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Thermo-viscoelasticity of two continua in a small deformation field*

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1. Introduction

In recent years a number of authors have formulated some theories to describe the thermo-elastic behaviour of two continua (diatomic structure). It is worthy mentioning the polarization gradient theories of Mindlin [1], Asker et al. [2] who formulated the idea in [1] with use of the lattice theory, Marinov et al. [3, 4] who utilized a mean continuum and a poiarization vector and Demiray [5]. However, the theory of thermo-mechanical behaviour of linear visco-elastic two continua is not developed. The present paper is an extention of our work [4]. The existence of a visco-elastic potential of linear two continua is shown.

2. Field equations

Let us consider at time t a material point P in undeformed body with uniform temperature. After the deformation, at time t, the material point P occupies position p while the temperature is in general no longer uniform. Denoted by x_j $(j=1, 2, 3)$ and x_i $(i=1, 2, 3)$ the co-ordinates of P and p in rectangular system used to describe the undeformed and deformed states respectively, Fig. 1.

The individual motion of each medium is represented by [4]

$$x_i^{(l)}=x_i^{(l)}(x_j,\ t) \quad (l=1,\ 2). \tag{2.1}$$

Displacement vectors of each medium in a small deformation field are

$$u_i^{(l)}=x_i^{(l)}(x_j,\ t) \qquad (l=1,\ 2). \tag{2.2}$$

Also the motion of two continua may be described if we consider a mean continuum with a polarization vector. The motion of the centre of masses and the polarization vector are determined by

$$x_i=\sum_l c^{(l)}x_i^{(l)}(x_j,\ t), \qquad (l=1,\ 2) \tag{2.3}$$

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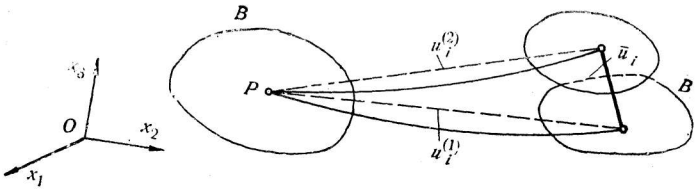

Fig. 1

(2.4) $$\bar{u}_i = u_i^{(1)} - u_i^{(2)} = x_i^{(1)}(x_j,\ t) - x_i^{(2)}(x_j,\ t),$$

where $c^{(l)} = \varrho^{(l)}/\varrho$, $\varrho = \sum_l \varrho^{(l)}$; $\varrho^{(l)}$ and ϱ are the partial and total mass densities.

The strain and the polarization measures are

(2.5) $$e_{ij}^{(l)} \quad u_{i,j}^{(l)} = x_{i,j}^{(l)} \quad u_i, \quad \bar{u}_{i,j} = x_{i,j}^{(1)} - x_{i,j}^{(2)} \quad \text{and} \quad x_{i,j}^{(l)} = \frac{\partial x_i^{(l)}}{\partial x_j}.$$

From the mathematical point of view, such a two-continuum may be considered as two media, coincident at an initial reference configuration B, provided that they interact with each other by internal forces $p_i^{(l)}$ [3, 4]. When the internal forces $p_i^{(l)}$ are central then we have

(2.6) $$\sum_l \varrho^{(l)} p_i^{(l)} = 0, \quad \sum_l \varrho^{(l)} e_{ijk} \bar{u}_j p_k^{(l)} = 0,$$

where e_{ijk} are permutation symbols.

The field equations, given in [4] are
balance of momentum,

2.7) $$\varrho \dot{v}_i = t_{ji,j} + \varrho F_i,$$

balance of moment of momentum,

(2.8) $$\varrho c^{(1)} c^{(2)} e_{ijk} \bar{u}_j \ddot{\bar{u}}_k = e_{ijk} x_{j,p} t_{pk} + \mathfrak{M}_{pi,p} + \varrho \mathcal{L}_i,$$

conservation of energy,

(2.9) $$\varrho \dot{\mathcal{E}} = t_{ji} \dot{x}_{i,j} + p_i \dot{\bar{u}}_i + g_{ji} \dot{\bar{u}}_{i,j} - q_{j,j}.$$

In these equations

$v_i = \dot{x}_i = \dot{u}_i$ — velocity vector of the center of masses,

$t_{ij} = \sum_l t_{ij}^{(l)}$ — total stress tensor (true tensor, $t_{ij} \neq t_{ji}$),

$t_{ij}^{(l)}$ = partial stress tensor (true tensor, $t_{ij}^{(l)} = t_{ji}^{(l)}$),

$F_i = \sum_l c^{(l)} F_i^{(l)}$ — total body force,

$F_i^{(l)}$ = partial body force,

$\mathfrak{M}_{ip} = e_{ijk} \bar{u}_j (c^{(2)} t_{pk}^{(1)} - c^{(1)} t_{pk}^{(2)}) = e_{ijk} \bar{u}_j g_{pk}$ — total mechanical tensor which results from polarization $\bar{u}_j$ (pseudotensor, $\mathfrak{M}_{ip} \neq \mathfrak{M}_{pi}$),

g_{pi} — total couple stress tensor (pseudotensor, $g_{pi} \neq g_{ip}$),

$\mathcal{E}_i = c^{(1)}c^{(2)}e_{ijk}\bar{u}_j(F_k^{(1)} - F_k^{(2)}) = e_{ijk}\bar{u}_j P_k$ — total mechanical force which results from polarization $\bar{u}_j$ (pseudotensor),

$\dot{\mathcal{E}}$ — time rate of change of the internal energy,

$p_i = \varrho c^{(1)} p_i^{(1)}$ — internal force and

$q_J = \sum_l q_i^{(l)}$ — total heat flux vector.

The local form of the entropy production inequality for the mean continuum is

$$\varrho\dot{\eta} - q_i/\theta^2\,\theta_{,i} + 1/\theta q_{i,i} \geq 0, \tag{2.10}$$

where θ denotes the temperature and η — the entropy production for the mean continuum.

Eq. (2.10) is the statement of the second law of thermodynamics. It only provides a thermodynamic constraint to all processes.

3. Thermomechanical constitutive equations

We consider a thermodynamic process initiated at time $t = t_0$ in a region B. It is assumed that at time $t = t_0 : \theta = \theta_0$ and $u_i = \bar{u}_i = t_{ij} = g_{ij} = p_i = 0$. Following Coleman [9] we define a thermodynamic process as the set of function u_i, $\bar{u}_i$ and θ such that

$$u_i = u_i(x_i, \tau), \qquad \bar{u}_i = \bar{u}_i(x_i, \tau), \quad \theta = \theta(x_i, \tau) \tag{3.1}$$

for all values of x_i in the region B and τ such that $t_0 \leq \tau \leq t$.

In the absence of any constitutive equations for the two-continuum, given $F_i(x_i, \tau)$ for all values of x_i in the region B and $t_0 \leq \tau \leq t$, the equations that are available for the determination of $u_i(t)$, $\bar{u}_i(t)$ and $\theta(t)$ are the field eqs. (2.7), (2.8) and (2.9) for the unknowns u_i, $\bar{u}_i$, t_{ij}, g_{ij}, p_i, θ, η, $\mathcal{E}$ and q_i. We have only 7 field equations for 30 unknowns.

It follows from eq. (2.9) that the polarization vector $\bar{u}_i$, the gradient of displacement vector $u_{ij} = u_{j,i}$, the gradient of polarization vector $\bar{u}_{ij} = \bar{u}_{j,i}$, the temperature θ and its gradient $\theta_{,i}$ will constitute our thermodynamtc variables. As our medium is behaving as viscoelastic, the proposed constitutive equations must involve the whole histories of $u_{ij}(\tau)$, $\bar{u}_i(\tau)$, $\bar{u}_{ij}(\tau)$ and $\theta(\tau)$ such that $t_0 \leq \tau \leq t$. This requirement is met by introducing into the constitutive equations the intrinsic co-ordinates ξ_α, which are simply functional of histories of $u_{ij}(\tau)$, $\bar{u}_i(\tau)$, $\bar{u}_{ij}(\tau)$ and $\theta(\tau)$.

$$\xi_\alpha = \xi_\alpha[u_{ij}(\tau), \bar{u}_i(\tau), \bar{u}_{ij}(\tau), \theta(\tau)], \quad t_0 \leq \tau \leq t. \tag{3.2}$$

Thus we can obtain the following constitutive equations for a thermo-viscoelastic two-continuum

$$\begin{aligned} t_{ij} &= t_{ij}(u_{ij}, \bar{u}_i, \bar{u}_{ij}, \theta, \xi_\alpha; \theta_{,i}), \\ g_{ij} &= g_{ij}(u_{ij}, \bar{u}_i, \bar{u}_{ij}, \theta, \xi_\alpha; \theta_{,i}), \\ p_i &= p_i(u_{ij}, \bar{u}_i, \bar{u}_{ij}, \theta, \xi_\alpha; \theta_{,i}), \\ \mathcal{E} &= \mathcal{E}(u_{ij}, \bar{u}_i, \bar{u}_{ij}, \theta, \xi_\alpha; \theta_{,i}), \end{aligned} \tag{3.3}$$

$$\eta=\eta(u_{ij},\ \bar{u}_i,\ \bar{u}_{ij},\ \theta,\ \xi_\alpha;\ \theta_{,i}),$$

$$q_i=q_i(u_{ij},\ \bar{u}_i,\ \bar{u}_{ij},\ \theta,\ \xi_\alpha;\ \theta_{,i})$$

with

(3.4) $$q_i(u_{ij},\ \bar{u}_i,\ \bar{u}_{ij},\ \theta,\ \xi_\alpha;\ 0)=0.$$

Now we have already 30 equations to solve for 30 unknows and we have for any random choice of deformation $u_i(x_i, \tau)$ polarization $\bar{u}_i(x_i, \tau)$ and temperature distribution $\theta(x_i, \tau)$ a unique admissible thermodynamic process in a region B.

It will be shown that t_{ij}, g_{ij}, p_i, $\mathcal{E}$ and η do not depend on $\theta_{,i}$. At this point we introduce the free energy

(3.5) $$\Psi=\mathcal{E}-\eta\theta.$$

For the free energy is valid

3.6) $$\Psi=\Psi(u_{ij},\ \bar{u}_i,\ \bar{u}_{ij},\ \theta,\ \xi_\alpha,\ \theta_{,i}).$$

Introducing into eq. (2.10) the time rate of change of the free energy Ψ we get

(3.7) $$\left(t_{ij}-\varrho\frac{\partial\Psi}{\partial u_{ij}}\right)\dot{u}_{ij}+\left(p_i-\varrho\frac{\partial\Psi}{\partial\bar{u}_i}\right)\dot{\bar{u}}_i+\left(g_{ij}-\varrho\frac{\partial\Psi}{\partial\bar{u}_{ij}}\right)\dot{\bar{u}}_{ij}$$

$$-\varrho\left(\eta+\frac{\partial\Psi}{\partial\theta}\right)\dot{\theta}-\varrho\frac{\partial\Psi}{\partial\xi_\alpha}\dot{\xi}_\alpha-\varrho\frac{\partial\Psi}{\partial\theta_{,i}}\dot{\theta}_{,i}-q_i/\theta\ \theta_{,i}\geq 0.$$

4. Non-isotermal constitutive equations

In this case thermal gradients are present. Because the inequality (3.7) may be satisfied for all constitutive variables we infer that

(4.1a, b, c, d) $$t_{ij}=\varrho\frac{\partial\Psi}{\partial u_{ij}},\quad p_i=\varrho\frac{\partial\Psi}{\partial\bar{u}_i},\quad g_{ij}=\varrho\frac{\partial\Psi}{\partial\bar{u}_{ij}},\quad \eta=-\frac{\partial\Psi}{\partial\theta},$$

(4.2e, f) $$\frac{\partial\Psi}{\partial\theta_{,i}}=0,\quad \theta\dot{\gamma}=-\varrho\frac{\partial\Psi}{\partial\xi_\alpha}\dot{\xi}_\alpha-q_i/\theta\,\theta_{,i}\geq 0,$$

where $\dot{\gamma}$ — time rate of change of the irreversible entropy per unit volume.

As a result of eq. (4.2e), it is noted that Ψ is not a function of $\theta_{,i}$. Similarly t_{ij}, p_i and g_{ij} are independent of $\theta_{,i}$. We consider now the inequality (4.2f). By prescribing u_i, $\bar{u}_i$ and θ for all values of τ in the interval $t_0\leq\tau\leq t$, the first term of inequality eq. (4.2f) is fixed. Furthermore, it is independent in $\theta_{,i}$. If $\theta_{,i}=0$, we obtain

(4.3) $$-\varrho\frac{\partial\Psi}{\partial\xi_\alpha}\dot{\xi}_\alpha\geq 0.$$

Similarly, since we can arbitrarily vary the second term of the inequality eq. (4.2f) by varying $\theta_{,i}$ for any fixed value of the first term, such as $\dot{\xi}_\alpha=0$, it follows that

(4.4) $$-q_i\theta_{,i} \geq 0.$$

On the basis of eqs. (4.1a, b, c, d) the following theorem is established: "In a linear two-continuum with the presence of temperal and spatial temperature variations the total stress tensor t_{ij}, the internal force p_i, the total couple stress tensor g_{ij} and the entropy η are derivable from a potential Ψ which is the free energy, where Ψ is a function of the gradient of displacement vector u_{ij}, the polarization vector $\bar{u}_i$, the gradient of polarization vector $\bar{u}_{ij}$, the temperature θ and the intrinsic co-ordinates ξ_α."

Following Coleman [9] we assume that the temperature variation from the reference temperature is small, i. e. $\theta=\theta_0+\zeta$, where ζ is small. Then eqs. (4.1d) and eqs. (4.2f) yield

(4.5) $$\varkappa=-\frac{\partial\Psi}{\partial\zeta},$$

(4.6) $$\theta\dot{\gamma}=-\varrho\frac{\partial\Psi}{\partial\xi_\alpha}\dot{\xi}_\alpha-1/\theta_0\zeta_{,i}\,q_i\geq 0,$$

where $\varkappa=\eta-\eta_0$ is the entropy increment associated with the above process.

Hence our irreversible thermodynamic processes are small departures from a thermodynamic equilibrium state. Then we can apply the principle of Onsager's reciprocal relations

(4.7) $$-Q_\alpha=-\varrho\frac{\partial\Psi}{\partial\xi_\alpha}=h_{\alpha\beta}\dot{\xi}_\beta+k_{\alpha_i}q_i$$

(4.8) $$-\frac{\zeta_{,i}}{\theta}=k_{i\alpha}\dot{\xi}_\alpha+b_{ij}q_j,$$

where $b_{ij}=b_{ji}$, $h_{\alpha\beta}=h_{\beta\alpha}$. Since q_i must vanish for $\zeta=0$, eq. (4.8) implies $k_{i\alpha}=0$. Thus eqs. (4.7) and (4.8) yield

(4.9) $$\varrho\frac{\partial\Psi}{\partial\xi_\alpha}+h_{\alpha\beta}\dot{\xi}_\beta=0,$$

(4.10) $$q_j=-\frac{1}{\theta b_{ij}}\zeta_{,i}=-b'_{ij}\zeta_{,i},$$

where $h_{\alpha\beta}$ and b'_{ij} must be positive definite matrices to ensure inequality eqs. (4.3) and (4.4).

We postulate now that the free energy Ψ is a smooth function and the reference state is defined by the condition $t_{ij}=p_i=g_{ij}=\xi_\alpha=\varkappa=\zeta=0$. By expanding the free energy in a Taylor series and neglecting the terms with powers higher than second order and also omitting the first order terms to satisfy the initial conditions, one obtains

$$\varrho\Psi=\frac{1}{2}a_{ijkl}u_{ij}u_{kl}+b_{ijkl}u_{ij}\bar{u}_{kl}+\frac{1}{2}c_{ijkl}\bar{u}_{ij}\bar{u}_{kl}+a_{ijk}u_{ij}\bar{u}_k$$

(4.11) $$+b_{ij\alpha}u_{ij}\xi_\alpha+C_{ijk}\bar{u}_{ij}\bar{u}_k+d_{ij\alpha}\bar{u}_{ij}\xi_\alpha+a_{ij}u_{ij}\zeta+b_{ij}\bar{u}_{ij}\zeta$$

$$+\frac{1}{2}c_{ij}\bar{u}_i\bar{u}_j+d_{i\alpha}\bar{u}_i\xi_\alpha+\frac{1}{2}e_{\alpha\beta}\xi_\alpha\xi_\beta+a_i\bar{u}_i\zeta+b_\alpha\xi_\alpha\zeta+\frac{1}{2}a\zeta^2.$$

Differentiating eq. (4.11) with respect to ξ_α and substituting into eq. (4.9) one yields

(4.12) $$h_{\alpha\beta}\dot{\xi}_\beta+e_{\alpha\beta}\xi_\beta+b_{ij\alpha}u_{ij}+d_{ij\alpha}\bar{u}_{ij}+d_{i\alpha}\bar{u}_i+b_\alpha\zeta=0,$$

where $b_{ij\alpha}$, $d_{ij\alpha}$, $d_{i\alpha}$, $e_{\alpha\beta}$, $h_{\alpha\beta}$ and b_α are material constants, however, they may depend on the reference temperature θ_0. Following Valanis [6,7] we assume that only the coefficients $h_{\alpha\beta}$ which describe the viscous process are extremely sensitive to small changes of the temperature

(4.13) $$h_{\alpha\beta}=h^o_{\alpha\beta}a_T[\zeta(t)]$$

and define a "reduced time" variable z by

(4.14) $$z=\int_0^t\frac{d\tau}{a_T[\zeta(t)]},$$

where $a_T[\zeta(\tau)]$ is a known function.

Introducing eq. (4.13) and eq. (4.14) into eq. (4.12) we get

(4.15) $$h^o_{\alpha\beta}\frac{d\xi_\beta}{dz}+e_{\alpha\beta}\xi_\beta+b_{ij\alpha}u_{ij}+d_{ij\alpha}\bar{u}_{ij}+d_{i\alpha}\bar{u}_i+b_\alpha\zeta=O\quad(\alpha,\beta\text{ not summed}).$$

Eq. (4.15) is a system of first order linear differential equation with constant coefficients with respect to intrinsic coordinates ξ_β. With the initial condition $\xi_\alpha(0)=0$ and $h^o_{\alpha\beta}$, $e_{\alpha\beta}$ to be positive definite symmetric matrix, the solution is

(4.16) $$\xi_\alpha=-\int_0^z\mathfrak{J}_{\alpha\beta}(z-\tau)[b_{ij\beta}u_{ij}(\tau)+d_{ij\beta}\bar{u}_{ij}(\tau)+d_{i\beta}\bar{u}_i(\tau)+b_\beta\zeta(\tau)]d\tau,$$

where the matrix $\mathfrak{J}_{\alpha\beta}(z)$ is symmetric with the character of fading memory

$$\mathfrak{J}_{\alpha\beta}(z)=\sum_{k=1}^{n}e^{-\lambda_k z}[A^{(1)}_{k\alpha\beta}+A^{(2)}_{k\alpha\beta}z+\ldots+A^{(m_k-1)}_{k\alpha\beta}z^{(m_k+1)}],$$

m_k is the possible multiplicity of the eigenvalue λ_k.

On the basis of eqs. (4.1a, b, c) we get

(4.17)
$$\frac{1}{\varrho}t_{ij}=a_{ijkl}u_{kl}+b_{ijkl}\bar{u}_{kl}+a_{ijk}\bar{u}_k+b_{ij\alpha}\xi_\alpha+a_{ij}\zeta,$$
$$\frac{1}{\varrho}p_i=a_{ikl}u_{kl}+c_{ikl}\bar{u}_{kl}+c_{ik}\bar{u}_k+d_{i\alpha}\xi_\alpha+a_i\zeta,$$
$$\frac{1}{\varrho}g_{ij}=b_{ijkl}u_{kl}+c_{ijkl}\bar{u}_{kl}+c_{ijk}\bar{u}_k+d_{ij\alpha}\xi_\alpha+b_{ij}\zeta.$$

Substituting eq. (4.16) into eq. (4.17) and assuming that

$$\mathfrak{J}_{\alpha\beta}(z)=\frac{d}{dt}\varkappa_{\alpha\beta}(z),$$

we obtain

(4.18) $$\frac{1}{\varrho}t_{ij}=\int_0^z A_{ijkl}(z-\tau)\frac{\partial u_{kl}}{\partial\tau}d\tau+\int_0^z B_{ijkl}(z-\tau)\frac{\partial\bar{u}_{kl}}{\partial\tau}d\tau$$

$$+\int_0^z A_{ijk}(z-\tau)\frac{\partial \bar{u}_k}{\partial \tau}\,d\tau+\int_0^z A_{ij}(z-\tau)\frac{\partial \zeta}{\partial \tau}\,d\tau,$$

$$\frac{1}{\varrho}p_i=\int_0^z A_{ikl}(z-\tau)\frac{\partial u_{kl}}{\partial \tau}\,d\tau+\int_0^z C_{ikl}(z-\tau)\frac{\partial \bar{u}_{kl}}{\partial \tau}\,d\tau$$

$$+\int_0^z C_{ik}(z-\tau)\frac{\partial \bar{u}_k}{\partial \tau}\,d\tau+\int_0^z A_i(z-\tau)\frac{\partial \zeta}{\partial \tau}\,d\tau,$$

$$\frac{1}{\varrho}g_{ij}=\int_0^z B_{ijkl}(z-\tau)\frac{\partial u_{kl}}{\partial \tau}\,d\tau+\int_0^z C_{ijkl}(z-\tau)\frac{\partial \bar{u}_{kl}}{\partial \tau}\,d\tau$$

$$+\int_0^z C_{ijk}(z-\tau)\frac{\partial \bar{u}_k}{\partial \tau}\,d\tau+\int_0^z B_{ij}(z-\tau)\frac{\partial \zeta}{\partial \tau}\,d\tau,$$

where $z=0$ is the time of initiation of the thermomechanical process and

$$A_{ijkl}(z)=a_{ijkl}+b_{ij\alpha}\varkappa_{\alpha\beta}(0)b_{klp}-b_{ij\alpha}\varkappa_{\alpha\beta}(z)b_{k\beta},$$
$$B_{ijkl}(z)=b_{ijkl}+b_{ij\alpha}\varkappa_{\alpha\beta}(0)d_{kl\beta}-b_{ij\alpha}\varkappa_{\alpha\beta}(z)d_{kl\beta},$$
$$C_{ijkl}(z)=c_{ijkl}+d_{ij\alpha}\varkappa_{\alpha\beta}(0)d_{kl\beta}-d_{ij\alpha}\varkappa_{\alpha\beta}(z)d_{kl\beta},$$
$$A_{ijk}(z)=a_{ijk}+b_{ij\alpha}\varkappa_{\alpha\beta}(0)d_{k\beta}-b_{ij\alpha}\varkappa_{\alpha\beta}(z)d_{k\beta},$$
$$A_{ikl}(z)=a_{ikl}+d_{i\alpha}\varkappa_{\alpha\beta}(0)b_{kl\beta}-d_{i\alpha}\varkappa_{\alpha\beta}(z)b_{kl\beta},$$
$$C_{ikl}(z)=c_{ikl}+d_{i\alpha}\varkappa_{\alpha\beta}(0)d_{kl\beta}-d_{i\alpha}\varkappa_{\alpha\beta}(z)d_{kl\beta},$$
$$C_{ijk}(z)=c_{ijk}+d_{ij\alpha}\varkappa_{\alpha\beta}(0)d_{k\beta}-d_{ij\alpha}\varkappa_{\alpha\beta}(z)d_{k\beta},$$
$$A_{ij}(z)=a_{ij}+b_{ij\alpha}\varkappa_{\alpha\beta}(0)b_{\beta}-b_{ij\alpha}\varkappa_{\alpha\beta}(z)b_{\beta},$$
$$C_{ik}(z)=c_{ik}+d_{i\alpha}\varkappa_{\alpha\beta}(0)d_{k\beta}-d_{i\alpha}\varkappa_{\alpha\beta}(z)d_{k\beta},$$
$$B_{ij}(z)=b_{ij}+d_{i\alpha}\varkappa_{\alpha\beta}(0)b_{\beta}-d_{ij\alpha}\varkappa_{\alpha\beta}(z)b_{\beta},$$
$$A_i(z)=a_i+d_{i\alpha}\varkappa_{\alpha\beta}(0)b_{\beta}-d_{i\alpha}\varkappa_{\alpha\beta}(z)b_{\beta}.$$

Since $\bar{u}_{kl}$, $\bar{u}_k$ are pseudotensors, hence $B_{ijkl}(z)$, $A_{ijk}(z)$ must be pseudotensors by the fact that t_{ij} is a true tensor. Also p_i, g_{ij} are pseudotensors, but ζ is a true scalar, this implies from eqs. $(4.18)_{2,3}$ that $A_i(z)$, $B_{ij}(z)$ should also be pseudotensors. Also the tensor $C_{ijk}(z)$ is a pseudotensor.

However, we shall consider only centrosymmetric-isotropic two-continuum (in such continuum all the material properties must be invariant with a central inversion and arbitrary rotation of co-ordinates), thus $B_{ijkl}(z)$, $A_{ijk}(z)$, $A_i(z)$ and $B_{ij}(z)$ must be equal to zero. Hence for a centrosymmetric-isotropic two-continuum, eqs. (4.18) become

$$\frac{1}{\varrho}t_{ij}=\int_0^z A_{ijkl}(z-\tau)\frac{\partial u_{kl}}{\partial \tau}\,d\tau+\int_0^z A_{ij}(z-\tau)\frac{\partial \zeta}{\partial \tau}\,d\tau, \tag{4.19}$$

$$\frac{1}{\varrho}p_i=\int_0^z C_{ik}(z-\tau)\frac{\partial \bar{u}_k}{\partial \tau}\,d\tau,$$

$$\frac{1}{\varrho}g_{ij}=\int_0^z C_{ijkl}(z-\tau)\frac{\partial \bar{u}_{kl}}{\partial \tau}\,d\tau.$$

The eqs. (4.19) may be written as

4.20
$$\frac{1}{\varrho}t_{ij}=\delta_{ij}\int_0^z A_1(z-\tau)\frac{\partial u_{kk}}{\partial \tau}\,d\tau+\int_0^z A_2(z-\tau)\frac{\partial u_{ij}}{\partial \tau}\,d\tau$$

$$+\int_0^z A_3(z-\tau)\frac{\partial u_{ji}}{\partial \tau}\,d\tau+\delta_{ij}\int_0^z a(z-\tau)\frac{\partial \zeta}{\partial \tau}\,d\tau,$$

$$\frac{1}{\varrho}p_i=\int_0^z \chi(z-\tau)\frac{\partial \bar{u}_i}{\partial \tau}\,d\tau,$$

$$\frac{1}{\varrho}g_{ij}=\delta_{ij}\int_0^z C_1(z-\tau)\frac{\partial \bar{u}_{kk}}{\partial \tau}\,d\tau$$

$$+\int_0^z C_2(z-\tau)\frac{\partial \bar{u}_{ij}}{\partial \tau}\,d\tau+\int_0^z C_3(z-\tau)\frac{\partial \bar{u}_{ji}}{\partial \tau}\,d\tau.$$

If we impose a constraint such that the geometrically independent polarization vector $\bar{u}_i$ is identified with the rotation vector due to the deformation that is $\bar{u}_i=\frac{1}{2}l_{ijk}u_{kj}$, then $\bar{u}_{ij}$ reduces to a new variable

$$\bar{u}_{ij}=\frac{1}{2}e_{jlm}u_{m,li}\equiv\eta_{ij}$$

with property $\eta_{ii}=0$.

Thus the equations $(4.20)_{1,3}$ directly reduce to the following equations

(4.21)
$$\frac{1}{\varrho}t_{ij}=\delta_{ij}\int_0^z \lambda(z-\tau)\frac{\partial e_{kk}}{\partial \tau}\,d\tau+\int_0^z \mu(z-\tau)\frac{\partial e_{ij}}{\partial \tau}\,d\tau+\int_0^z a(z-\tau)\frac{\partial \zeta}{\partial \tau}\,d\tau,$$

$$\frac{1}{\varrho}p_i=\int_0^z \chi(z-\tau)\frac{\partial \bar{u}_i}{\partial \tau}\,d\tau,$$

$$\frac{1}{\varrho}g_{ij}=\int_0^z \eta(z-\tau)\frac{\partial \eta_{ij}}{\partial \tau}\,d\tau+\int_0^z \eta'(z-\tau)\frac{\partial \eta_{ji}}{\partial \tau}\,d\tau,$$

where $t_{ij}=t_{ji}$ and $g_{ij}\neq g_{ji}$.

Equations (4.21) are explicit forms of the constitutive equations of linear thermo-viscoelastic two-continuum with 6 material functions. If we impose that $|\bar{u}_i|=0$, then eqs. (4.21) directly reduce to equations of the classical thermo-viscoelastic continuum with couple stress — eqs. (61a, b) from [7].

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