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Analysis of overconstrained spatial mechanisms

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Introduction

Matrix methods for analysis of the mechanisms permit for the determining of links position to be composed a system of nonlinear equations in such a form:

$$(1) \qquad f_i(x_1, x_2, \dots, x_m;\ q_1, q_2, \dots, q_s)=0,$$

$$(i=1, 2, \dots, j),\ f_i \in C^1,$$

where $x_1, x_2, \dots, x_m$ are variable parameters, determining the instantaneous links position;

$q_1, q_2, \dots, q_s$ are constructive parameters (geometry and mounting) of the mechanism. When $j > m-1$ there exist redundant constraints in the mechanism and the transmission of motion is possible when definite correlations between the constructive parameters are established.

Overconstrained mechanisms are sensitive to technological and mounting errors. Such errors disturb the correlations between the constructive parameters; transmission of motion is possible only with elasticity distortions of links or in the presence of windage in the kinematic pairs. That is why high requirements of precisions must be present to overconstrained mechanisms. But that may find their field of application having in mind the possibility of increasing the construction hardness due to the existing additional bearings.

F. M. Dimentberg's works [2,3] have contributed a lot to the overconstrained mechanisms' analysis and caused the publishing of many other works in this field. Considerable successes in solving these problems Pamidi, Soni, Dukkipati [7], Waldron [8, 9] have achieved.

The method of analysis used by F. M. Dimentberg and his followers have been based on being identically equal to zero the resultant, composed for two polynomials. The authors of this article suggest a method based upon: a) the conventional unclosure of the mechanism [4, 5]; b) the usage of the theorem for the existence of an inplicit functions set and separating from it the dependent and independent functions. The conventional unclosure of the mechanism allows us to decrease the number of equations in system (1) and the containing in it variable parameters.

Nature of the method used

Let us suppose that system (1) is satisfied in the point

$$X^0=(x_1^0, x_2^0, \dots, x_m^0) \tag{2}$$

at a vector-combination of constructive parameters

$$Q=(q_1, q_2, \dots, q_s). \tag{3}$$

If the mechanism really has an instantaneous degree of freedom equal to one, it means that in the vicinity of the point (2) exists a set of local one-value functions

$$\{x_2(x_1, Q), x_3(x_1, Q), \dots, x_m(x_1, Q)\} \in C^1, \tag{4}$$

where x_1 is a variable parameter determining the position of the driving link 1.

From the theorem for the existence of an implicit functions set it follows that the function (4) exists if between j functions f_i $(i=1, 2, \dots, j)$ exist $K=m-1$ such functions, for which the Jacobian

$$\frac{D(f_1, f_2, \dots, f_k)}{D(x_2, x_3, \dots, x_m)} \neq 0. \tag{5}$$

Without effecting the generality of the ratiocinations, we assumed that the functions of the separated underset have numbers 1, 2, 3, ..., k.

Let us consider now the system of functions

$$\{f_1, f_2, \dots, f_k, f_d\} \in C^1, \; m \leq d \leq j, \tag{6}$$

where d is the variable number of the function, joined to the system of k functions. As (6) is a system of $m=k+1$ dependent functions, as is known from the linear algebra, for it the Jacobian

$$\frac{D(f_1, f_2, \dots, f_k, f_d)}{D(x_1, x_2, \dots, x_m)} = \Delta_m = 0, \tag{7}$$

where Δ_m is the determinant with an order m in the Jacobian's matrix for the system (1).

In an expanded form the equation (7) may be represented like this:

$$\varphi_d(x_1, x_2, \dots, x_m;\; q_1, q_2, \dots, q_s) = 0. \tag{8}$$

The number of equations (8) is determined by the number of systems of functions (6) which may be composed by variations of the index d in the function f_d.

For an analytical determining of the relations to be found between the constructive parameters of the mechanism it is advisable to do the following:

1) to separate from the system (1) an under-system of $k=m-1$ equations, for which the functional determinant $\Delta_k \neq 0$, and to determine a system of local one-value functions (4);

2) to substitute the expressions (4) in all the equations with number $d\,(m \leq d \leq j)$ of the system (1), whereupon they will accept the form

$$f_d(x_1, x_2(x_1), \dots, x_m(x_1), Q) = F_d(x_1, Q) = 0; \tag{9}$$

3) to determine the relations between the constructive parameters of the mechanism — the components of the vector Q — from the condition that the

functions F_d must be identically equal to zero. Because of this is advisable to represent the function F_d in the form of a power polynomial

$$F_d = A_n y^n + A_{n-1} y^{n-1} + \cdots + A_o, \tag{10}$$

where $y = \operatorname{tg}\frac{x_1}{2}$; $A_n, A_{n-1}, \ldots, A_o$ — constant coefficients, expressed by the constructive parameters of the mechanism. The unknown relations between the constructive parameters will be determined by the equations

$$A_n = A_{n-1} = \cdots = A_o = 0. \tag{11}$$

The presence of the equations (8), supplementing the system (1), allows a considereble simplifying of the solution. Although the equations $f_d = 0$ and $\varphi_d = 0$ are dependent, it is convenient to use them simultaneously. This permits us to simplify the excluding of $x_2, x_3, \ldots, x_m$, when determining the power polynomial (10) of x_1.

This way of solving permits finding the whole totality of relations between the constructive parameters, determining not one but p mechanisms, having one degree of freedom at the presence of redundant constraints.

It is an interesting solution a partial but simple one, as a result of which are obtained a part only of the unknown relations between the constructive parameters, i. e. a part only of the p group mechanisms.

Expanding the (8) equation, an expression can be obtained with the following form

$$a_{11}\psi_1(x_1, x_2, \ldots, x_m) + a_{12}\psi_2(x_1, x_2, \ldots, x_m) + \cdots = 0, \tag{12}$$

where $a_{11}, a_{12}, \ldots$ are coefficients, expressed by the constructive parameters. Partial relations between the constructive parameters are determined by the equations:

$$a_{11} = a_{12} = \cdots = 0. \tag{13}$$

The method presented above is illustrated at the end of this article in the example of the four-bar spatial linkage.

Numerical method

When overconstrained mechanisms with a complicated structure (five or six links) are analysed, the vector Q of the constructive parameters can be determined by a numerical method by computer application. We must use such a calculation procedure for this:

a) to separate from (1) an underset consisting of $k = m - 1$ independent equations;

b) to give the vector Q of the constructive parameters and the values of $x_1^{(1)}, x_1^{(2)}, \ldots$ in the given range for variation of x_1;

c) to choose from the separated underset values for $X^1, X^2, \ldots$, corresponding to $x_1^{(1)}, x_1^{(2)}, \ldots$, to put them in all the equations with number $d\,(m \leq d \leq j)$ of system (1) and find the disparity in these equations as differencies between their left and right parts;

d) using an optimisation method, find the corrections for the vector Q which will make the disparities negligibly little.

When dividing the value range for x_1 it is neccessary to have in mind that the number of equation which the disparities are determined from must be less than the number of all possible constructive parameters of the mechanism.

This allows us a part of the constructive parameters (one or two) to be made variable, and the rest to be calculated at each correction procedure of Q.

As in the case of the analytical solution, instead of the equation $f_d=0$ equation $\Delta_m=0$ may be used.

Example. Application of the method for analysing the four-bar linkage with redundant constraints

In the linkage here considered (Fig. 1) A_{01} and A_{03} are revolute pairs. Let us research the conditions for turning of the mechanism links, supposing that A_{12} and A_{23} are revolute (R) or cylindrical (C) pairs. With such a structure there exist redundant constraints for this mechanism.

In order to compose the system of equations (1) we shall exclude link 2. For coordination of the unclosed kinematic chain motions we shall use the following equations [5]:

$$(\bar{R}^{(3)}-\bar{R}^{(1)})(\bar{a}^{(1)}\times\bar{a}^{(3)})=h_2\sin a_2; \tag{14}$$

$$\bar{a}^{(1)}\,.\,\bar{a}^{(2)}=\cos\alpha_2. \tag{15}$$

In these equations $\bar{R}^{(3)}$ and $\bar{R}^{(1)}$ are radius-vectors of the coordinate systems $S_3^{(i)}$ and $S_1^{(i)}$ origins $O_3^{(i)}$ and $O_1^{(i)}$ conducted from the system $S_0^{(j)}$ origin $Q_0^{(j)}$; $\bar{\alpha}^{(3)}$ and $\bar{a}^{(1)}$ are unit vectors of the axes $Z_3^{(i)}$ and $Z_1^{(i)}$; α_2 is the angle of skew of the kinematic pairs axes of the link 2; h_2 — the shortest distance between these axes. Using four-order matrixes and homogeneous coordinates, we shall obtain a system of two equations of the following kind:

$$\begin{gathered}a_{i1}\cos x_1+a_{i2}\sin x_1+a_{i3}\cos x_2+a_{i4}\sin x_2+a_{i5}\cos x_1\cos x_2\\ +a_{i6}\cos x_1\sin x_2+a_{i7}\sin x_1\cos x_2+a_{i8}\sin x_1\sin x_2+a_{i9}=0,\\ (i=1,2),\end{gathered} \tag{16}$$

where x_1 and x_2 are correspondingly the angles φ_{01} and φ_{03} in Fig. 1.

Here:

$$a_{11}=h_0\cos\alpha_0\sin\alpha_1\cos\alpha_3+h_1\sin\alpha_0\cos\alpha_1\cos\alpha_3-h_3\sin\alpha_0\sin\alpha_1\sin\alpha_3$$

$$a_{12}=-a_{01}\sin\alpha_0\sin\alpha_1\cos\alpha_3$$

$$a_{13}=-h_0\cos\alpha_0\cos\alpha_1\sin\alpha_3+h_1\sin\alpha_0\sin\alpha_1\sin\alpha_3-h_3\sin\alpha_0\cos\alpha_1\cos\alpha_3$$

$$a_{14}=a_{03}\sin\alpha_0\cos\alpha_1\sin\alpha_3$$

$$a_{15}=h_0\sin\alpha_0\sin\alpha_1\sin\alpha_3-h_1\cos\alpha_0\cos\alpha_1\sin\alpha_3-h_3\cos\alpha_0\sin\alpha_1\cos\alpha_3$$

$$a_{16}=-a_{01}\sin\alpha_1\sin\alpha_3+a_{03}\cos\alpha_0\sin\alpha_1\sin\alpha_3$$

$$a_{17}=a_{01}\cos\alpha_0\sin\alpha_1\sin\alpha_3-a_{03}\sin\alpha_1\sin\alpha_3$$

$$a_{18}=-h_1\cos\alpha_1\sin\alpha_3-h_3\sin\alpha_1\cos\alpha_3$$

$$\begin{aligned}a_{19}=h_0\sin\alpha_0\cos\alpha_1\cos\alpha_3+h_1\cos\alpha_0\sin\alpha_1\cos\alpha_3-h_2\sin\alpha_2+\\ +h_3\cos\alpha_0\cos\alpha_1\sin\alpha_3\end{aligned}$$

$$a_{21}=-\sin\alpha_0\sin\alpha_1\cos\alpha_3$$

$$a_{22}=0$$

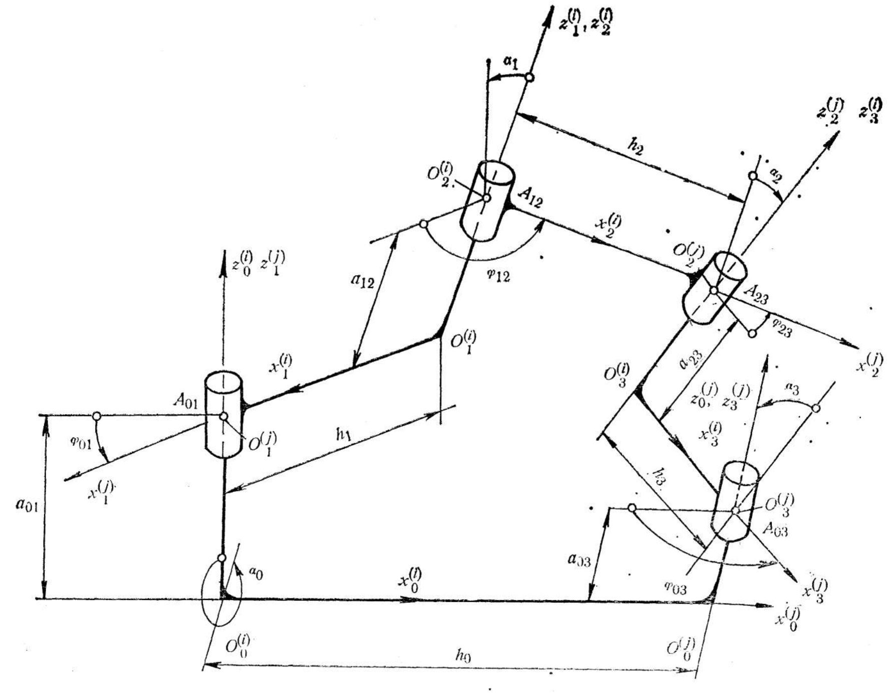

Fig. 1

$a_{23} = \sin\alpha_0 \cos\alpha_1 \sin\alpha_3$

$a_{24} = 0$

$a_{25} = \cos\alpha_0 \sin\alpha_1 \sin\alpha_3$

$a_{26} = 0$

$a_{27} = 0$

$a_{28} = \sin\alpha_1 \sin\alpha_3$

$a_{29} = \cos\alpha_0 \cos\alpha_1 \cos\alpha_3 - \cos\alpha_2.$

Necessary and sufficient conditions ensuring the turning of the mechanism inks in the whole variation range of x_1 is the identical equality to zero of the unctional determinant of second order

$$\frac{D(f_1, f_2)}{D(x_1, x_2)} = 0. \tag{17}$$

By determining from the equations (16) $x_2 = \varphi(x_1)$ and developing the Jacobian (17) the following polynomial can be found:

$$A_{13}y^{13} + A_{12}y^{12} + \cdots + A_0 = 0, \tag{18}$$

where $y = \operatorname{tg}\frac{x_1}{2}$; $A_{13}, A_{12}, \ldots, A_0$ are constant coefficients expressed by the constructive parameters of the mechanism. Because of their bulk they are not given in this article.

The desirable relations between the constructive parameters determined from the equalities:

$$A_{13}=A_{12}=\cdots=A_0=0, \tag{19}$$

are given in Table 1. In this table are not given degenerate mechanisms as well as spherical ones.

At a partial solution of the problem where one can determine as a result only a part of the relations desirable between the constructive parameters we can limit ourselves with (17) in the form

$$b_1\psi_1(x_1, x_2)+b_2\psi_2(x_1, x_2)+\cdots+b_{19}\psi_{19}(x_1, x_2)=0. \tag{20}$$

Here:

$$b_1=\sin^2\alpha_0\,(h_1\sin\alpha_3\cos\alpha_3-h_3\sin\alpha_1\cos\alpha_1),$$

$$b_2=\sin^2\alpha_1\,(h_0\sin\alpha_3\cos\alpha_3-h_3\sin\alpha_0\cos\alpha_0),$$

$$b_3=\sin^2\alpha_1\,(h_3\sin\alpha_0-h_0\cos\alpha_0\sin\alpha_3\cos\alpha_3),$$

$$b_4=a_{01}\sin^2\alpha_0\sin\alpha_1\cos\alpha_1\sin\alpha_3\cos\alpha_3,$$

$$b_5=a_{10}\sin\alpha_0\cos\alpha_0\sin^2\alpha_1\sin\alpha_3\cos\alpha_3,$$

$$b_6=a_{03}\sin\alpha_0\cos\alpha_0\sin^2\alpha_1\sin\alpha_3\cos\alpha_3,$$

$$b_7=\sin^2\alpha_3\,(h_0\sin\alpha_1\cos\alpha_1-h_1\sin\alpha_0\cos\alpha_0),$$

$$b_8=b_9=h_0\sin\alpha_0\sin^2\alpha_1\sin^2\alpha_3,$$

$$b_{10}=\sin^2\alpha_3\,(h_1\sin\alpha_0-h_0\sin\alpha_1\cos\alpha_0\cos\alpha_1),$$

$$b_{11}=\sin^2\alpha_3\,(a_{03}\sin\alpha_0\cos\alpha_0\sin\alpha_1\cos\alpha_1-a_{01}\sin\alpha_0\sin\alpha_1\cos\alpha_1)$$

$$b_{12}=\sin^2\alpha_1\sin^2\alpha_3\,(a_{03}+a_{03}\cos^2\alpha_0-2a_{01}\cos\alpha_0),$$

$$b_{13}=\sin^2\alpha_3\,(h_1\sin\alpha_0-h_0\cos\alpha_0\sin\alpha_1\cos\alpha_1),$$

$$b_{14}=a_{01}\sin\alpha_0\cos\alpha_0\sin\alpha_1\cos\alpha_1\sin^2\alpha_3,$$

$$b_{15}=-a_{10}\sin^2\alpha_0\sin^2\alpha_1\sin^2\alpha_3,$$

$$b_{16}=-a_{03}\sin^2\alpha_0\sin^2\alpha_1\sin^2\alpha_3,$$

$$b_{17}=a_{03}\sin^2\alpha_0\sin\alpha_1\cos\alpha_1\sin\alpha_3\cos\alpha_3,$$

$$b_{18}=\sin\alpha_0\sin^2\alpha_1\cos\alpha_3\,(a_{03}\sin\alpha_3-a_{01}\cos\alpha_0\sin\alpha_3),$$

$$b_{19}=a_{03}\sin\alpha_0\cos\alpha_0\sin\alpha_1\cos\alpha_1\sin^2\alpha_3.$$

It can be found from the equality

$$b_1=b_2=\cdots=b_{19}=0, \tag{21}$$

some partial solution which coincide with the cases 1) to 4) in Table 1.

In order to determine the position functions $\varphi_{12}(\varphi_{01})$, $\varphi_{23}(\varphi_{01})$, $a_{12}(\varphi_{01})$ and $a_{23}(\varphi_{01})$ it is necessary twice making an unclosure in the pairs A_{12} and A_{23}; a_{12} and a_{23} are translational movements along the pair axes $Z_2^{(i)}$ and $Z_3^{(i)}$ correspondingly. The obtained position functions will give us full information about the mechanism structure.

Table 1

No	Relations between the constructive parameters	Structure of the mechanism	Position functions
1	$\alpha_1=0,\ \alpha_3=0,$ $\alpha_0=\alpha_2$	A_{12} — cylindrical A_{23} — cylindrical	$\varphi_{03}=\arccos\left(\frac{h_0-h_2}{h_3}+\frac{h_1}{h_3}\cos\varphi_{01}\right)$
2	$\alpha_0=0,\ \alpha_3=0,$ $\alpha_1=\alpha_2$	A_{12} — prismatical A_{23} — cylindrical	$\varphi_{03}=\arccos\left(\frac{h_1-h_2}{h_3}+\frac{h_0}{h_3}\cos\varphi_{01}\right)+\varphi_{01}$
3	$\alpha_0=0,\ \alpha_1=0$ $\alpha_2=\alpha_3$	A_{12} — cylindrical A_{23} — prismatical	$h_0\cos\varphi_{03}+h_1\cos(\varphi_{03}-\varphi_{01})+h_2-h_3=0$
4	$\alpha_0=0,\ \alpha_2=0,$ $\alpha_1=\alpha_3=90^0,$ $a_{01}=a_{03}$	A_{12} — prismatical A_{23} — prismatical	$\varphi_{03}=\varphi_{01}$
5	$h_1=h_3,\ h_0=h_2,$ $\alpha_1=\alpha_3,\ \alpha_0=\alpha_2,$ $a_{01}=a_{03}=a_{12}=a_{23}=0$ Skew Benet's mechanism	A_{12} — revolute A_{23} — revolute	$\varphi_{03}=\arccos\left[\frac{(\cos\alpha_1\cos\alpha_0-1)\cos\varphi_{01}-\sin\alpha_1\sin\alpha_0}{\cos\alpha_1\cos\alpha_0-1-\sin\alpha_1\sin\alpha_0\cos\varphi_{01}}\right]$
6	$h_1=h_3,\ h_0=h_2,$ $\alpha_1=\alpha_3,\ \alpha_0=\alpha_2,$ $a_{01}=a_{03}=\alpha_{12}=a_{23}=0$ straight Benet's mechanism	A_{12} — revolute A_{23} — revolute	$\varphi_{03}=\arccos\left[\frac{(\cos\alpha_1\cos\alpha_0+1)\cos\varphi_{01}-\sin\alpha_1\sin\alpha_0}{\cos\alpha_1\cos\alpha_0+1-\sin\alpha\,\sin\alpha_0\cos\varphi_{01}}\right]$

Condition for existence of a crank in four-bar linkage with redundant constraints

At the synthesis of a mechanism with redundant constraints it is necessary to ensure the existence of cranks, favourable conditions for force transfer and so on. This imposes new limits on the constructive parameters besides those determined by the links turning conditions.

For determining link 1 crank existence conditions, we shall represent the position function in the form of a polynomial:

$$M_2 Z^2 + M_1 Z + M_0 = f(\varphi_{01}, Z) = 0, \tag{22}$$

where M_2, M_1, M_0 are functions of φ_{01} $(0 \leq \varphi_{01} \leq 2\pi)$.

For existence of real values of Z in the range $0 \leq \varphi_{01} \leq 2\pi$ it is necessary

$$M_1^2 - 4 M_0 M_2 > 0. \tag{23}$$

The left part of this unequation can be represented in the form of a polynomial of fourth degree form $\operatorname{tg} \frac{\varphi_{01}}{2}$.

$$N_4 s^4 + N_3 s^3 + N_2 s^2 + N_1 s + N_0 > 0, \quad (-\infty < s < +\infty), \tag{24}$$

where $s = \operatorname{tg} \frac{\varphi_{01}}{2}$.

The unequality (24) will be fulfilled if in the equation

$$N_4 S^4 + N_3 s^3 + N_2 s^2 + N_1 s + N_0 = 0 \tag{25}$$

all roots are complex. This permits obtaining unequations which must be imposed on the constructive parameter of the mechanism.

Such a method for determining the conditions for existence of a crank in the RSSR mechanism with two revolute and two spherical pairs has been suggested by S. H. Kislitsin and his joint authors [1]. We shall note that by observing the unequation (23) in the mentioned range $\frac{\partial f}{\partial z} \neq 0$ and links 2 and 3 will not have special positions. The function $Z(\varphi_{01})$ in the range $0 \leq \varphi_{01} \leq 2\pi$ will be locally one-valued.

For the overconstrained mechanism, mentioned in position 2 of Table 1, the polynomial (24) is of second degree only and in this case for crank existence it is sufficient

$$h_3^2 - (h_1 - h_2 \pm h_0)^2 > 0. \tag{26}$$

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