БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ
ТЕОРЕТИЧНА И ПРИЛОЖНА МЕХАНИКА
София 1978, год. IX, № 1

BULGARIAN ACADEMY OF SCIENCES
THEORETICAL AND APPLIED MECHANICS
Sofia, 1978, Y ar IX, No 1

Exact finite elements for thin elastic shells of revolution

Zenon Waszczyszyn (Cracow, Poland)

1. Introductory remarks

In order to approximate the displacement vector **u** polynomials with n independent coefficients (which are referred to degrees of freedom of the finite element — DFE) are usually applied in the finite element method (FEM). The accuracy of approximation can be raised either by increasing the number of DFE (i. e., by increasing the order of approximating polynomials) or by increasing the number of finite elements (usually local refinement of the finite element mesh).

Besides such a standard approach exact analytical solutions for shape functions have also been used This is limited to a rather small class of plates and shells [1, 2, 3, 4].

The above mentioned idea of application of exact solutions for shells of revolution is explored in the presented report. Contrary to the previous papers those solutions are obtained numerically integrating the complete set of equations of the linear theory of thin shells. The basis matrix method [5] is applied to compute both the stiffness matrix **K** and the vector of equivalent forces **F** without assuming any shape functions.

The advantage of such a procedure is that DFE is constant and equals n, where n is now the number of coefficients of the state vector of the canonic set of ordinary differential equations of the shell (e. g. for the Kirchhoff-Love theory $n=8$). Various shapes of the middle surface as well as variable element properties along the generatrix (thickness, material coefficients) can be easily taken into account. The only limitation is that the length of the finite element cannot exceed the length of the short shell.

2. Basis matrix method and computation of the FE matrices

The symmetric boundary-value problem is considered given below by the following linear ordinary differential equation

$$\mathbf{y}'=\mathbf{A}\mathbf{y}+\mathbf{p} \quad (\mathbf{y}, \mathbf{p} \in R^n) \tag{1}$$

with two-point boundary conditions

(2) $$\mathbf{B}_0\mathbf{y}_0+\mathbf{B}_L\mathbf{y}_L=\mathbf{g}$$

Elements of the matrix **A** and the vector **p** are to be assumed as continuous functions of the independent variable $\xi \in [\xi_0, \xi_L]$.

The integral of Eq. (1) can be written in the form

(3) $$\mathbf{y}(\xi)=\mathbf{Y}(\xi)\mathbf{c}+\mathbf{z}(\xi),$$

where $\mathbf{Y}(\xi)$ is the basis matrix, that is the solution of the matrix homogeneous equation

(4) $$\mathbf{Y}'=\mathbf{AY}.$$

The vector $\mathbf{z}(\xi)$ is any integral of Eq. (1) and the vector **c** is to be calculated from boundary conditions (2).

The algorithm of calculation of the matrices **Y** and **z** worked out in [5] depends on a sequence of solutions of an initial-value problem and on some arithmetic transformations of the obtained results. This algorithm can be modified to compute the FE matrices.

Let us assume that functions **y** and **p** are the state and load vectors respectively

(5) $$\mathbf{y}=\{\mathbf{q}, \bar{\mathbf{Q}}\}, \quad \mathbf{p}=\mathbf{f}(\mathbf{P}, \vartheta).$$

The subvectors **q** and $\bar{\mathbf{Q}}$ are the generalized displacements and forces correspondingly, **P** is the surface load vector and ϑ is temperature.

Eq. (4) can be numerically integrated with the unit initial values at point ξ_0:

(6) $$\mathbf{Y}_0=\mathbf{I}\xrightarrow{NI}\mathbf{Y}_L=\mathbf{Y}(\xi_L)=\begin{bmatrix}\mathbf{q}_q & \mathbf{q}_Q\\ \mathbf{Q}_q & \mathbf{Q}_Q\end{bmatrix},$$

where all submatrices in $\mathbf{Y}_L$ are of size $(n/2\times n/2)$. In the same way Eq. (1) can be integrated with the initial value $\mathbf{y}_0=\mathbf{0}$:

(7) $$\mathbf{y}_0=\mathbf{z}_0=\mathbf{0}\xrightarrow{NI}\mathbf{z}_L=\{\mathbf{q}_p, \bar{\mathbf{Q}}_p\}.$$

The stiffness matrix can be calculated setting successive unit displacements along constrains according to the components of the vectors $\mathbf{q}_0$ and $\mathbf{q}_L$. To obtain it the following set of matrix equations must be solved:

(8) $$(\mathbf{B}_0+\mathbf{B}_L\mathbf{Y}_L)\mathbf{D}_0=\mathbf{I}, \quad \mathbf{D}_L=\mathbf{Y}_L\mathbf{D}_0,$$

where the boundary condition matrices $\mathbf{B}_0$ and $\mathbf{B}_L$ correspond to the clambed contours (ends) of the element:

(9) $$\mathbf{B}_0=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}, \quad \mathbf{B}_L=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{I} & \mathbf{0}\end{bmatrix}.$$

The solution of Eq. (8) is as follows:

(10) $$\mathbf{D}_0=\begin{bmatrix}\boldsymbol{\delta}_{00} & \boldsymbol{\delta}_{0L}\\ \mathbf{r}_{00} & \mathbf{r}_{0L}\end{bmatrix}, \quad \mathbf{D}_L=\begin{bmatrix}\boldsymbol{\delta}_{L0} & \boldsymbol{\delta}_{LL}\\ \mathbf{r}_{L0} & \mathbf{r}_{LL}\end{bmatrix}.$$

The submatrices in (10) can be identified as displacements δ_{ij} and reactions $\mathbf{r}_{ij}$ for $i, j=0, L$.

The stiffness matrix is composed from submatrices $\mathbf{r}_{ij}$

(11) $$\bar{\mathbf{K}}=\begin{bmatrix}-\mathbf{r}_{00} & -\mathbf{r}_{0L}\\ \mathbf{r}_{L0} & \mathbf{r}_{LL}\end{bmatrix},$$

where the minus at $\mathbf{r}_{0j}$ corresponds to the sign convention adopted.

The matrix (one-column vector) of equivalent nodal forces **F** is calculated on the base of (7) satisfying kinematic boundary conditions

(12) $$(\mathbf{B}_0+\mathbf{B}_L\mathbf{Y}_L)\mathbf{d}_0=-\mathbf{B}_L\mathbf{z}_L\,,\quad \mathbf{d}_L=\mathbf{Y}_L\mathbf{d}_0+\mathbf{z}_0\,.$$

From the solution

(13) $$\mathbf{d}_0=\{\boldsymbol{\delta}_{0p},\mathbf{r}_{0p}\},\quad \mathbf{d}_L=\{\boldsymbol{\delta}_{Lp},\mathbf{r}_{Lp}\}$$

the following vector is combined

(14) $$\overline{\mathbf{F}}=\{-\mathbf{r}_{0p},\mathbf{r}_{Lp}\}.$$

3. Exact finite element for shells of revolution

Variables in the linear theory of thin shells of revolution can be separated expanding all quantities in trigonometric series of the circumferential independent variable θ. If the load has a plane of symmetry the series are one of the two types:

(15) $$c=\sum_j c^j(\xi)\cos j\theta,\quad s=\sum_j s^j(\xi)\sin j\theta.$$

In such a way the following considerations are limited to the analysis of ordinary differential equations to calculate the functions $c^j(\xi)$ and $s^j(\xi)$.

A canonic set of equations for arbitrary harmonics has been derived in [6] on the base of Sanders' theory [7]. In this case $n=8$ and vectors **q** and $\overline{\mathbf{Q}}$ are as follows:

(16) $$\mathbf{q}=\{u, v, w, \beta\},\quad \overline{\mathbf{Q}}=\{\overline{N}, \overline{S}, \overline{T}, \overline{M}\}\,.$$

Components of the vector $\overline{\mathbf{Q}}$ are the equivalent boundary forces per uni length of the coordinate line on the middle surface. If these forces are used then the matrix $\overline{\mathbf{K}}$ will be a nonsymmetric one, in general. The above is one of the reason to introduce the integral nodal forces (Fig. 1)

$$\mathbf{Q}_i=c\pi\varrho_i\overline{\mathbf{Q}}_i=\{N, S, T, M\},$$

where ϱ_i is the radius of the contours $i=0, L$ and the coefficient c equals 2 for $j=0$ and 1 for $j\geq 1$.

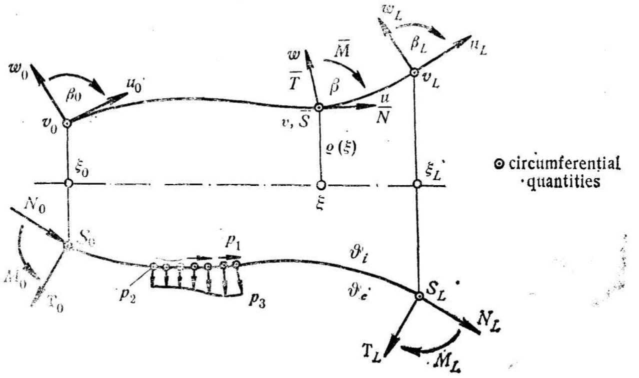

Fig. 1

Finally the stiffness matrix **K** and the vector of equivalent nodal forces **F** have the following form:

$$\mathbf{K}=c\pi\begin{bmatrix} -\varrho_0\mathbf{r}_{00} & -\varrho_0\mathbf{r}_{0L} \\ \varrho_L\mathbf{r}_{L0} & \varrho_L\mathbf{r}_{LL} \end{bmatrix}, \quad \mathbf{F}=c\pi\begin{bmatrix} -\varrho_0\,\mathbf{r}_{0p} \\ \varrho_L\,\mathbf{r}_{Lp} \end{bmatrix}. \tag{18}$$

The worked out element can be called the exact finite element (EFE) since all shell equations are fulfilled. Apart from the Sanders theory EFE are computed for the Reissner-type theory where shear deformations are considered. In such a case, as well as in the case of sandwich-type cross-section with deformable core, the number of DFE $n=10$.

4. Final remarks

The main advantage of EFE is the limitation of the number of DFE to $n=8$ or 10. Shape functions are completely eliminated from considerations and due to numerical integration all shell equations are satisfied.

In the method applied the BV problem is changed into a sequence of initial-value problems. Stability requirements influence the length L of EFE. In applications one assumes that the finite element is a short shell with the length $L \leq L_{cr}$, where the critical value L_{cr} is evaluated according to [8]:

$$L_{cr} \approx 5\frac{\sqrt{R_{\min}h}}{\sqrt[3]{3(1-\nu^2)}}. \tag{19}$$

Conical and hyperbolic EFE have been used in [9] to compute the basis of high chimneys. Now both full-walled elements on elastic foundation and sandwich elements are tested.

References

[1] Meyer, R. R., M. B. Harmon. Conical Segment Method for Analyzing Open Crown Shells of Revolution for Edge Loading, AIAA J., No 4, **1**, 1963, pp. 886—891.

[2] Popov, E. P., J. Penzien. Zung-Au Lu. Finite Element Solution for Axisymmetrical Shells, Proc. ASCE, EM5, **90**, 1964, pp. 119—145.

[3] Pardoen, G. S. Asymmetric Bending of Circular Lates Using the Finite Element Method, Comp., Structure, **5**, 1975, pp. 197—202.

[4] Lakis, A. A. Cylindrical Finite Element for Analysis of Cylindrical Structures, Dept. Mech. Engng. McGill Univ., R, No 70-2, Montreal, Nov. 1969.

[5] Moszyński, K. Solving of Ordinary Differential Equations on Computers (in Polish) Wyd. NT, Warsaw, 1971.

[6] Waszczyszyn, Z., A. Młodzianowski. Exact Finite Element for the Thin Elastic Shell of Revolution (in Polish), Rozpr. Inż., No 1, **25**, 1977, pp. 97—114.

[7] Sanders, J. L., Jr. An Improved First-Approximation Theory for Thin Shells, NASA, TR R-24, 1959.

[8] Kalnins, A. Analysis of Shells of Revolution Subjected to Symmetrical and Nonsymmetrical Loads, J. Appl. Mech., No 3, **31**, 1964, pp. 467—476.

[9] Gumiński, A. J., Krupiński, A. Młodzianowski, Z. Waszczyszyn. Computer Analysis of the Basis of a High Chimney (in Polish), Proc. XXI Confer., Krynica — 1975, **1**, pp. 131—141.