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On a Certain Quasi-Static Problem of Thermodiffusion in an Elastic Cylinder

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The thermodiffusion phenomenon in a long elastic cylinder is investigated in this paper. The following assumptions are true:

— the surface of the cylinder is free;

— the temperature of the surface is determined function of angle φ and time;

— chemical potential is a constant on the surface of the cylinder;

— we consider the plane state of the strains;

— we neglect body forces;

— we neglect elastic dilatation in heat and diffusive equations;

— we neglect inertia terms in equations of motion.

The constitutive equations for the considered body read [1]

$$\begin{aligned} &\sigma_{ij}=2_\mu\varepsilon_{ij}+(\lambda\varepsilon_{kk}+\gamma_T^\Theta+\gamma_M M)\delta_{ij}, \\ &C=\gamma_M\varepsilon_{kk}+\frac{d}{a}\,\Theta+\frac{1}{a}\,M, \qquad \delta_{ij}=\begin{cases}1 & \text{for } i=j\\ 0 & \text{for } i\neq j.\end{cases}\end{aligned} \tag{1}$$

Here σ_{ij} and ε_{ij} denote the stress tensor and strain tensor, respectively, $\Theta=T-T_p$, where T is the absolute temperature of the solid, T_p is the reference temperature of a natural state with zero strains and zero stresses and C stands for a concentration field. The function M is to be identified with the chemical potential of the solid. Moreover, $\lambda=\lambda_0-a\gamma_M^2$; λ_0, μ — Lamé constants; a — coefficient of diffusion, $\gamma_M=\frac{1}{a}(3\lambda_0+2_\mu)\alpha_c$, $\gamma_T=(3\lambda_0+2_\mu)\,\alpha_T+d\gamma_M$, where α_c is the coefficient of linear diffusive expansion and α_T is the coefficient of linear thermal expansion; d — coefficient of thermodiffusion.

The system of field equations describing the phenomenon of thermodiffusion in a two-component solid body [1] in the considered case has the following form:

$$\left(\nabla^2-\frac{1}{\varkappa_M}\frac{\partial}{\partial t}\right)M-\delta_M\frac{a\partial\Theta}{\partial t}=0;\quad \left(\nabla^2-\frac{1}{\varkappa_T}\frac{\partial}{\partial t}\right)\Theta-\delta_T\frac{\partial M}{\partial t}=0; \tag{2}$$

$$\begin{aligned} &(\lambda+\mu)\frac{\partial e}{\partial r}+\mu\left[\nabla^2u_r-\frac{1}{r^2}\left(u_r+2\frac{\partial u_\varphi}{\partial\varphi}\right)\right]=\frac{\partial}{\partial_r}[\gamma_T\Theta+\gamma_M M], \\ &(\lambda+\mu)\frac{\partial e}{r\partial\varphi}+\mu\left[\nabla^2u_\varphi-\frac{1}{r^2}\left(u_\varphi-2\frac{\partial u_r}{\partial\varphi}\right)\right]=\frac{\partial}{r\partial\varphi}[\gamma_T\Theta+\gamma_M M],\end{aligned} \tag{3}$$

where $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$ — a Laplace operator in a polar system of coordinates r, φ; $\varkappa_M$ — coefficient of diffusion, $\varkappa_T$ — thermal diffusivity, $\delta_M=d/\varkappa_M$, $\delta_T=\frac{d}{\varkappa_T}(ac^{\varepsilon,c}+d^2)^{-1}$, $c^{\varepsilon,c}$ denote the specific heat at constant strain and concentration; u_r, u_φ — displacements, $e=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_\varphi}{r\partial\varphi}$ — dilatation.

The boundary and initial conditions in the considered problem have the following form:

$$M(r,\varphi,0)=0=\Theta(r,\varphi,0), \tag{4}$$

$$M(R,\varphi,t)=M_0,\quad \Theta(R,\varphi,t)=T(\varphi)\,e^{-\beta_T t}, \tag{5}$$

$$\sigma_{rr}(R,\varphi,t)=0=\sigma_{r\varphi}(R,\varphi,t);\quad |\sigma_{\alpha\beta}(0,\varphi,t)|<\infty, \tag{6}$$

where R — the radius of the cylinder; β_T — a constant.

Using the strain-displacement relation

$$2\varepsilon_{ij}=u_{i,j}+u_{j,i}$$

and the relation $(1)_1$ one can easily present the conditons (6) in terms of displacements u_r, u_φ.

Introducing the boundary value for the chemical potential in the form

$$M(R,\varphi,t)=M(\varphi)\,e^{-\beta_M t},\quad \beta_M \text{ — a constant}, \tag{$(5)_1'$}$$

we see that the system of eqs. (2) with the conditions (4), $(5)_1'$ and $(5)_2$ is symmetrical. Solving these equations and putting $\beta_M=0$, $M(\varphi)\equiv M_0$ we obtain expressions for Θ and M.

If the functions $T(\varphi)$ and $M(\varphi)$ satisfy the Dirichlet conditions with respect to $\varphi\in(-\pi,\pi\rangle$, we can write the Fourier expansions for them. Putting these expansions into eqs. (2) and conditions (4), $(5)_1'$ and $(5)_2$ we obtain a initial-boundary value problem as presented below:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}-\begin{Bmatrix}\varkappa_T^{-1}\\ \varkappa_M^{-1}\end{Bmatrix}\frac{\partial}{\partial t}\right)\begin{Bmatrix}\Theta_n^{c,s}\\ M_n^{c,s}\end{Bmatrix}-\frac{\partial}{\partial t}\begin{Bmatrix}\delta_T M_n^{c,s}\\ \delta_M\Theta_n^{c,s}\end{Bmatrix}=0, \tag{7}$$

$$M_n^{c,s}(R,t)=M_n^{c,s}\exp(-\beta_M t),\quad \Theta_n^{c,s}(R,t)=T_n^{c,s}\exp(-\beta_T t), \tag{8}$$

$$M_n^{c,s}(r,0)=0=\Theta_n^{c,s}(r,0),\quad n=0,1,\dots, \tag{9}$$

where $\Theta_n^{c,s}(r,t)$, $M_n^{c,s}(r,t)$, $T_n^{c,s}$, $M_n^{c,s}$ — coefficients of the Fourier expansions of the functions $\Theta(r,\varphi,t)$, $M(r,\varphi,t)$, $T(\varphi)$ and $M(\varphi)$, respectively. We obviously have

$$\Theta_0^s=M_0^s=T_0^s=M_0^s\equiv 0.$$

Applying the finite Hankel trasform [2] (with respect to $r\in\langle 0,R\rangle$) to (7) we obtain a system of two ordinary differential equations of the first order with respect to t ($n=0,1,\dots$). A characteristic equation of this system has the form

$$s^2+\frac{\mu_{ni}^2\alpha_1}{R^2\alpha_2}s+\frac{\mu_{ni}^4}{R^4\alpha_2}=0, \tag{10}$$

where $\alpha_1=\frac{1}{\varkappa_T}+\frac{1}{\varkappa_M}$, $\alpha_2=\frac{1}{\varkappa_T\varkappa_M}-\delta_T\delta_M$, μ_{ni} — the positive roots of the equation $J_n(z)=0$. The equation (10) has two different roots. Making use of the relations [1]

$$\delta_T = \frac{d}{\varkappa_T(ac^{\varepsilon,c}+d^2)}, \quad \delta_M = \frac{d}{\varkappa_M}$$

we obtain

(11) $$\alpha_2 = \frac{1}{\varkappa_T \varkappa_M}\left(1 - \frac{d^2}{ac^{\varepsilon,c}+d^2}\right) > 0,$$

because $a>0$ and $c^{\varepsilon,c}>0$ [1]. This inequality is very important for the thermodiffusion problem because it shows that the boundary conditions for temperature and chemical potential are independent. The relation presented above always holds.

It is easy to resolve the initial-boundary value problem (7) — (9) and the form of results is not so complicated. But to solve the equations (3) with boundary conditions (6) is not as simple as above. The solution of eqs. (3) have a very complicated form. In this short communication we show the stress, displacement and concentration field for the axisymmetrical problem only; the expressions for temperature and chemical potential we show for the non-axially-symmetric problem, which are not so complicated.

In the presented expressions the material coefficients are restricted in the following way:

(12) $$d \ll \varkappa_M, \quad d \ll \varkappa_T ac^{\varepsilon,c}, \quad \beta_T \ll \frac{\varkappa_X}{R^2} \quad (X = T, M).$$

In real processes we often encounter such cases of the considered problems. This might be for example the very slow cooling down of the cylinder surface and the small influence of diffusion on temperature and vice versa.

Applying the relations due to summation of some Fourier-Bessel series shown in [3], we obtain in this case

(13) $$\Theta(\varrho,\varphi,t) \approx e^{-\beta_T t} \sum_{n=0}^{\infty} \varrho^n (T_n^c \cos n\varphi + T_n^s \sin \varphi) - \frac{2d\varkappa_M M_0}{ac^{\varepsilon,c}(\varkappa_T - \varkappa_M)} \times \sum_{i=1}^{\infty} \frac{J_0(\varrho\mu_{0i})}{\mu_{0i} J_1(\mu_{0i})} e^{-\mu_{0i}^2 \overline{F}_0},$$

(14) $$M(\varrho,\varphi,t) \approx M_0 \left[1 - 2\sum_{i=1}^{\infty} \frac{J_0(\varrho\mu_{0i})}{\mu_{0i} J_1(\mu_{0i})} e^{-\mu_{0i}^2 \overline{F}_0}\right] - \frac{2d\varkappa_T}{\varkappa_T - \varkappa_M} e^{-\beta_T t} \sum_{n=0}^{\infty}\sum_{i=1}^{\infty} \frac{\mu_{ni} J_n(\varrho\mu_{ni})[T_n^c \cos n\varphi + T_n^s \sin n\varphi]}{(\mu_{ni}^2 - \beta_T R^2 \alpha_1) J_{n+1}(\mu_{ni})} e^{-\mu_{ni}^2 \overline{F}},$$

(15) $$u_r^0(\varrho,t) \approx \frac{R\varrho}{2(1-c^2)}[m_T T_0 e^{-\beta_T t} + m_M M_0] - 2RM_0\left[\frac{d\varkappa_M m_T}{ac^{\varepsilon,c}(\varkappa_T - \varkappa_M)} + m_M\right] \sum_{i=1}^{\infty} \frac{J_1(\varrho\mu_{0i}) + \frac{\varrho c^2}{1-c^2} J_1(\mu_{0i})}{\mu_{0i}^2 J_1(\mu_{0i})} e^{-\mu_{0i}^2 \overline{F}_0} - 2RT_0 e^{-\beta_T t}\left[\frac{d\varkappa_T m_M}{\varkappa_T - \varkappa_M} + m_T\right] \sum_{i=1}^{\infty} \frac{J_1(\varrho\mu_{0i}) + \frac{\varrho c^2}{1-c^2} J_1(\mu_{0i})}{(\mu_{0i}^2 - \beta_T R^2 \alpha_1) J_1(\mu_{0i})} e^{-\mu_{0i}^2 \overline{F}_0},$$

$$(16)\qquad \sigma^0_{rr}(\varrho, t) \approx 4\mu M_0 \left[\frac{d\varkappa_M m_T}{ac^{\varepsilon,c}(\varkappa_T-\varkappa_M)} + m_M\right] \sum_{i=1}^{\infty} \frac{J_1(\varrho\mu_{0i}) - \varrho J_1(\mu_{0i})}{\varrho\mu_{0i}^2 J_1(\mu_{0i})} e^{-\mu_{0i}^2 \bar{F}_0}$$

$$+\frac{4\mu d\varkappa_T}{\varkappa_T-\varkappa_M} m_M T_0 e^{-\beta_T t} \sum_{i=1}^{\infty} \frac{J_1(\varrho\mu_{0i}) - \varrho J_1(\mu_{0i})}{\varrho(\mu_{0i}^2 - \beta_T R^2 a_1) J_1(\mu_{0i})} e^{-\mu_{0i}^2 \bar{F}_0},$$

$$(17)\qquad \sigma^0_{\varphi\varphi}(\varrho, t) \approx -4_\mu M_0 \left[\frac{d\varkappa_M m_T}{ac^{\varepsilon,c}(\varkappa_T-\varkappa_M)} + m_M\right] \sum_{i=1}^{\infty} \left\{ \frac{e^{-\mu_{0i}^2 \bar{F}_0}}{\varrho\mu_{0i}^2 J_1(\mu_{0i})}\right.$$

$$\left.\times[J(\varrho\mu_{0i}) - \varrho\mu_{0i} J_0(\varrho\mu_{0i}) + \varrho J_1(\mu_{0i})]\right\} - \frac{4\mu d\varkappa_T}{\varkappa_T-\varkappa_M} m_M T_0 e^{-\beta_T t}$$

$$\times \sum_{i=1}^{\infty} \frac{J_1(\varrho\mu_{0i}) - \varrho\mu_{0i} J_0(\varrho\mu_{0i}) + \varrho J_1(\mu_{0i})}{\varrho(\mu_{0i}^2 - \beta_T R^2 a_1) J_1(\mu_{0i})} e^{-\mu_{0i}^2 \bar{F}_0},$$

$$(18)\qquad c^0(\varrho, t) \doteq T_0 e^{-\beta_T t} \left\{ \frac{d}{a} + \frac{\gamma_M m_T}{1-c^2} - \frac{2d\varkappa_T}{\varkappa_T-\varkappa_M} \left[\frac{2c^2\gamma_M m_M}{1-c^2}\right.\right.$$

$$\left.\left.\times \sum_{i=1}^{\infty} \frac{e^{-\mu_{0i}^2 \bar{F}_0}}{\mu_{0i}^2 - \beta_T R^2 a_1} + \left(\gamma_M m_M + \frac{1}{a}\right) \sum_{i=1}^{\infty} \frac{\mu_{0i} J_0(\varrho\mu_{0i}) e^{-\mu_{0i}^2 \bar{F}_0}}{(\mu_{0i}^2 - \beta_T R^2 a_1) J_1(\mu^0{}_i)}\right]\right\}$$

$$+ M_0 \left\{ \frac{1}{a} + \frac{\gamma_M m_M}{1-c^2} - \frac{4c^2\gamma_M}{1-c^2} \left[\frac{d\varkappa_T m_T}{ac^{\varepsilon,c}(\varkappa_T-\varkappa_M)} + m_M \right] \sum_{i=1}^{\infty} \frac{1}{\mu_{0i}^2} e^{-\mu_{0i}^2 \bar{F}_0}\right.$$

$$\left. - 2\left[\gamma_M m_M + \frac{1}{a} + \frac{d\varkappa_M \gamma_M m_T}{ac^{\varepsilon,c}(\varkappa_T-\varkappa_M)}\right] \sum_{i=1}^{\infty} \frac{J_0(\varrho\mu_{0i})}{\mu_{0i} J_1(\mu_{0i})} e^{-\mu_{0i}^2 \bar{F}_0} \right\}.$$

These relations hold for $t > \frac{3R^2}{2\varkappa_T}$.

Here $\varrho = \frac{r}{R}$, $\bar{F}_0 = \frac{\varkappa_M t}{R^2}$, $m_T = \frac{\gamma_T}{\varrho_0 c_1^2}$, $m_M = \frac{\gamma_M}{\varrho_0 c_1^2}$, $c_1^2 = \frac{\lambda + 2\mu}{\varrho_0}$, $c_2^2 = \frac{\mu}{\varrho_0}$, $c^2 = \frac{c_2^2}{c_1^2}$, ϱ_0 — density, X° denote the solution for the axisymmetrical problem. ($X^\circ = u, \sigma$, etc.).

Now we can easily trace the influence of nonmechanical fields on each of the other nonmechanical ones and also on mechanical fields.

References

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