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Toroidal sandwich shells of uniform strength in deformed state

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1. Introduction

Elastic toroidal shells subjected to in-plane bending have been analysed by E. Reissner [1, 2]. Problems of perfectly plastic toroidal shells have been developed by P. G. Hodge and J. Skrzypek [3, 4]. It has been noticed by L. G. Brazier that flattening of the original cross-section of pipe-bend with constant thickness causes a curved tube to be more flexible in bending than a straight tube of the same cross-section [5]. Shells which are fully-stressed in deformed state, analysed in the present paper, exhibit feature of optimality from the viewpoint of the Brazier effect.

Ideal toroidal sandwich shells are analysed as a problem of the theory of shells of revolution with non-uniform wall thickness, although the circumferential displacement is not axi-symmetric (Fig. 1). Distance between the working sheets of the shell $2\bar{D}$ is assumed constant and small when compared to the radii of principal curvatures. Thicknesses of working sheets are to be found from the condition of uniform strength in deformed state.

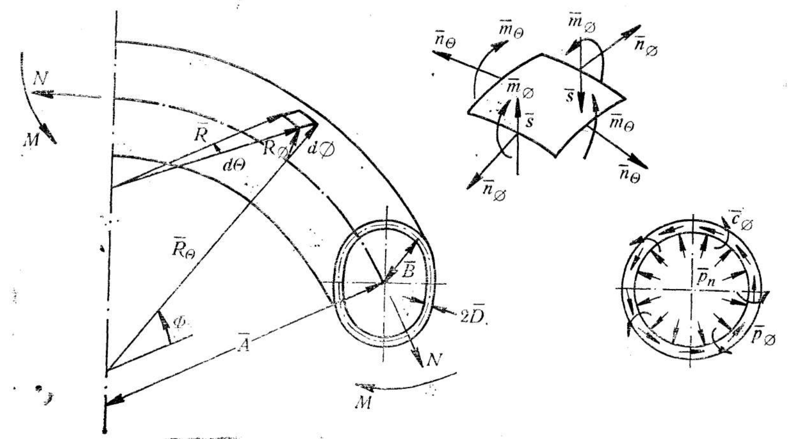

Fig. 1

Geometrically nonlinear theory of finite displacements but small strains has been applied under the Kirchhoff hypothesis and the Bernoulli assumption of plane cross-section of the toroid as a whole.

The material obeys the Hooke law; condition of uniform strength is based on the Huber-Mises-Hencky strength theory. Plane stress has been assumed in the working sheets. The influence of shearing stresses has been neglected. The shearing force, however, has been taken into account in equilibrium equations. Problem has been solved using consistently the material description.

2. Basic relations

Let us consider a general axisymmetric deformation of the element of the shell, meridional and circumferential, admissing the non-zeroth circumferential displacements (Fig. 2). Due to an appropriate rigid translation of the coordinate system we can express all quantities as functions of the angular coordinate Φ only. Dimensional quantities are denoted by a bar over the corresponding symbols. Let k be the parameter of change of curvature

(1) $$k = d\vartheta/d\theta - 1.$$

Principal strains in the sandwich sheets, meridional and circumferential, expressed in terms of undeformed geometry and the middle surface displacements $\bar{u}_r$, $\bar{u}_z$ take the form

(2)
$$\varepsilon_\Phi^{\pm} = (d\bar{l}_\Phi/d\bar{L}_\Phi)^{\pm} - 1 = \frac{\text{э}_\Phi \pm \bar{D}\bar{\varkappa}_\Phi}{1 \pm \bar{D}/\bar{R}_\Phi},$$
$$\text{э}_\Phi = \frac{\sin\Phi - \bar{u}_r/\bar{R}_\Phi}{\sin\varphi} - 1, \quad \bar{\varkappa}_\Phi = \frac{\varphi' - 1}{\bar{R}_\Phi},$$
$$\varepsilon_\theta^{\pm} = (d\bar{l}_\theta/d\bar{L}_\theta)^{\pm} - 1 = \frac{\text{э}_\theta \pm \bar{D}\bar{\varkappa}_\theta}{1 \pm \bar{D}/\bar{R}_\theta},$$
$$\text{э}_\theta = \frac{\bar{u}_r(1+k) + \bar{R}_\theta \cos\Phi k}{\bar{R}_\theta \cos\Phi}, \quad \bar{\varkappa}_\theta = \frac{\cos\varphi(1+k) - \cos\Phi}{\bar{R}_\theta \cos\Phi},$$

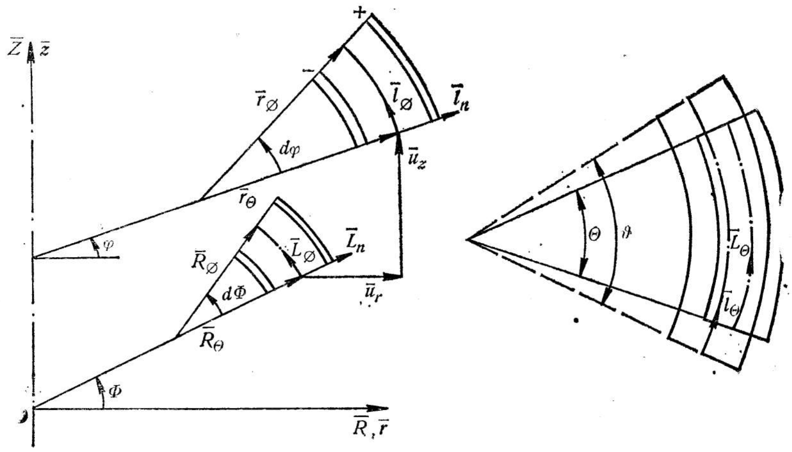

Fig. 2

whereas displacements and the angle of slope are defined by the formulae

$$\overline{u}_r^{\pm}=\overline{u}_r\pm\overline{D}(\cos\varphi-\cos\Phi),$$

$$(3)\qquad \overline{u}_z^{\pm}=\overline{u}_z\pm\overline{D}(\sin\varphi-\sin\Phi),$$

$$\varphi=\operatorname{arctg}\frac{\sin\Phi-\overline{u}_r'/R_\Phi}{\cos\Phi+\overline{u}_z'/R_\Phi}.$$

Upper and lower signs in relations (3) refer to the exterior and interior sheets, respectively. Infinitesimal strain rates and increments of displacements are easily obtained by differentiating (2) and (3) with respect to time.

To derive equations of equilibrium the principle of virtual work has been used [2]:

$$(4)\qquad \delta W_e=\delta W_i.$$

Introducing generalized stresses and resultant shearing force, all per unit length, $\overline{n}_\Phi$, $\overline{m}_\Phi$, $\overline{n}_\theta$, $\overline{m}_\theta$, $\overline{s}$, as well as reducing the external forces applied to sandwich sheets to the middle surface pressure, normal and tangential, and couple, all per unit surface, $\overline{p}_n$, $\overline{p}_\Phi$, $\overline{c}_\Phi$, we come finally to the following form of the equations of equilibrium:

$$(\overline{R}_\theta\cos\Phi\overline{n}_\Phi)'+\overline{R}_\Phi\sin\varphi(1+k)\overline{n}_\theta+\varphi'\overline{R}_\theta\cos\Phi\overline{s}=-\overline{R}_\Phi\overline{R}_\theta\cos\Phi\overline{p}_\Phi,$$

$$(5)\qquad (\overline{R}_\theta\cos\Phi\overline{s})'-\overline{R}_\Phi\cos\varphi(1+k)\overline{n}_\theta-\varphi'\overline{R}_\theta\cos\Phi\overline{n}_\Phi=-\overline{R}_\Phi\overline{R}_\theta\cos\Phi\overline{p}_n,$$

$$(\overline{R}_\theta\cos\Phi\overline{m}_\Phi)'+\overline{R}_\Phi\sin\varphi(1+k)\overline{m}_\theta+\overline{R}_\Phi\overline{R}_\theta\cos\Phi\overline{s}=-\overline{R}_\Phi\overline{R}_\theta\cos\Phi\overline{c}_\Phi.$$

Primes denote here differentiation with respect to the Lagrangian angular coordinate Φ.

As stress-strain relations we apply the Hooke law

$$(6)\qquad \varepsilon_\Phi^{\pm}=\frac{1}{\overline{E}}(\overline{\sigma}_\Phi^{\pm}-\nu\overline{\sigma}_\theta^{\pm}),\quad \varepsilon_\theta^{\pm}=\frac{1}{\overline{E}}(\overline{\sigma}_\theta^{\pm}-\nu\overline{\sigma}_\Phi^{\pm}).$$

Thicknesses of working sheets $\overline{j}^{\pm}=\overline{j}^{\pm}(\Phi)$ are to be found from the condition of uniform strength

$$(7)\qquad (\overline{\sigma}_\Phi^{\pm})^2-\overline{\sigma}_\Phi^{\pm}\overline{\sigma}_\theta^{\pm}+(\overline{\sigma}_\theta^{\pm})^2=\overline{\sigma}_0^2,$$

$\overline{E}$, ν denote the Young moduli and the Poisson ratio and $\overline{\sigma}_0$ — allowable stress of sandwich sheet material.

3. Reduction of governing equations in dimensionless form

For convenience we introduce dimensionless formulae for all quantities applied. Let $\overline{D}^*$ denote the characteristic dimension of the shell, $\overline{n}_0=2\overline{\sigma}_0\overline{D}^*$, $\overline{m}_0=2\overline{\sigma}_0\overline{D}\overline{D}^*$, characteristic values of force and moment per unit length. Dimensionless quantities have been defined as follows:

$$\sigma_0=\overline{\sigma}_0/\overline{E}\quad D=\overline{D}/\overline{D}^*\quad j^{\pm}=\overline{j}^{\pm}/\overline{D}^*\quad u_r,u_z,u_\theta=(\overline{u}_r,\overline{u}_z,\overline{u}_\theta)/\overline{D}^*,$$

$$R_\Phi,R_\theta,R=(\overline{R}_\Phi,\overline{R}_\theta,\overline{R})/\overline{D}^*\quad r_\Phi,r_\theta,r=(\overline{r}_\Phi,\overline{r}_\theta,\overline{r})/\overline{D}^*,$$

$$L_\Phi, L_\theta, L_n = (\bar{L}_\Phi, \bar{L}_\theta, \bar{L}_n)/\bar{D}^* \quad l_\Phi, l_\theta, l_n = (\bar{l}_\Phi, \bar{l}_\theta, \bar{l}_n)/\bar{D}^*,$$
$$n_\Phi, n_\theta = (\bar{n}_\Phi, \bar{n}_\theta)/\bar{n}_0 \quad m_\Phi, m_\theta = (\bar{m}_\Phi, \bar{m}_\theta)/\bar{m}_0,$$
$$s = \bar{s}/\bar{n}_0 \quad p_n, p_\Phi = (\bar{p}_n, \bar{p}_\Phi)(\bar{D}^*/\bar{n}_0) \quad c_\Phi = \bar{c}_\Phi(\bar{D}^*/\bar{m}_0). \tag{8}$$

Applying consistently the assumption of small strains as well as the assumption of small thickness of the shell and introducing convenient parametrization of the equation (7)

$$\bar{\sigma}^\pm_\Phi = \frac{2\bar{\sigma}_0}{\sqrt{3}} \sin\left(\omega^\pm + \frac{\pi}{3}\right) \quad \bar{\sigma}^\pm_\theta = \frac{2\bar{\sigma}_0}{\sqrt{3}} \sin \omega^\pm \tag{9}$$

we come, after elimination of strains, to the final system of eight governing equations: six differential and two non-differential:

$$\left.\begin{aligned}
&u_r = \sigma_0 R\left[(2-\nu)\sin\omega^+/\sqrt{3} - \nu\cos\omega^+\right] - Rk - D(\cos\varphi - \cos\Phi)\\
&u_r = \sigma_0 R\left[(2-\nu)\sin\omega^-/\sqrt{3} - \nu\cos\omega^-\right] - Rk + D(\cos\varphi - \cos\Phi)\\
&u_r' = R_\Phi(\sin\Phi - \sin\varphi) - (\sigma_0/2)R_\Phi\sin\varphi\,[(1-2\nu)(\sin\omega^+\\
&\qquad + \sin\omega^-)/\sqrt{3} + \cos\omega^+ \cos\omega^-]\\
&\varphi' = 1 + (\sigma_0/2D)R_\Phi[(1-2\nu)(\sin\omega^+ - \sin\omega^-)/\sqrt{3} + \cos\omega^+ - \cos\omega^-]\\
&u_z' = R_\Phi(\cos\Phi - \cos\varphi) + (\sigma_0/2)R_\Phi\cos\varphi\,[(1-2\nu)(\sin\omega^+\\
&\qquad + \sin\omega^-)/\sqrt{3} + \cos\omega^+ + \cos\omega^-]
\end{aligned}\right\} \tag{10a}$$

$$\left.\begin{aligned}
&j^{\pm\prime} = -j^\pm\left[\frac{(1 \pm D\cos\varphi/R)'}{1 \pm D\cos\varphi/R} + \omega^{\pm\prime}\operatorname{ctg}\left(\omega^\pm + \frac{\pi}{3}\right) + \frac{R'}{R}\right.\\
&\qquad \left. + \frac{R_\Phi \sin\varphi \sin\omega^\pm(1 \pm D\varphi'/R_\Phi)}{R\sin(\omega^\pm + \pi/3)(1 \pm D\cos\varphi/R)}\right] - \frac{\sqrt{3}[s(\varphi' \mp R_\Phi/D) + R_\Phi(p_\Phi \mp c_\Phi)]}{2\sin(\omega^\pm + \pi/3)(1 \pm D\cos\varphi/R)}\\
&s' = (1/\sqrt{3})\,\varphi'[j^+ \sin(\omega^+ + \pi/3)(1 + D\cos\varphi/R) + j^- \sin(\omega^- + \pi/3)(1 - D\cos\varphi/R)]\\
&\quad - R'/R + (R_\Phi\cos\varphi/\sqrt{3}R)[j^+\sin\omega^+(1 + D\varphi'/R_\Phi) + j^-\sin\omega^-(1 - D\varphi'/R_\Phi)] - R_\Phi p_n
\end{aligned}\right\} \tag{10b}$$

Eqs. (10a) and (10b) together with the following six boundary conditions

$$\varphi(0) = \varphi(\pi) - \pi = u_z(0) = u_z(\pi) = s(0) = s(\pi) = 0 \tag{11}$$

determine eight unknown functions u_r, u_z, φ, j^+, j^-, s, ω^+, ω^-. They can be solved numerically as a two-point BVP.

System of coupled Eqs. (10) can be separated, however, into two subsystems (10a) and (10b). To this end let us eliminate u_r, ω^+ and ω^- from the first four of Eqs. (10a) to yield the equation

$$\frac{dX}{d\Phi} = \pm(R_\Phi/R)\sin X \sin(\Phi + X); \quad X(0) = X(\pi) = 0, \tag{12}$$

where $X = (\varphi - \Phi)/2$. It follows from Eqs. (10) through (12) that it must always occur

$$\Phi \equiv \varphi \wedge \omega^+ \equiv \omega^- \equiv \omega. \tag{13}$$

Inserting (13) again into (10a) we separate them as an independent subsystem

$$u_r=\sigma_0 R\,[(2-\nu)\sin\omega/\sqrt{3}-\nu\cos\omega]-Rk$$

$$\omega'=\left(\frac{R'}{R}\right)\frac{(1+\nu)(\cos\omega-\sin\omega/\sqrt{3})-k/\sigma_0}{\nu\sin\omega+(2-\nu)\cos\omega/\sqrt{3}} \tag{14}$$

$$u_z'=-\sigma_0 R'\operatorname{ctg}\Phi\,[(1-2\nu)\sin\omega/\sqrt{3}+\cos\omega],$$

which can be solved separately using the third and fourth of the boundary conditions (11). First two boundary conditions are satisfied automatically due to (13), so that the subsystem of three differential equations (10b) is to be solved together with the last two only boundary conditions (11). Additionally tnside the region of admissible loads the following physical restrictions on positive thicknesses

$$\inf j^+\geq 0\ \wedge \inf j^-\geq 0 \tag{15}$$

must be satisfied. Finally either the optimal distribution of total thickness between two sheets: exterior and interior (satisfying the condition of minimum volume), or the boundary of admissible loads

$$\inf j^+=0\ \wedge \inf j^-=0 \tag{16}$$

is to be found.

4. Solution for the case of circular toroid

In the case if the generating curve for toroid in original configuration is assumed to be a circle

$$R_\Phi=B\quad R=A+B\cos\Phi \tag{17}$$

the subsystem (14) can be solved analytically. To this end let us eliminate ω from the first two Eqs. (14) to yield

$$\left[-(2-\nu)\frac{u_r'}{R'}+(1-2\nu)\frac{u_r}{R}+(1-2\nu)\,k\right]^2+3\left[\nu\frac{u_r'}{R'}+\frac{u_r}{R}+k\right]^2=\alpha^2, \tag{18}$$

where $\alpha=2\sigma_0(1-\nu^2)$.

It is easy to prove that for the original geometry described by (17) the equation (18) is satisfied if and only if the generating curve for toroid in deformed state is geometrically similar to the original one. In other words, the circular cross section before deformation remains circular also after deformation, with the change in radius only. Furthermore, in this case the stress function ω must be constant throughout the shell

$$\omega=\text{const}, \tag{19}$$

so that the state of stresses in working sheets is uniform. Consequently, the bending state in the wall may appear only as a result of different thicknesses of sandwich sheets. The complete solution of equation (18) takes now the simple form

$$u_r=\sigma_0\,[(1-2\nu)\sin\omega/\sqrt{3}+\cos\omega]\,(A+B\cos\Phi), \tag{20}$$

where the constant ω is interrelated with the parameter of main curvature k by

$$k=\sigma_0(1+\nu)(\sin\omega/\sqrt{3}-\cos\omega). \tag{21}$$

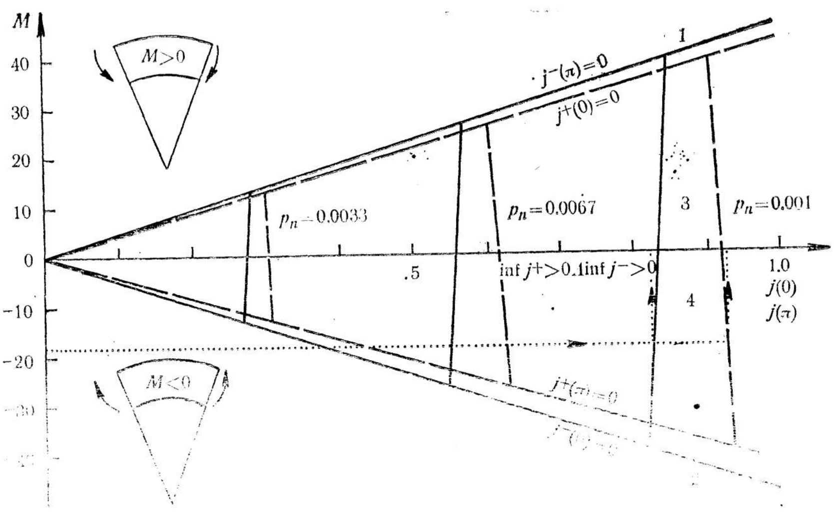

Fig. 3

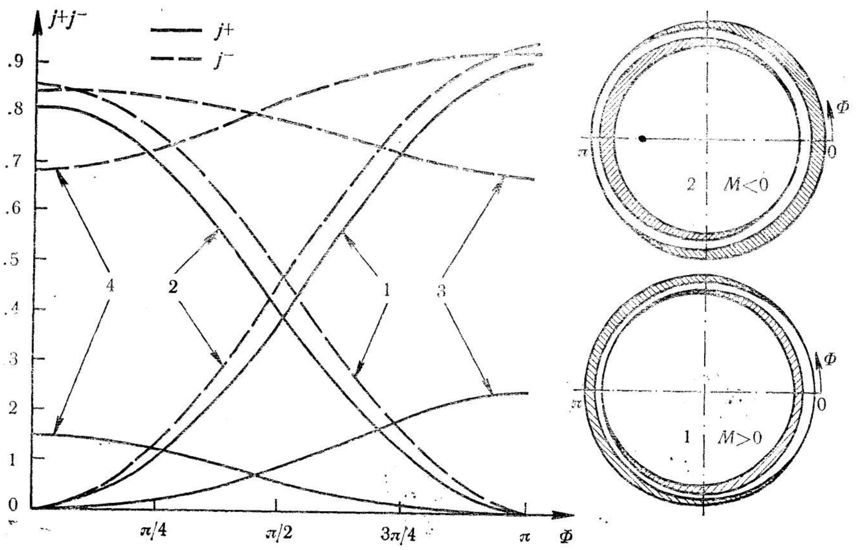

Fig. 4

Introducing (21) into (10b) we come to the problem of solving of the system of three quasi-linear differential equations which has been solved by the use of an appropriate numerical technique for two point BVP. The program has been made in algol 1204, providing not only the optimal distribution of thicknesses of sandwich sheets and associated with the distributions of internal forces, but also the range of admissible loads (e. g. bending moment and pressure) for which the solution exists.

Fig. 3 shows the region of admissible moment obtained from the conditions (16) for the data

$$A=200 \quad B=20 \quad \nu=0.5 \quad p_\Phi=c_\Phi=0. \tag{22}$$

The dotted diagram presents the procedure for finding the appropriate total thickness for the given bending moment and pressure. The typical distributions of total thicknesses between sandwich sheets for points on the boundaries of the region (1 and 2), or for points inside the region (3 and 4) (with condition of minimum volume considered) have been shown on Fig. 4.

It follows from Fig. 3 that the value of admissible bending, due to the membrane state in sandwich sheets, is limited by the value of internal pressure applied. The range of bending can be extended by inserting concentrated masses at the points $\Phi=0$ and $\Phi=\pi$. In other words, toroid may be reinforced by introducing two longitudinal ribs carrying the most of the bending moment. Instead of the last two boundary conditions (11) we may assume nonzeroth values of shearing force at points considered, which determine cross-sections of ribs. This problem will be discussed separately.

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