БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ
ТЕОРЕТИЧНА И ПРИЛОЖНА МЕХАНИКА
София, 1978, год. IX, № 1

BULGARIAN ACADEMY OF SCIENCES
THEORETICAL AND APPLIED MECHANICS
Sofia, 1978, Year IX, No 1

Application of the finite difference method at arbitrary irregular grids to solution of various problems of applied mechanics

T. Liszka, J. Orkisz

1. Introduction

It has been observed, that the finite element method (FEM) can be easier applied to treat irregular domain than the finite difference technique (FDM).

By using an arbitrary irregular mesh of nodal points one can preserve the basic advantages of the classical FDM and avoid its main difficulties. An arbitrary grid permits the satisfaction of boundary conditions in the case of an irregular shape of a domain and enables a local condensation of a mesh. On the other hand, several new problems mainly associated with the automatic generation of FD formulae may arise.

The basic assumptions of the method were suggested by Jensen [1]. A six-point control scheme, combined with the two-dimensional Taylor's series expansion, was adopted for obtaining FD formulae with derivatives up to the second order. Kączkowski and Tribiłło [2] extend this approach for higher order derivative terms. Perrone and Kao [3] have suggested a nine-point control scheme with an averaging process to improve the accuracy of obtained FD formulas.

In the present paper we continue investigations of irregular FD techniques [4—5]. A new "optimal" way of generation of the FD formulae has been introduced. For two dimensional, second order problems the nine-point control scheme has been used, but the procedure remains applicable for wider problems.

The set of computer programs in Algol 1204 was tested by calculating several both linear and non-linear problems of mechanics. The programs are fully automatic like those based on the FEM and are specially designed for the solution of large systems by using a small computer "Odra 1204".

2. Automatic selection of "stars"

All the points included to the control scheme we call "a star" of nodes. The most important problems caused by irregular mesh appear during the automatic selection of stars. The number and the positions of the nodes

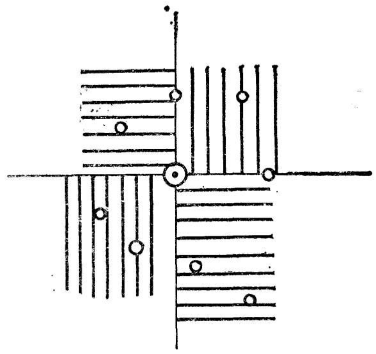

Fig. 1

in each star are the decisive factors affecting FD formulae approximations.

The "eight segments" criterion suggested by Perrone and Kao [3] results in well selected stars, but it appears to be too rigorous, overly complicated and too time consuming for the computer. In the present paper the analogous, however much simpler and quicker procedure was applied with good results. The domain around the central point is divided into four quadrants and two nearest points in each of them are included into the star (Fig. 1). The control for the node can be performed only by: distance computing, comparing the signs of the local coordinates of the node.

3. Difference coefficients for irregular meshes

For any sufficiently differentiable function $f(x,y)$ in a given domain the Taylor series expansion around a point (x_0, y_0) can be written as

$$f=f_0+h\frac{\partial f_0}{\partial x}+k\frac{\partial f_0}{\partial y}+\frac{h^2}{2}\frac{\partial^2 f_0}{\partial x^2}+\frac{k^2}{2}\frac{\partial^2 f_0}{\partial y^2}+hk\frac{\partial^2 f_0}{\partial x\partial y}+O(\Delta^3), \tag{1}$$

where

$$f=f(x,y),\ f_0=f(x_0,y_0),\ h=x-x_0,\ k=y-y_0,\ \Delta=\sqrt{h^2+k^2}.$$

Writing eq. (1) for each of m nodes in the star, we arrive to the set of linear equations

$$[A]\{Df\}-\{f\}=\{0\},$$

where

$$[A]=\begin{bmatrix} h_1 & k_1 & h_1^2/2 & k_1^2/2 & h_1k_1 \\ h_2 & & & & \vdots \\ \vdots & & & & \vdots \\ h_m & k_m & . & . & . \end{bmatrix}$$

$$\{f\}^T=\{f_1-f_0, f_2-f_0, \dots, f_m-f_0\}$$

for the five unknown derivatives at the point (x_0, y_0)

$$\{Df\}^T=\left\{\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}, \frac{\partial^2 f_0}{\partial x^2}, \frac{\partial^2 f_0}{\partial y^2}, \frac{\partial^2 f_0}{\partial x\partial y}\right\}.$$

The main difficulties in successful application of considered attempt is how to avoid a singularity or an ill condition in matrix $[A]$ of eq. (2) and in obtaining acceptable derivatives. By selecting more nodes in a star ($m>5$) it is more probable to have sufficient amount of equations to obtain good approximation of derivatives. Then the Eqs. (2) become an overdetermined set of linear equations. Its solution is obtained by minimization of a norm

$$B=\sum_{i=1}^{m}\left[\left(f_0-f_i+\frac{\partial f_0}{\partial x}h_i+\frac{\partial f_0}{\partial y}k_i+\cdots\right)\frac{1}{\Delta_i^3}\right]^2=\min.$$

Writing

$$\frac{\partial B}{\partial\{Df\}}=0$$

we arrive to a set of five equations with five unknowns.

The weight coefficients $(1/\Delta_i^3)$ are inversely related to the error term in eq. (1) considering that the nearest points have more influence on the results. For the regular square mesh and 9 point star this method produces values of the derivatives one order more accurate when compared to classical FD formulae (Fig. 2). In the case of irregular mesh it results also in better solution (see section 7).

The value of m depends on the mesh, especially in the typical (regular) regions. As far as possible for saving computer time, we assume square or rectangular mesh, so that nine-point stars are suggested.

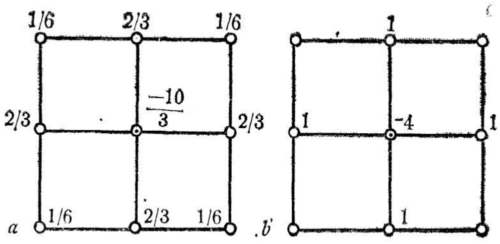

Fig. 2. FD formulas for $\nabla^2(*h^2)$
a) obtained, b) classical one

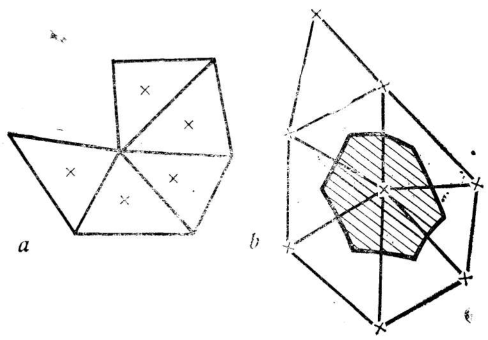

Fig. 3

4. Assignment of an area

In nonhomogeneous problems it is necessary to define the domain assigned to each node. This problem becomes especially important for analysis based on a variational approach (energy minimization).

The simplest proposals assign each node of the circle with radius dependent on the star size or the area of polygon circumscribing the star. Then the reasonable coefficient is applied to ensure the proper value of the sum of the areas, which should be equal to the area of the whole domain. Two interesting solutions are based on the triangulation of the entire domain (Fig. 3).

Subsequently the nodes are located in the centers of gravity of the triangles or in the apexes of triangles (then 1/3 of an area of surrounging triangles is assigned to the node). The independence of the area assignment and the star shape is the main imperfection of these two approaches.

5. Application to non-elliptic equations

For time-dependent problems we usually arrive to parabolic or hyperbolic differential equations. At this point it becomes necessary to ensure the stability of the generated formulae. So far there are no theoretical investigations in this problem.

Furthermore, various FD formulae, generated simply by Taylors expansions, produce often non-stable approximations.

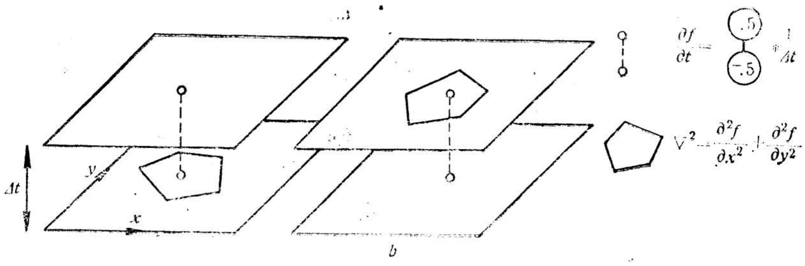

Fig. 4

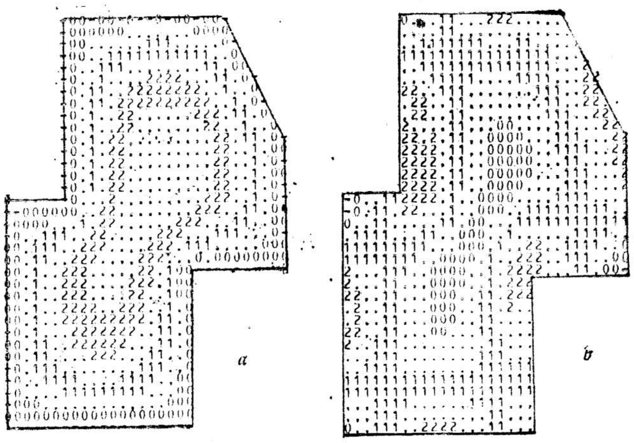

Fig. 5, a,b

Applying the mesh invariant in time, it is possible to avoid these difficulties In this case we can introduce into calculations the stable formulae for the time derivatives.

The expressions for the space derivatives (in x, y coordinates) may now be generated automatically.

As an illustration of the method discussed two possible approximations for the time dependent problem of heat transfer

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial t} \tag{5}$$

have been tested (Fig. 4).

The first scheme becomes stable only for reasonably small value Δt (mesh size in time). They should be evaluated according to the smallest space size of analysed mesh (if not, the results become unstable around the finest regions of the grid). The second scheme (Fig. 4,b) need more calculations (solving of the set of linear equations for each time step). It makes it possible, however, to assume larger time mesh spaces.

6. Non-linear problems

The FDM is universally applicable to both linear and non-linear problems. For nonlinear problems various iterative procedures can be applied. The Newton-Raphson and selfcorrecting procedures converge faster than other

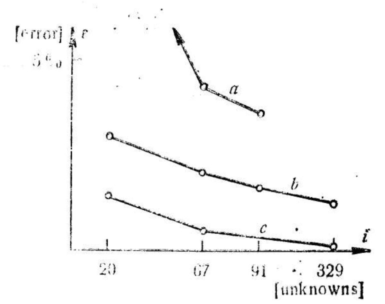

Fig. 6

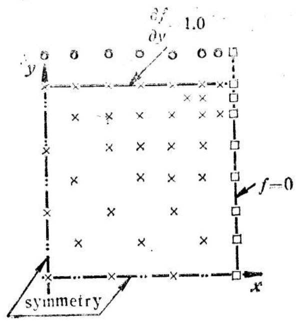

Fig. 7

approaches. They have been adopted for the FDM at arbitrary irregular grids. Designation of the "slope matrix" was made by means of previously calculated coefficients of the FD formulae.

Using FD formulation for some nonlinear problems (like elastic-plastic torsion of a bar) leads to increased errors in the vicinity of an elastic-plastic boundary (e. g. line 3, Fig. 9). They are caused by discontinuous slope of proper solution at this boundary. The arbitrary mesh enables to improve these results by inserting additional nodes along the elastic-plastic boundary and introducing an additional boundary value problem. If the assumed elastic boundary is not correct, then the mesh should be redefined during the computation.

7. Numerical results

As an example the Poisson's equation

$$\nabla^2 f = -1 \tag{6}$$

for the domain shown in Fig. 5 has been solved. The solution (Fig. 5, a) can be a stress function for the torsion of a prysmatic bar. The stress distribution is mapped in Fig. 5, b.

Fig. 6 illustrates the convergence of the solution with the number of nodes for 3 analogous programs:

a) selection of the star based on distance criterion only and generating FD formulae by the averaging process [3];

b) selection based on distance criterion and minimization procedure used for FD formulae;

c) selection of stars by "four quadrants" criterion and minimization procedure.

The CPU time consumption for the best program (c version) for selection of stars and generation of FD formulae for 329 inner nodes was 177 seconds (i. e. about 0.8 CPU seconds of CYBER 70 computer).

Temperature distribution problem for the square bar with uniformly heated boundary along $y=\pm 1$ line and constant zero temperature along $x=\pm 1$ line (Fig. 7) has been examined. The diagram (Fig. 8) shows the influence of additional nodes outside the boundary.

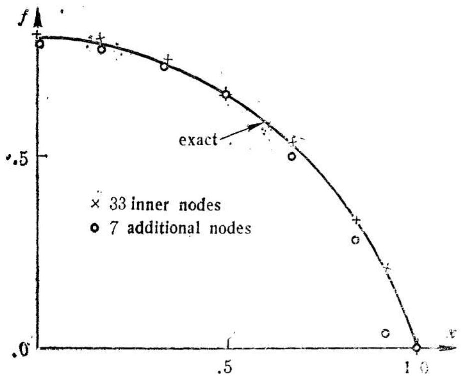

Fig. 8

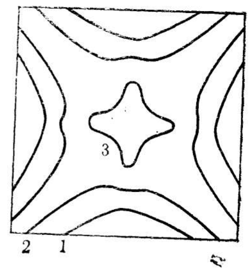

Fig. 9

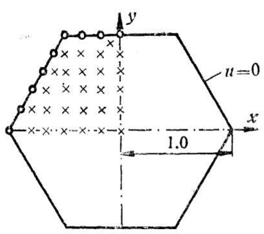

Fig. 10

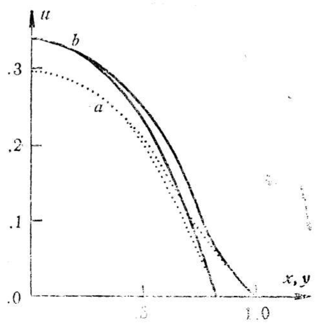

Fig. 11. Deflection of an ideal membrane: *a*) linear approximation, *b*) non-linear case

A physically nonlinear problem of a torsion of an elastic-plastic bar has been solved. Fig. 9 illustrates the elastic-plastic boundaries for several steps of the torsion process. Geometrical nonlinearity has been analysed on the case of a perfect membrane (Figs. 10, 11).

Presented solutions show the applicability of the FDM at arbitrary grids to wide class of problems of applied mechanics.

8. Conclusions

Future development of the presented set of programs will enable considerable increase of application of the method. Further research is also required for analysis based on a variational approach, mainly for precise defining of the assigned areas for integration. The basic advantage of this approach is in the fact that in the energy expression the highest order of the derivatives is only one-half of that obtained in the differential approach. For the present, the method permits to solve various real problems of applied mechanics. Due to the successful solution of the difficulties discussed, FDM at irregular meshes is universal enough to be competitive for the FEM, especially for the problems with physical or geometrical nonlinearities, optimization and time and/or temperature dependence.

References

1. Jensen, P. S. Comp. & Struct., No. 1, 1971, 17.
2. Tribiłło, R., Z. Kączkowski. Arch. Inż. Lądowej, No. 2, 1975 (in Polish).
3. Perron, N., R. Kao. Comp. & Struct., No. 5, 1975, 45.
4. Liszka, T., J. Orkisz. XVIII Polish Conf. Mech. of Solids. Wisła, Sept. 1976.
5. Liszka, T., J. Orkisz. III Polish Conf. Comp. Meth. in Mechanics, Opole, May 1977.