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Free convection from a partly insulated plate

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Introduction

Similarity solutions of free convective flow past a semi-infinite vertical plate were presented by Ostrach [1], Sparrow and Gregg [2], Takhar [3]. Kuiken [4] treated it as a singular perturbation problem whereas Ackroyd [5] solved it in terms of co-ordinate expansions. Takhar [6] solved the present problem numerically. Here a modified version of temperature distribution in Karman-Pohlhausen method is employed and again the problem is solved. This results in a different value of the aspect ratio a/l for different fluids, where a is the distance of the effective origin from the leading edge and l is the ength of the uniformly heated part of the plate.

Analysis

Here the origin is taken at the lower edge of the plate, x-axis along the plate and the y-axis normal to the plate with the associated velocities u and v in these directions. Then the problem is governed witq usual assumptions by the following equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}v\frac{\partial u}{\partial y}=ga(T-T_\infty)+\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=k\frac{\partial^2 T}{\partial y^2}. \tag{3}$$

All the terms have their usual meaning. The boundary conditions are

$$\text{i)}\quad u=0,\ v=0\ \text{at}\ y=0;\quad \text{ii)}\quad T=T_w,\ 0<x<l;$$

$$\text{iii)}\quad \frac{\partial T}{\partial y}=0,\ x>l;\quad \text{iv)}\quad T\to T_\infty,\ \frac{\partial T}{\partial y}=0\ \text{as}\ y\to\infty.$$

Solution

Assuming cubic profiles for the velocity and temperature, it can be shown in view of the boundary conditions that

$$u=u_1(1-\eta)^2, \quad \theta=(1+B\eta)\,(1-\eta)^2, \tag{4}$$

where $\eta=y/\delta$ is the similarity variable and δ is the momentum or thermal boundary layer thickness, u_1 is the scaling velocity. Integrating (2) and (3) over a cross-section of the boundary layer and using (4), we get

$$\frac{1}{105}\frac{d}{dx}(u_1^2\,\delta)=g\alpha\delta(T_w-T_\infty)\,(B+4)12-\nu u_1/\delta, \tag{5}$$

$$\frac{1}{210}\frac{d}{dx}(u_1\delta(T_w-T_\infty)\,(2B+7))=-k(T_w-T_\infty)\,(B-2)/\delta. \tag{6}$$

Two cases arise here depending upon the thermal specifications of the plate. i) Uniformly heated plate: For the uniformly heated length of the plate we have $T_w-T_\infty=\text{constant}=\Delta T$ for $0<x<l$. Then equations (5) and (6) reduce to

$$\frac{1}{105}\frac{d}{dx}(u_1^2\,\delta)=\frac{B+4}{12}(g\alpha\Delta T)\,\delta-\nu u_1/\delta, \tag{7}$$

$$\frac{1}{210}\frac{d}{dx}(u_1\delta(2B+7))=-k(B-2)/\delta. \tag{8}$$

From this Takhar [6] derived Squires' [7] results for $B=0$. Keeping B unknown, as suggested by Maslen [7], the solutions of (7) and (8) are

$$\delta=\left[\frac{3360\,(2-B)}{(B+4)(2B+7)}\right]^{1/4}\left[\sigma+\frac{10\,(2-B)}{3\,(2B+7)}\right]^{1/4}.\,\sigma^{-1/2} \left[\frac{g\alpha\Delta T x^3}{4\nu^2}\right]^{-1/4}(4)^{-1/4}\,x, \tag{9}$$

$$u_1=\left[\frac{70\,(B+4)\,(2-B)}{3\,(2B+7)}\right]^{1/2}\left[\sigma+\frac{10\,(2-B)}{3\,(2B+7)}\right]^{-1/2}.\,\nu \left[\frac{g\alpha\Delta T x^3}{4\nu^2}\right]^{1/2}(4)^{1/2}x^{-1}, \tag{10}$$

where $\sigma=\nu/k$ is the Prandtl number. If instead of taking $B=0$, we now satisfy the momentum equation at the wall, i. e.

$$g\alpha\,(T_w-T_\infty)\delta=\frac{4\nu u_1}{\delta}, \tag{11}$$

we get the condition, on using (9) and (10), as

$$\frac{10(2-B)}{2B+7}=\sigma(B+1). \tag{12}$$

ii) Insulated flat plate: For the insulated part of the plate $x>l$ we have the boundary condition $(\partial T/\partial y)_{y=0}=0$ which leads to $B=2$. Then equations (5) and (6) reduce to

$$\frac{1}{105}\frac{d}{dx}(u_1^2\,\delta)=\frac{1}{2}g\alpha\delta(T_w-T_\infty)-\frac{\nu u_1}{\delta}, \tag{13}$$

(14) $$u_1\delta(T_w - T_\infty) = c\,\frac{\nu}{ga},$$

where c is an unknown constant. From (11) and (14) we get

(15) $$u_1^2 = c\delta/4.$$

Then from (13) and (15) we get

(16) $$\frac{1}{420}\frac{d}{dx}(c\delta^2) = \frac{\nu u_1}{\delta} = \left(\frac{c\nu^2}{4\delta}\right)^{1/2}$$

and on integration it gives

(17) $$\delta^{5/2} = 525\,\nu(x-a)/2\sqrt{c}.$$

Hence

(18) $$\delta = [525\,\nu\,(x-a)/2\sqrt{c}\,]^{2/5} \text{ and } u = \frac{\sqrt{c\delta}}{2},$$

where a and c are constants. They are determined by matching the two solutions at $x=1$, where the momentum and heat fluxes are continuous.
Hence,

(19) $$\int_0^\infty u^2 dy = u_1^2\,\delta/105,$$

(20) $$\int_0^\infty u(T-T_\infty)dy = u_1\delta(T_w - T_\infty)(1/30+B/105).$$

For the uniformly heated part of the plate $0<x<l$,

(21) $$\int_0^\infty u^2 dy = \left[\frac{(3360)^{1/4}}{105} - \frac{70}{3}\left(\frac{ga\Delta T}{\nu^2}\right)^{3/4}(B+4)^{3/4}\left(\frac{2-B}{2B+7}\right)^{5/4}\right]$$
$$\cdot\left[\sigma + \frac{10\,(2-B)}{3(2B+7)}\right]^{-3/4}\nu^2\sigma^{-1/2}l^{5/4},$$

(22) $$\int_0^\infty u(T-T_\infty)dy = (70/30)^{1/2}(1/30+B/105)(2-B/2B+7^{3/4}(3360)^{1/4}$$
$$\times(B+4)^{1/4}(\sigma+(10/3)(2-B/2B+7))^{-1/4}\,\nu\sigma^{-1/2}\,\Delta T(ga\Delta T/\nu^2)^{1/4}l^{3/4}.$$

For the insulated part of the plate, $x>l$, we have

(23) $$\int_0^\infty u^2 dy = (c/420)[525\,\nu(l-a)/2\sqrt{c}]^{4/5},$$

(24) $$\int_0^\infty u(T-T_\infty)dy = 11\,c\nu/(210\,ga).$$

From (22) and (24) we obtain the value of c and substituting it in (23) we get

$$(25) \qquad 1-\frac{a}{l}=\left[\frac{1232(B+4)(2-B)}{315\,(2B+7)^2}\left(\frac{48}{11}\frac{2B+7}{B+4}\right)^{1/4}\right]\cdot\sigma^{-1/4}$$

$$\left(\sigma+\frac{10}{3}\frac{2-B}{2B+7}\right).$$

From (12) and (25) we get

$$(26) \qquad 1+\frac{a}{l}=\frac{1232}{1050}\left[\frac{16}{11}(2B+7)\right]^{1/4}\left(\frac{B+1}{2B+7}\right).$$

For $\sigma=0.733$ we get from (12) $B=0.84$. For this value of B from (2σ) we get $\frac{a}{l}=0.53$. Similarly for $\sigma=7$ we get $\frac{a}{l}=0.83$. We can also show that as $\sigma\to\infty$, $(B+1)\to 6/\sigma$ and $a/l\to\left\{1-\frac{1232}{875}\left(\frac{80}{11}\right)^{1/4}\frac{1}{\sigma}\right\}\to 1$ and as $\sigma\to 0$, $B\to 2$ and $a/l\to\left(1-\frac{112}{175}\right)=0.36$. Thus the motion of the 'effective leading edge' is quite modest. According to the present modified approach the effective leading edge never goes forward of about 1/3rd the way back the heated part of the plate and if $\sigma\to\infty$ it approaches ($a=1$) more slowly even though still as σ^{-1}. This behaviour of the aspect ratio gives better agreement with the Takhar's [3] more exact findings obtained by matching Ostrach's [1] numerical solution for a uniformly heated plate and Takhar's [6] numerical solution for an insulated flat plate by keeping the fluxes of heat and momentum continuous.

We can now show that the local Nusselt number is given by

$$(27) \qquad Nu=\frac{2}{\sqrt{12}}\left(\frac{Gr}{4}\right)^{1/4}\sigma^{1/2}\left[\sigma-\frac{10}{3}\frac{B-2}{2B+7}\right]^{-1/2}\left[\frac{4}{7}(B+2)\right.$$

$$\left.\times\left(1-\frac{3\sigma}{10}\frac{2B+7}{B-2}\right], \quad Gr=\frac{g\alpha\Delta Tx^3}{\nu^2}.$$

Defining the flow parameters $s=Fa\,.\,Re^2/(Gr/4)^{3/4}$, where $Fa=T_w/\frac{1}{2}\varrho u_1^2$, $Re=u_1\varkappa/\nu$, we have

$$(28) \qquad S=2\left(\frac{280}{27}\right)^{1/4}(B+4)^{3/4}\left(\frac{2-B}{2B+7}\right)^{1/4}\sigma^{1/2}\left[\sigma+\frac{10(2-B)}{3(B2+7)}\right]^{-3/4}.$$

Thus for air $\sigma=0.733$, $B=0.84$, $\frac{a}{l}=0.53$.

Hence $S=48.11$, $Nu/(Gr/4)^{1/4}=0.7476$. For water $\sigma=7$, $B=-0.497$, $a/l=0.83$. Hence $S=7.7240$, $Nu/(Gr/4)^{1/4}=1.1457$.

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