AND APPLIED MECHANICS
Sofia · 1977, Year VIII, No. 1

The flow field induced by an oscillating porous sphere

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1. Introduction

It has been experimentally established that when a fluid is set into oscillation, as in the presence of a sound source or an oscillating boundary, steady streaming motions are generated under certain conditions. Streaming of this kind had already been reported by Faraday [1] and Dvorak [2] in the last century. A theoretical explanation (not based on the ideas of boundary layer theory) of some of the phenomena associated with the above experiments was given for the first time by Rayleigh [3]. Carrière [4], Andrade [5] and Schlichting [6] studied the time independent streaming induced by oscillating solid cylinders and spheres in an incompressible fluid. Then their experiments led to a contradiction.

The shape and direction of the induced steady circulating flows depend upon the geometry of the oscillating body. Carrière studying the streaming near a circular cylinder, observed flow patterns to be directed toward the cylinder along the line of propagation, while Andrade and Schlichting studying the same problem observed a qualitatively similar steady flow, but with a motion in the opposite direction. After some years when the foregoing experiments were re-examined it was found that the Reynolds number was $Re \sim 10$ for the experiments of Carrière, while $Re \sim 1000$ for the experiments of Schlichting and Andrade. So Schlichting's boundary layer treatment is valid for large Re and cannot be expected to describe the flow in the experiments of Carrière.

In a theoretical work, carried out at the same time, Schlichting found out a solution periodic in time by the approximation method of the two dimensional non-steady boundary layer equations. It was found that only the boundary conditions at the cylinder can be satisfied, and that at a large distance the tangential component of velocity is finite but not zero. Unable to satisfy that condition he relaxes it to a stipulation that the tangential velocity remains bounded at the edge of the boundary layer. This result indicates that a potential flow which is periodic with respect to time induces a steady secondary motion at a large distance from the wall of the body. The case of an oscillating circular cylinder has been investigated theoretically and by Andres & Ingard [7], Holtsmark et al. [8] and Raney et al. [9]. Following the same method of analysis, Lane [10] has studied the problem of an oscillating sphere.

If a body of typical dimension a oscillates with velocity $U_\infty \cos \omega t$ in a fluid, of kinematic viscosity ν, which is otherwise at rest we can display the four dimensionless parameters which have appeared

$$\varepsilon = \frac{U_\infty}{\omega a}, \quad \bar{M}^2 = \frac{\omega a^2}{\nu}, \quad R_e = \frac{U_\infty a}{\nu}, \quad R_{e_s} = \frac{U_\infty^2}{\omega \nu}.$$

We are concerned furthermore with the situation $\varepsilon \ll 1$. Physically this condition implies that the amplitude of the oscillation is small compared with a. It was Stuart [11] who recognized that the parameter R_{e_s} is nothing else than the Reynolds number based on a velocity $O(\varepsilon U_\infty)$, which is the order of magnitude of the induced mean velocity. From the usual concepts of boundary layer theory Stuart [12] and Riley [13] showed that if $R_{e_s} \gg 1$, in addition to the unsteady boundary layer there exists a second boundary layer of thickness $O\left(\frac{R_{e_s}^{1/2}}{a}\right)$ in which the steady streaming decays to zero. Wang [14] examines the case of an oscillating sphere when $R_{e_s} = O(1)$, but he does not introduce that parameter explicitly. Wang's analysis is incomplete because the structure of the outer steady streaming is not determined. Recently Riley [15—19] has systematically applied the method of higher-order boundary-layer theory to study many problems related with the vibrating bodies in an unbounded fluid. However, many points remain to be answered.

In this paper we shall study the steady streaming induced by a porous sphere oscillating along a diameter in a viscous fluid, which is otherwise at rest. The assumptions we shall make are small amplitude of oscillation and small parameter of frequency M. Since $\varepsilon \ll 1$ and $\varepsilon = \frac{R_e}{\bar{M}^2}$ we have $R_e \ll \bar{M}^2$.

2. Formulation of the problem

Consider a porous sphere with a radius a immersed in an infinite expanse of homogeneous viscous fluid initially at rest. Adopt spherical polar coordinates at the center of the sphere with the axis Oz pointing upstream parallel to the transverse vibration with speed $U_\infty \cos \omega t$ at infinity. The velocity components (v_r', v_θ') are related to the stream function ψ' by

$$v_r' = \frac{1}{r'^2 \sin \theta} \cdot \frac{\partial \psi'}{\partial \theta}, \quad v_\theta' = -\frac{1}{r' \sin \theta} \cdot \frac{\partial \psi'}{\partial r'}. \tag{1}$$

Non-dimensional variables are now introduced according to the scheme

$$r' = ar, \; t' = \omega^{-1}\tau, \quad v_r' = U_\infty \cdot v_r, \quad v_\theta' = U_\infty \cdot v_\theta, \quad \psi' = U_\infty a^2 \psi, \quad p' = p \varrho U_\infty^2 R_e^{-1},$$

where p is the pressure, U_∞ is a typical velocity, ω^{-1} is a typical time and the radius a of the sphere is a typical length. The governing equation of motion in terms of non-dimensional variables is

$$\bar{M}^2 \frac{\partial (D^2 \psi)}{\partial \tau} + R_e \left\{ \frac{1}{r^2} \cdot \frac{\partial (\psi, D^2 \psi)}{\partial (r, \mu)} + \frac{2}{r^2} \cdot D^2 \psi \, . \, L\psi \right\} = D^4 \psi, \tag{2}$$

where

$$D^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{(1-\mu^2)}{r^2} \cdot \frac{\partial^2 \psi}{\partial \mu^2}, \quad L\psi = \frac{\mu}{(1-\mu^2)} \cdot \frac{\partial \psi}{\partial r} + \frac{1}{r} \cdot \frac{\partial \psi}{\partial \mu}, \quad \mu = \cos \theta. \tag{3}$$

The appropriate boundary conditions which must be satisfied by the solutions inside and outside the sphere are:
(i) normal velocity is continuous on the boundary:

$$v_r|_{r=1+0}=\hat{v}_r|_{r=1-0}, \quad [\psi=\hat{\psi} \text{ at } r=1] \tag{4}$$

(ii) tangential velocity, i. e. velocity of slip vanishes on the boundary:

$$v_\theta|_{r=1+0}=\hat{v}_\theta|_{r=1-0}=0, \quad \left[\frac{\partial\psi}{\partial r}=\frac{\partial\hat{\psi}}{\partial r}=0 \text{ at } r=1\right]. \tag{5}$$

The boundary condition satisfied by ψ as $r\to\infty$ is

$$\psi\sim\frac{1}{2}r^2(1-\mu^2)\,e^{ir}. \tag{6}$$

Here and what follows the real part of any complex quantity is to be understood.

Uniqueness of the solution for small R_e is ensured by the additional conditions that the velocity must be a finite continuous function of position at the center $r=0$

$$\hat{\psi}=\frac{\partial\hat{\psi}}{\partial r}=0. \tag{7}$$

As we have already indicated we concern ourselves entirely with the case for which $\varepsilon\ll 1$ and $\bar{M}\ll 1$.

If $\psi(r, \theta, R_e)$ is an exact solution to equation (2) we assume that as $R_e\to 0$ the space round the sphere divides into two separate but overlapping regions.

The inner (Stokes) and outer (Oseen) expansions for the region $r\geq 1$ are assumed to be of the form

$$\psi=\psi_1+R_e\psi_2+\cdots \tag{8}$$

$$\Psi=\Psi_1+R_e\Psi_2+\cdots \tag{9}$$

In the Oseen region we introduce strained co-ordinates

$$\varrho=Mr; \quad \Psi=M^2\psi.$$

For the region $r\leq 1$ we seek the solution of the form

$$\hat{\psi}=\hat{\psi}_1+R_e\hat{\psi}_2+\cdots \tag{10}$$

The expressions obtained from (8) and (9) by truncating a given number of terms satisfy approximately the exact equation of motion and the boundary conditions, with residuals which are bounded uniformly in separate but overlapping regions to known order in R_e. There is no guarantee that for any given positive R_e the series, formed by taking an infinite number of terms in the expansions, will even converge, let alone converge to the true solution. Thus they are at best exact solutions of approximate equations, and are only asymptotic representations ot the solution of the problem as $R_e\to 0$. This particular gloss runs through and through mathematical physics, and though it must be recognized, it is not a limitation peculiar to this analysis.

3. Construction of the solution

A. First-order solution

Substituting (8) in equation (2) and equating the terms free of R_e on both sides we have

$$\bar{M}^2 \frac{\partial}{\partial \tau}(D^2\psi_1) = D^4\psi_1. \tag{11}$$

The boundary conditions for ψ_1 are (4) — (6). If we seek a solution for ψ_1 of the form

$$\psi_1 = F(r, \bar{M})(1-\mu^2)\,e^{i\tau} \tag{12}$$

we find

$$\left[\frac{\partial^2}{\partial r^2} - \left(\frac{2}{r^2} + M^2\right)\right]\left(\frac{\partial^2 F}{\partial r^2} - \frac{2}{r^2}F\right) = 0, \tag{13}$$

where $M^2 = i\bar{M}^2$.

The general solution of this equation is

$$F(r, M) = A_1 r^2 + A_2 \cdot \frac{1}{r} + \frac{B_1}{M^2}\left(\frac{1}{r} - M\right)e^{Mr} + \frac{B_2}{M^2}\left(\frac{1}{r} + M\right)e^{-Mr},$$

where A_1, A_2, B_1, B_2 are constants.
Using boundary conditions (4) — (6) we obtain

$$F(1) = m_0, \quad F'(1) = 0, \quad F(\infty) = \frac{1}{2}r^2,$$

where m_0 (whose range will be determined later) is some unknown constant called permeability [20, 21].
Therefore

$$\psi_1 = \frac{1}{2}\left[r^2 - \frac{M^2(1-m_0) + (3-m_0)(1+M)}{rM^2} + \frac{(3-m_0)}{M}\left(\frac{1}{Mr} + 1\right)e^{M(1-r)}\right](1-\mu^2)\,e^{i\tau}. \tag{14}$$

So in (14) the first two terms (which correspond respectively to a free stream and a doublet) are irrotational, and the third one is rotational.

The solution of equation (11) that satisfies conditions (4), (5) and (7) is

$$\hat{\psi}_1 = \frac{m_0}{(M^2-3M+3)e^{M} - (M^2+3M+3)e^{-M}}\left\{\frac{1}{2}[(M^2-M+1)e^{M} - (M^2+M+1)e^{-M}]r^2 + \left(\frac{1}{r} - M\right)e^{Mr} - \left(\frac{1}{r} + M\right)e^{-Mr}\right\}(1-\mu^2)\,e^{i\tau}. \tag{15}$$

If in (14) and (15) we keep r fixed and let $\bar{M} \to 0$, we find that

$$\psi_1(r, \mu, \tau) = \frac{1}{2}\left[r^2 - \frac{(3-m_0)}{2}\cdot r + \frac{m_0+1}{2r}\right](1-\mu^2)\,e^{i\tau} + O(\bar{M}) \tag{16}$$

and

$$\hat{\psi}_1(r, \mu, \tau) = \frac{m_0}{2}r^2(2-r^2)(1-\mu^2)\,e^{i\tau} + O(\bar{M}). \tag{17}$$

If now we keep $\varrho = Mr$ fixed and let $\bar{M} \to 0$ we obtain the first approximation of the outer region

(18) $$\Psi_1(\varrho,\mu,\tau)=\frac{1}{2}\left\{\varrho^2-\frac{M(3-m_0)}{\varrho}+(3-m_0)M\left(1+\frac{1}{\varrho}\right)e^{-\varrho}\right\}(1-\mu^2)\,e^{i\tau}+O(\bar{M}).$$

B. Second-order solution

If we substitute (10) in (2) for the $O(R_e)$ term we have the equation

(19) $$\bar{M}^2\frac{\partial D^2\hat{\psi}_2}{\partial\tau}+\frac{1}{r^2}\left[\frac{\partial(\hat{\psi}_1, D^2\hat{\psi}_1)}{\partial(r,\mu)}+2D^2\hat{\psi}_1 L\hat{\psi}_1\right]=D^4\hat{\psi}_1,$$

which is valid inside the sphere.

Since we expect $\hat{\psi}_2$ to contain terms independent of τ we write

(20) $$\hat{\psi}_2=\frac{1}{2}(\hat{\psi}_{20}+e^{2i\tau}\hat{\psi}_{22})\mu(1-\mu^2).$$

The solutions for $\hat{\psi}_{20}$ and $\hat{\psi}_{22}$ that satisfy the conditions (5) and (7) are

$$\hat{\psi}_{2n}=C_n\cdot\frac{r^3}{3}(3r^2-5)\quad(n=0,2).$$

Therefore

(22) $$\hat{\psi}_2=\frac{1}{6}(3r^5-5r^3)(C_0+C_2e^{2i\tau}),$$

where the constants C_0 and C_2 are to be determined later by the condition (4).

Further we concentrate on the second-order solution in the Stokes region. We again, as above, decompose second approximation near the sphere, as

(23) $$\psi_2=\frac{1}{2}(\psi_{20}+\psi_{22}e^{2i\tau}).$$

Substituting (23) into (2) and using ψ_1 given by (16) we see that the equations satisfied by $\psi_{2n}(r,\mu)$, $(n=0,2)$ are

(24) $$D^4\psi_{2n}=-\frac{3(3-m_0)}{4r^3}\left[2r-(3-m_0)+\frac{m_0+1}{r^2}\right]\mu(1-\mu^2)+O(M).$$

The boundary conditions of this equation are

(25) $$\frac{\partial\psi_{2n}}{\partial r}=0\quad\text{at}\quad r=1,$$

(26) $$\psi_{2n}=\hat{\psi}_{2n}\quad\text{at}\quad r=1$$

and

(27) $$\lim_{r\to\infty}\psi=\lim_{\varrho\to 0}\Psi.$$

If we now seek a solution of this equation in the form

$$\psi_{2n}=f_n(r)\mu(1-\mu^2)$$

we obtain the following equation for f_n

(28) $$f_n^{\mathrm{IV}}-\frac{12}{r^2}f_n^{\mathrm{II}}+\frac{24}{r^3}f_n^{\mathrm{I}}=-\frac{3(3-m_0)}{4}\left[\frac{2}{r^2}-\frac{3-m_0}{r^3}+\frac{m_0+1}{r^5}\right].$$

The general solution of this equation is

$$f_n(r)=A_{on}+B_{on}\cdot\frac{1}{r^2}+C_{on}\cdot r^3+D_{on}r^5-\frac{(3-m_0)}{32}\left[2r^2-(3-m_0)r-\frac{m_0+1}{2}\right].$$

In view of boundary equations (25) and (26) we have

$$-2B_{on}+3C_{on}+5D_{on}=\frac{(3-m_0)}{16}(1+m_0), \tag{29}$$

$$A_{on}+B_{on}+C_{on}+D_{on}=-\frac{(3-m_0)}{16}-\frac{2}{3}m_k\cdot(k=1,2), \tag{30}$$

where $m_1=C_0$ and $m_2=C_2$ are permeability constants of second order.

To complete the solution of our problem to the second order we must, of course, consider the outer region, far from the sphere. If we suppose that

$$\Psi_2=\frac{1}{2}(\Psi_{20}+\Psi_{22}\,e^{2i\tau}) \tag{31}$$

and substitute into equations (9) and (2) we obtain

$$Д^4\Psi_{2n}-nД^2\Psi_{2n}=-\frac{(3-m_0)}{2}\left(1+\frac{3}{\varrho}+\frac{3}{\varrho^2}\right)e^{-\varrho}\mu(1-\mu^2)+O(\bar{M}), \tag{32}$$

where $Д^2=\frac{\partial^2}{\partial\varrho^2}+\frac{(1-\mu^2)}{\varrho^2}\cdot\frac{\partial^2}{\partial\mu^2}$ and $(n=0,\ 2)$.

The function Ψ_2 should satisfy the matching condition (27) and requirement of zero velocities at infinity. Thus

$$\Psi_{2n}=O(\varrho^2)\quad\text{as}\quad\varrho\to\infty. \tag{33}$$

The solutions of (32) which hold the conditions (33) and the appropriate matching conditions are

$$\Psi_{20}(\varrho,\mu)=-\frac{(3-m_0)}{2}\left[\frac{1}{2}-\frac{3}{\varrho^2}+\left(1+\frac{3}{\varrho}+\frac{3}{\varrho^2}\right)e^{-\varrho}\right]\mu(1-\mu^2)+O(\bar{M}), \tag{34}$$

$$\Psi_{22}(\varrho,\mu)=-\frac{(3-m_0)}{2}\mu(1-\mu^2)\left[\frac{3}{2\varrho^2}+\left(1+\frac{3}{\varrho\sqrt{2}}+\frac{1}{2\varrho^2}\right)e^{-\varrho\sqrt{2}}-\left(1+\frac{3}{\varrho}+\frac{3}{\varrho^2}\right)e^{-\varrho}\right]. \tag{35}$$

Hence

$$C_{00}=D_{00}=0, \qquad C_{02}=D_{02}=0,$$

$$B_{00}=-\frac{(3-m_0)}{32}(1+m_0), \qquad A_{02}=\frac{(3-m_0)}{32}(m_0-1)-\frac{2}{3}m_2,$$

$$A_{00}=\frac{(3-m_0)}{12}(m_0-1)-\frac{2}{3}m_1. \qquad B_{02}=-\frac{(3-m_0)}{32}(1+m_0).$$

Thus we have found the solutions:

1) Interior expansion valid inside the sphere:

$$\hat{\psi}=\frac{m_0}{2}r^2(2-r^2)(1-\mu^2)\,e^{i\tau}+$$

$$+\frac{R_e\cdot r^3}{6}[(3r^2-5)m_1+(3r^2-5)m_2\cdot e^{2i\tau}]\mu(1-\mu^2)+O(\bar{M})+\cdots \tag{36}$$

2) An inner expansion (Stokes expansion) which holds near the surface of the sphere

$$\psi=\frac{1}{4}\left[2r^2-(3-m_0)r+\frac{m_0+1}{r}\right](1-\mu^2)\,e^{i\tau}- \tag{37}$$

$$-R_e\frac{(3-m_0)}{64}\left\{\left[2r^2-(3-m_0)r+\frac{m_0+1}{r^2}-\frac{m_0+1}{r}-(m_0-1)+\frac{64}{3(3-m_0)}m_1\right]+\right.$$

$$\left.\left[2r^2-(3-m_0)r+\frac{m_0+1}{r^2}-\frac{m_0+1}{r}-(m_0-1)+\frac{64}{3(3-m_0)}m_2\right]e^{2i\tau}\right\}\mu(1-\mu^2)+O(\bar{M})+\cdots$$

3) An outer expansion (Oseen expansion) which holds in the region far away from the sphere:

$$\Psi=\frac{1}{2}\left[\varrho^2-\frac{M(3-m_0)}{\varrho}+(3-m_0)M\left(\frac{1}{\varrho}+1\right)e^{-\varrho}\right](1-\mu^2)\,e^{i\tau}-$$

$$-R_e\frac{(3-m_0)}{4}\left\{\left[\frac{1}{2}-\frac{3}{\varrho^2}+\left(1+\frac{3}{\varrho^2}+\frac{3}{\varrho}\right)e^{-\varrho}\right]+\right. \tag{38}$$

$$\left.+\left[\frac{3}{2\varrho^2}+\left(1+\frac{1}{\varrho\sqrt{2}}+\frac{3}{2\varrho^2}\right)e^{-\varrho\sqrt{2}}-\left(1+\frac{3}{\varrho}+\frac{3}{\varrho^2}\right)e^{-\varrho}\right]e^{2i\tau}\right\}+O(\bar{M})+\cdots$$

3. Range of porosity parameters

Following Leonov [20] we can obtain the lower range of porosity parameter m_0 by putting the normal velocity equal to zero (or $\psi_1=\hat{\psi}_1=0$) which gives $m_0=0$. To obtain the upper range of the parameter m_0 we have to calculate the difference

$$\Delta p\,|_{r=1}=p\,|_{r=1+0}-\hat{p}\,|_{r=1-0}. \tag{39}$$

In the case for the first-order porosity the distribution outside and inside of the sphere are given respectively by the equations

$$\frac{\partial p}{\partial r}=\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}D^2\psi_1$$

$$\frac{\partial p}{\partial\theta}=-\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial r}D^2\psi_1 \tag{40}$$

and

$$\frac{\partial\hat{p}}{\partial r}=\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}D^2\hat{\psi}_1,$$

$$\frac{\partial\hat{p}}{\partial\theta}=-\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial r}D^2\hat{\psi}_1. \tag{41}$$

After some algebra we have found the expression for pressure outside and inside the sphere respectively:

$$p=p_0-\frac{\cos\theta}{2r^2}(3-m_0)\,e^{i\tau},$$

$$\hat{p}=p_0-10m_0r\cos\theta\,e^{i\tau}, \tag{42}$$

where p_0 is an arbitrary constant.

Therefore

$$\Delta p\,|_{r=1} = \cos\theta\left[\frac{3-m_0}{2} - 10m_0\right] e^{i\tau} = 0,$$

when $m_0 = \frac{1}{7}$ and we have $0 \leq m_0 \leq \frac{1}{7}$.

It is evident that in the case of $m_0 = 0$ the sphere will behave as a solid sphere without porosity.

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Received 5. X. 1976