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Effect of Outflow Velocity and Physico-Chemical Properties on the Stability of a Cylindrical Jet in a Liquid-Liquid System

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The problem about the stability of a jet in a two-phase system has been the object of a number of investigations, generally subdivided into two basic groups. One of these involves studies in which the effect of viscosity is neglected. Plateau [1] first stated the idea that a liquid cylinder subjected to the action of superficial forces becomes unstable when its length surpasses its circumference. Rayleigh [2] proposed a linear setting of the problem of stability of a non-viscous liquid jet flowing into a gas. Later, the problem thus stated was solved [3, 4, 5]. Analysis of the stability of non-viscous jets accounting for non-linear effects was made in [7, 8, 9, 10], while in [11, 12, 13] the influence of the properties and pressure of the surrounding gaseous medium was taken into consideration.

The second group of investigations account in the analysis for the viscosity effect [6, 14, 15]. Tomotika [6] studied the capillary instability of a viscous liquid jet, surrounded by another viscous liquid, assuming the relative flow velocity of both fluids to be zero. On the basis of Tomotika's results, L'ishevskii [14] and Sheele and Meister [15] considered the case of a relative velocity different from zero, for both fluids. The numerical analysis of the obtained relationships, however, was made without accounting for the influence of viscosity.

The present work had the purpose of studying the stability of a viscous cylindrical laminar jet, flowing into another viscous liquid, in the case of uniform velocity profiles of the main flows of both fluids.

Let us assume a jet flowing with a mean velocity of U_N out of a circular nozzle into the semi-space of another immiscible liquid. Due to the axisymmetry of the problem, the equations of motion for the jet and the surrounding liquid will be written in a cylindrical co-ordinate system with the X-axis along the direction of the jet axis. As conventionally done in the analysis of hydrodynamic stability [16], the solution of these equations will be expressed as the sum of two terms, the one describing the main laminar flows and the other — the disturbed flow. We assume further that the disturbances are symmetrical with

respect to the X-axis. Following the standard procedure and on replacing in the equations of Navier-Stokes, neglecting the terms which are non-linear with respect to the disturbed flow, one obtains:

$$
\begin{aligned}
&\frac{\partial \tilde{u}_j}{\partial t}+\frac{\partial(U_j\tilde{u}_j)}{\partial x}+V_j\frac{\partial \tilde{u}_j}{\partial r}+\tilde{v}_j\frac{\partial U_j}{\partial r}=-\frac{1}{\varrho_j}\frac{\partial \tilde{p}_j}{\partial x}+\nu_j\nabla^2\tilde{u}_j,\\
&\frac{\partial \tilde{v}_j}{\partial t}+U_j\frac{\partial \tilde{v}_j}{\partial x}+\tilde{u}_j\frac{\partial V_j}{\partial x}+\frac{\partial(V_j\tilde{v}_j)}{\partial r}=-\frac{1}{\varrho_j}\frac{\partial \tilde{p}_j}{\partial r}+\nu_j\left(\Delta^2\tilde{v}_j-\frac{\tilde{v}_j}{r^2}\right),\\
&\frac{\partial(\tilde{u}_j r)}{\partial x}+\frac{\partial(\tilde{v}_j r)}{\partial r}=0 \qquad (j=1,\ 2).
\end{aligned}
\tag{1}
$$

System (1) is a fundamental system of linear partial differential equations for determining the disturbed hydrodynamic parameters in the jet and in the surrounding medium (marked by a wave). The necessary boundary conditions are obtained from the non-slip conditions at the interface [17] and should be imposed both upon the main and upon the disturbed flows. When applied for a disturbed motion, they have the form

$$\tilde{u}_1=\tilde{u}_2 \qquad \text{for } r=\delta, \tag{2}$$

$$\tilde{v}_1=\tilde{v}_2 \qquad \text{for } r=\delta, \tag{3}$$

$$\tilde{p}_{rx_1}=\tilde{p}_{rx_2} \qquad \text{for } r=\delta, \tag{4}$$

$$\tilde{p}_{rr_1}=\tilde{p}_{rr_2}=\tilde{p}_T \qquad \text{for } r=\delta, \tag{5}$$

wherein

$$\tilde{p}_T=-T\left(\frac{\tilde{\delta}}{\delta^2}+\frac{\partial^2\tilde{\delta}}{\partial x^2}\right),$$

while $\tilde{\delta}$ is determined from the following equation:

$$\frac{\partial \tilde{\delta}}{\partial t}+U_j\frac{\partial \tilde{\delta}}{\partial x}+\tilde{u}_j\frac{\partial \delta}{\partial x}\Big|_{r=\delta}=\tilde{v}_j\Big|_{r=\delta}, \tag{7}$$

in which one may equivalently use either U_1, $\tilde{u}_1$, $\tilde{v}_1$ or U_2, $\tilde{u}_2$, $\tilde{v}_2$, because according to (2) and (3) at the jet surface $\tilde{v}_1=\tilde{v}_2$, $\tilde{u}_1=\tilde{u}_2$ and analogously for the main flows.

The final stating of the problem about the jet stability acquires — on linear approximation — the following form: finding a solution of system (1) with boundary conditions (2)—(7). The hydrodynamic values of the main (undisturbed) flows also participate in (1)—(7).

Following Rayleigh [2] the modern linear theory of stability of liquid jets is based on the assumption that both the jet and its surrounding fluid are in uniform rectilinear flow. The results thus derived for jet stability are in good qualitative agreement with those experimentally obtained. Besides these results appropriately modified can be applied when studying the effects of both the velocity and the jet profile relaxation in case of real jets.

Assuming together with [2, 14, 15] that the velocity profiles in the (cylindrical) jet and the surrounding phase are rectilinear in (1) and (7) we have $U_1=U_N=\text{const}$; $U_2=U_a=\text{const}$.

Applying the well-known method of linear stability we shall represent the stream function for the disturbed motion (in case of symmetrical waves) in the form $\tilde{\psi}_j=\Phi_j(r)\,e^{ikz+\alpha t}$.

Thus from (1) and (7) we obtain the following equation given in dimensionless form and related the growth rate of the disturbance α to liquid properties, jet velocity, jet diameter and wave number of the disturbance:

$$(8)\qquad \begin{vmatrix} \varepsilon K_0(\varepsilon) & \varkappa_2 K_0(\varkappa_2) & \varepsilon I_0(\varepsilon) & \varkappa_1 I_0(\varkappa_1) \\ K_1(\varepsilon) & K_1(\varkappa_2) & -I_1(\varepsilon) & -I_1(\varkappa_1) \\ 2\mu\varepsilon^2 K_1(\varepsilon) & \mu(\varepsilon^2+\varkappa_2^2)K_1(\varkappa_2) & -2\varepsilon^2 I_1(\varepsilon) & -(\varkappa_1^2+\varepsilon^2)I_1(\varkappa_1) \\ F_1 & F_2 & F_3 & F_4 \end{vmatrix} = 0$$

where

$$(9)\qquad F_1=\left(\alpha_1+i\varepsilon\gamma\sqrt{\frac{We_1}{2}}\right)^2\varrho K_0(\varepsilon)+4\mu\varepsilon Z_1\sqrt{\frac{We_1}{2}}\left(\alpha_1+i\varepsilon\gamma\sqrt{\frac{We_1}{2}}\right)[\varepsilon K_0(\varepsilon)+K_1(\varepsilon)]+\varepsilon(\varepsilon^2-1)K_1(\varepsilon)$$

$$(10)\qquad F_2=4\mu\varepsilon Z_1\sqrt{\frac{We_1}{2}}\left(\alpha_1+i\varepsilon\gamma\sqrt{\frac{We_1}{2}}\right)[\varkappa_2 K_0(\varkappa_2)+K_1(\varkappa_2)]+\varepsilon(\varepsilon^2-1)K_1(\varkappa_2)$$

$$(11)\qquad F_3=\left(\alpha_1+i\varepsilon\sqrt{\frac{We_1}{2}}\right)^2 I_0(\varepsilon)+4\varepsilon Z_1\sqrt{\frac{We_1}{2}}\left(\alpha_1+i\varepsilon\sqrt{\frac{We_1}{2}}\right)[\varepsilon I_0(\varepsilon)-I_1(\varepsilon)]$$

$$(12)\qquad F_4=4\varepsilon Z_1\sqrt{\frac{We_1}{2}}\left(\alpha_1+i\varepsilon\sqrt{\frac{We_1}{2}}\right)[\varkappa_1 I_0(\varkappa_1)-I_1(\varkappa_1)]$$

$$(13)\qquad \varkappa_1^2=\varepsilon^2+\frac{\alpha_1}{Z_1\sqrt{2}}+i\varepsilon\frac{\sqrt{We_1}}{2Z_1};\qquad \varkappa_2^2=\varepsilon^2+\frac{\alpha_1\varrho}{\mu Z_1\sqrt{2}}+\frac{i\varepsilon\sqrt{We_1}}{2Z_1}\frac{\gamma\varrho}{\mu}.$$

In [14] and [15] analogue equations have been obtained. But these equations have been solved only for some limiting cases which are to be mentioned further on. The main advantage of (8)—(13) is that it takes into consideration both the influence of the jet velocity and the viscosity on the jet stability.

Equation (8) is a non-linear algebraic equation, from which α_1 is determined as a function of ε. The main difficulty in solving this equation comes from the presence of α_1 as an argument of the Bessel functions. This equation acquires a simpler form in some limiting cases of the dimensionless parameters.

If we assume that a non-viscous liquid jet flows into a non-viscous medium, then the condition is fulfilled that

$$(14)\qquad \varkappa_2^1\gg\varepsilon^2,\ \varkappa_2^2\gg\varepsilon^2$$

and equation (8) may be reduced to

$$(15)\qquad \left(\alpha_1+i\varepsilon\sqrt{\frac{We_1}{2}}\right)^2\frac{I_0(\varepsilon)}{I_1(\varepsilon)}+\left(\alpha_1+i\varepsilon\gamma\sqrt{\frac{We_1}{2}}\right)\varrho\frac{K_0(\varepsilon)}{K_1(\varepsilon)}+\varepsilon(\varepsilon^2-1)=0.$$

At zero velocities of the main flows ($\gamma=0$, $We_1=0$) one obtains precisely the equation deduced by Christiansen [5] and cited also by Sheele and Meister [18] as a limiting case of Tomotika's solution. If furthermore the non-viscous jet flows into a gas the density of the surrounding fluid is significantly lower than the jet density and hence, approximately $\varrho=0$. On this assumption we find from (15) the equation

$$(16)\qquad \left(\alpha_1+i\varepsilon\sqrt{\frac{We_1}{2}}\right)^2\frac{I_0(\varepsilon)}{I_1(\varepsilon)}+\varepsilon(\varepsilon^2-1)=0,$$

which at a velocity of jet outflow $U_N=0$ ($We_1=0$, respectively) represents Rayleigh's result for a non-viscous liquid jet, flowing into a gas.

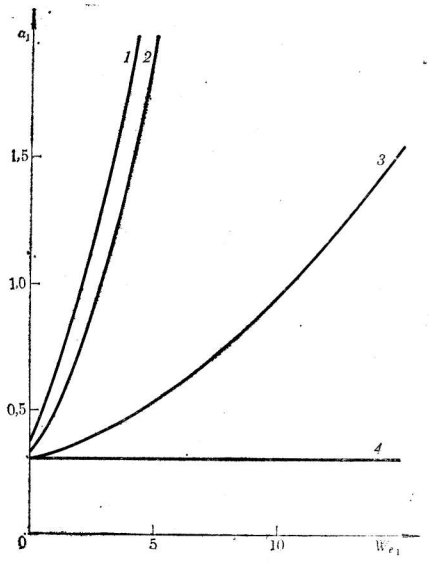

Fig. 1. Juxtaposing the solutions for the boundary cases, with numerical solution of equation (8), *1* — eq. (16); *2* — eq. (15); *3* — eq. (8); *4* — eq. (19).

In the case of a highly viscous liquid jet, flowing into a gas, the following condition is fulfilled:

$$\mu_1 \gg \mu_2,\ \varrho_1 \gg \varrho_2 . \tag{17}$$

With these assumptions, the values μ and ϱ are close to zero, so that the long waves will be most unstable, i. e. $\varepsilon \ll 1$. This offers the possibility of using the expansion of the Bessel functions at values of the argument smaller than unity [19]:

$$I_1(\varepsilon) \approx \frac{\varepsilon}{2};\ I_0(\varepsilon) \approx 1;\ I_1(\varkappa_1) \approx \frac{\varkappa_1}{2} \tag{18}$$

on which we obtain from (8)

$$\left(\alpha_1 + i\varepsilon \sqrt{\frac{We_1}{2}}\right)^2 + 3\varepsilon^2 Z_1 \left(\alpha_1 + i\varepsilon \sqrt{\frac{We_1}{2}}\right) - 2\varepsilon^2(\varepsilon^2 - 1) = 0. \tag{19}$$

When the velocity $U_N = 0$, i. e. $We_1 = 0$, equation (19) becomes identical with the limiting approximation of Tomotika's equation, obtained by Sheele and Meister.

The numerical solution of equations (15), (16) and (19) offers no difficulties. For the solution of the general equation (8) we employed the method of successive approximations. As initial step we took the solution of equation (15). Our solution was carried out with an accuracy up to 0.01% on a IBM 360-40 digital computer.

Fig. 1 shows a comparison of the numerical results on the maximum increase rate of disturbance, as a function of We_1 from equations (8), (15), (16), (19) for $\mu = 1.6$, $\varrho = 1.6$, $\gamma = 0$, $Z_2 = 0.236 \times 10^{-2}$. It is evident that at values of We_1 close to zero, the influence of viscosity is small and the approximate equations furnish results which approach the solution of the general equation (8). With the increase of We_1 approximations (15) and (16) in which the influence of viscosity has been neglected give a much greater jet instability than does the solution of the general equation (8), while equation (19) practically does not account for the effect of We_1. Except in the cases when We_1 is close to zero, the influence of viscosity can no more be neglected, due to which under these conditions the limiting approximations are rendered inapplicable.

The Ohnesorge number may be calculated both for the jet and for the surrounding liquid. The following relationship is valid:

$$Z_2 = Z_1 \frac{\mu}{\sqrt{\varrho}} . \tag{20}$$

It becomes evident from the above relationship that μ cannot be varied independently of Z_2 and Z_1. The change of μ may be realized in two ways: by fixing Z_2 which would mean a change of at the expense of the jet viscosity or by fixing Z_1 which is associated with a change of the viscosity of the surrounding liquid. The influence of jet viscosity upon the jet stability, expressed in terms of the dimensionless simplex μ and the Ohnesorge number Z_1, may be seen in Fig. 2 which shows the alteration of the increase rate of disturbance (as calculated from equation (8)) as a function of We_1 at a constant value of

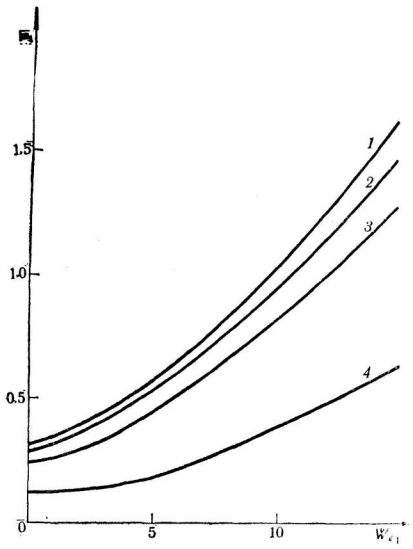

Fig. 2. Alteration of the disturbance increase rate α_1 calculated according to equation (8) in dependence with We_1 at different μ and Z_1, $1-\mu=360$, $Z_2=70.5.10^{-2}$; $2-\mu=50$, $Z_2=11.7.10^{-2}$; $3-\mu=10$, $Z_2=2.36.10^{-2}$; $4-\mu=0.5$; $Z_2=0.117,10^{-2}$

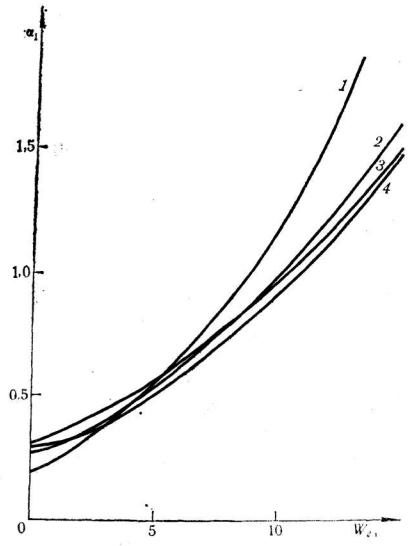

Fig. 3. Alteration of the disturbance increase rate α_1 calculated according to equation (8) in dependence with We_1 at different Z_2, $1-\mu=20$, $Z_1=0.93.10^{-4}$; $2-\mu=0.2$, $Z_1=0.93.10^{-1}$; $3-\mu=0.02$, $Z_2=0.93.10^{-1}$; $4-\mu=0.0033$, $Z_1=0.535$.

$Z_2=0.236\times10^{-2}$. The decrease of μ and the respective increase of Z_1 cause stabilization of the jet — more pronounced at greater We_1 values.

The increase of viscosity of the surrounding medium (the growih of μ and Z_2, respectively) has a stabilizing effect upon the jet only at values of the Weber number close to zero. With the increase of We_1 one observes an interesting effect, illustrated by Fig. 3, which shows α_1 plotted against We_1 at different values of μ and Z_2 and a constant values of $Z_1=0.185\times10^{-2}$. Irrespective of the jet stabilization at low Weber numbers, when a certain value of this number is surpassed, the increase of μ begins to have a destabilizing effect. Presumably, in those cases the neglection of the strong interaction between the main laminar flows is not justified.

Notation

a — nozzle radius, m
i — imaginary unit
$I_n(x)$ — modified Bessel function of the first order
$K_n(x)$ — modified Bessel function of the second order
$k=\frac{2\pi}{\lambda}$ — wave number, m^{-1}
p_{rx_j} — tangential component of the stress tensor, N/m^2
p_{rr_j} — normal component of the stress tensor, N/m^2
p_T — superficial tension, N/m^2
U_j — velocity component of the main flow along the X-axis
U_a — mean velocity of the surrounding liquid

U_N — mean jet velocity
$\widetilde{u}_j$ — velocity component of the disturbed flow along the X-axis
T — superficial tension coefficient, N/m
t — time, s
V_j — velocity component of the main flow along the z-axis
$\widetilde{v}_j$ — velocity component of the disturbed flow along the z-axis
$We_j=\frac{U_N^2 \varrho_j 2a}{T}$ — Weber number
$Z_j=\frac{\mu_j}{\sqrt{\varrho_j 2aT}}$ — Ohnesorge number

Greek symbols

α — increase rate of disturbance, s^{-1}
α_1 — dimensionless increase rate of disturbance
$\gamma=\frac{U_a}{U_N}$ — dimensionless ratio of the mean velocities
δ — undisturbed jet radius, m
$\widetilde{\delta}$ — disturbance at the jet surface, m
$\varepsilon=Ka$ — dimensionless wave number
$\varkappa_j$ — complex argument of the Bessel functions
λ — disturbance wave length, m^{-1}
μ_j — viscosity, $N.s/m^2$
ν_j — kinematic viscosity, m/s
ϱ_j — density, kg/m^3
$\varrho=\frac{\varrho_2}{\varrho_1}$ — dimensionless density

Subscripts

$j=1$ for the jet and 2 for the surrounding liquid

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