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On the Triangular Finite Element TUBA 3

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The triangular plate bending finite element with twenty-one degrees of freedom is fully compatible and consistent in its convergence. The lateral displacement w along the edges is a fifth-order curve, and the slopes normal to the edges $w,_n$ vary as a curve of fourth order. In ref. 1, considerable simplifications of the derivation of the finite element TUBA 6 (Fig. 1) have been achieved by the application of homogeneous triangular co-ordinates ζ. This is reflected also in very economical computation time for all mathematical operations.

In refs. 2, 3 and 4, it is shown that mid-side nodes 4, 5 and 6 (Fig. 1) with which only three freedoms are connected cause a significant increase in the band width of the complete structure stiffness matrix. Also, they have a disproportionally larger effect upon the dimensions of the above matrix than the corner nodes. Considerably more efficient element with eighteen degrees of freedom may be obtained without much loss of refinement by eliminating the mid-side nodes (ref. 2). In ref. 1, the new element is denoted as TUBA 3. We think that some mistakes have been mad› in the row matrix $[\alpha_4]$ (equation (40) of ref. 1). In the paper by R. G. Vos and W. P. Vann (ref. 5), the element TUBA 3 is used and also the wrong expression for $[\alpha_4]$. It can be seen in Fig. 2 of the same paper that the numerical results for the element TUBA 3 are erratic at coarse meshes and convergence is not monotonic with the mesh refinement. The authors attribute this to the inconsistent formulation of the element.

The purpose of this note is, a correct row matrix $[\alpha_4]$ to be obtained, following in general ref. 1.

The deflection w of the triangular plate element prescribed by complete polynomial function of order m = 5 may be written as

$$w=[\omega]\{p\}, \tag{1}$$

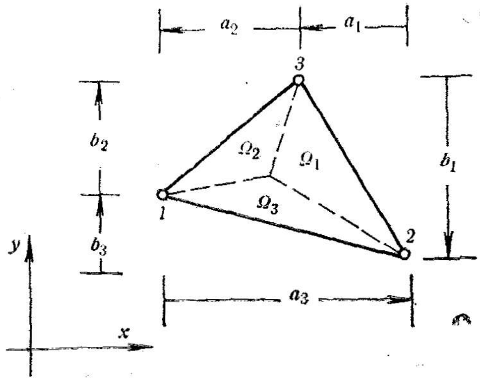
where $[\omega]$ is the (1×21) row matrix

$$\begin{aligned}[\omega]=[&\zeta_1^5 \quad \zeta_2^5 \quad \zeta_3^5 \quad \zeta_1^4\zeta_2 \quad \zeta_1^4\zeta_3 \quad \zeta_2^4\zeta_3 \quad \zeta_1\zeta_2^4 \quad \zeta_1\zeta_3^4\\ &\zeta_2\zeta_3^4 \quad \zeta_1^3\zeta_2^2 \quad \zeta_1^3\zeta_3^2 \quad \zeta_2^3\zeta_3^2 \quad \zeta_1^2\zeta_2^3 \quad \zeta_1^2\zeta_3^3 \quad \zeta_2^2\zeta_3^3\\ &\zeta_1^3\zeta_2\zeta_3 \quad \zeta_1\zeta_2^3\zeta_3 \quad \zeta_1\zeta_2\zeta_3^3 \quad \zeta_1^2\zeta_2^2\xi_3 \quad \zeta_1\zeta_2^2\zeta_3^2 \quad \zeta_1^2\zeta_2\zeta_3^2],\end{aligned} \tag{2}$$

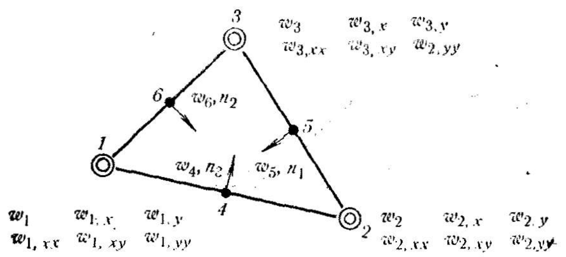
and $\{p\}$ is the (21×1) column vector

$$\{p\}^T=\langle p_1 p_2 \dots p_{21}\rangle. \tag{3}$$

Fig. 1. $\zeta_1=\Omega_1/\Omega \quad \zeta_2=\Omega_2/\Omega \quad \zeta_3=\Omega_3/\Omega$
$\Omega=\Omega_1+\Omega_2+\Omega_3$



$$\{\varrho_T\}^T=\langle w_1 \quad w_2 \quad w_3 \quad w_{1,x} \quad w_{1,y} \quad w_{2,x} \quad w_{2,y} \quad w_{3,x} \quad w_{3,y} \quad w_{1,xx} \quad w_{1,xy} \quad w_{1,yy} \quad w_{2,xx} \quad w_{2,xy} \quad w_{2,yy} \quad w_{3,xx} \quad w_{3,xy} \quad w_{3,yy} \quad w_{4,n3} \quad w_{5,n1} \quad w_{6,n2}\rangle$$



Then in ref. 1, a nodal parameter vector $\{\varrho_{TN}\}$ with dimensions (21×1) is defined. It includes the set of displacements w, the natural derivatives $w,_{ij}$ and $w,_{ij^2}$, and the normal derivatives $w,_{\nu}$ at the nodal points as shown in Fig. 2. In this case the following transformation rule holds

(4) $$\{p\}=[A]\{\varrho_{TN}\}.$$

The transformation matrix $[A]$ proves to be very advantageous from programming point of view, since only its last three rows depend on the element shape.

The final deformation vector $\{\varrho_T\}$ refers all nodal parameters to the global system O_{xyz}, with the exception of the slopes normal to the edges as shown in Fig. 1. The transformation may be written in the form

(5) $$\{\varrho_{TN}\}=[X_T]\{\varrho_T\},$$

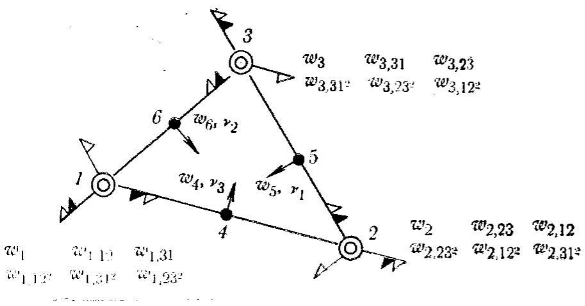
where the (21×21) matrix $[X_T]$ is already dependent upon the element dimensions a_i and b_i $(i=1, 2, 3)$.

Having $[X_T]$, the deflection w may be obtained from the final expression

(6) $$w=[\omega][A][X_T]\{\varrho_T\}.$$

If we want to eliminate the three freedoms at mid-points of the element TUBA 3, this may be done by imposing a cubic variation on the normal slope $w,_n$ along the element edges. The normal slope is expressed by the first and second derivatives at the adjoining vertices. It is evident that the compatibility of the normal slope function is retained.

In ref. 1, to obtain the deflection w without normal slope freedoms, the sets $\{\bar{\varrho}_{TN}\}$ and $\{\bar{\varrho}_T\}$ of dimensions (18×1) are introduced instead of victors $\{\varrho_{TN}\}$ and $\{\varrho_T\}$ by omitting the last three entries. Then the last (3×3) diagonal submatrix in $[X_T]$ is suppressed and the new transformation matrix is denoted as $[\bar{X}_T]$. Now, equation (5) applies for the new entities too.



$$\{\varrho_{TN}\}^T = \langle w_1 \quad w_2 \quad w_3 \quad w_{1,12} \quad w_{1,31} \quad w_{2,23} \quad w_{2,12} \quad w_{3,31} \quad w_{3,23} \quad w_{1,12^2} \quad w_{1,31^2} \quad w_{2,23^2} \quad w_{2,12^2} \quad w_{3,31^2} \cdot w_{3,23^2} \quad w_{1,23^2} \quad w_{2,31^2} \quad w_{3,12^2} \quad w_{4,\nu_3} \quad w_{5,\nu_1} \quad w_{6,\nu_2} \rangle$$

Fig. 2.

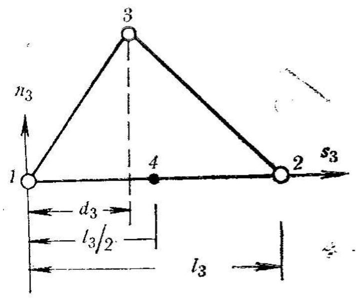


Fig. 3. The eccentricity parameter μ_3 is $\mu_3 = 2d_3/l_3 - 1$

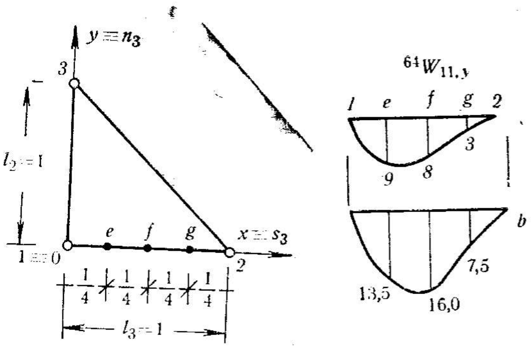


Fig. 4. $a_1 = -1$ $a_2 = 0$ $a_3 = 1$
$b_1 = -1$ $b_2 = 1$ $b_3 = 0$
$\mu_3 = -1$ $\Omega = 1/2$
along the edge l_3:
$\xi_3 = 0$ $\xi_2 = 1 - \xi_1$

It may be written

$$\{\varrho_{TN}\} = [C]\{\bar{\varrho}_{TN}\} = \begin{bmatrix} [I_{18}] \\ [C_0] \end{bmatrix} \{\bar{\varrho}_{TN}\}. \tag{7}$$

The transformation (3×18) matrix $[C_0]$ expresses the relations between the normal slope freedoms at the mid-points and the first and second natural derivatives at the corners. In ref. 1, the result for the normal slope w_{4,ν_3} is shown with no successive mathematical operations given. This will be done here.

As shown in ref. 2, the normal slope having a cubic variation along an arbitrarily oriented edge l_3 (Fig. 3) may be written in the form

$$w_{,n_3}(s_3) = c_0 + c_1 s_3 + c_2 s_3^2 + c_3 s_3^3, \tag{8}$$

and its first derivative is

$$w_{,n_3 s_3}(s_3) = c_1 + 2c_2 s_3 + 3c_3 s_3^2. \tag{9}$$

The substitution of the boundary conditions

$$w,_{n_3}(0)=w_{1,n_3}, \quad w,_{n_3 s_3}(0)=w_{1,n_3 s_3},$$
$$w,_{n_3}(l_3)=w_{2,n_3}, \quad w,_{n_3 s_3}(l_3)=w_{2,n_3 s_3},$$

into (8) and (9) yields

$$c_0=w_{1,n_3}, \quad c_1=w_{1,n_3 s_3},$$
$$c_2=3(-w_{1,n_3}+w_{2,n_3})/l_3^2-(2w_{1,n_3 s_3}+w_{2,n_3 s_3})/l_3, \tag{10}$$
$$c_3=2(w_{1,n_3}-w_{2,n_3})/l_3^3+(w_{1,n_3 s_3}+w_{2,n_3 s_3})/l_3^2.$$

The normal slope at the mid-point 4 is obtained by substituting (10) into (8) and putting $s_3=l_3/2$

$$w_{4,n_3}=(w_{1,n_3}+w_{2,n_3})/2+(w_{1,n_3 s_3}-w_{2,n_3 s_3})l_3/8. \tag{11}$$

It is easy to confirm that

$$w_{1,n_3}=[-(1+\mu_3)w_{1,12}-2w_{1,31}]\,l_3/4\Omega,$$
$$w_{2,n_3}=[(1-\mu_3)w_{2,12}+2w_{2,23}]\,l_3/4\Omega,$$
$$w_{1,n_3 s_3}=(w_{1,31^2}-\mu_3 w_{1,12^2}-w_{1,23^2})/4\Omega, \tag{12}$$
$$w_{2,n_3 s_3}=(w_{2,31^2}-\mu_3 w_{2,12^2}-w_{2,23^2})/4\Omega,$$

where Ω is the triangle area. Substituing (12) into (11) we find the désired result for w_{4,n_3}. By definition we have

$$w_{4,\nu_3}=w_{4,n_3}\,l_3/4\Omega. \tag{13}$$

In ref. 1, the following vector is defined

$$\{\varrho_{TN12}\}^T=\langle w_{1,12} \quad w_{1,31} \quad w_{2,23} \quad w_{2,12}$$
$$w_{1,12^2} \quad w_{1,31^2} \quad w_{1,23^2} \quad w_{2,23^2} \quad w_{2,12^2} \quad w_{2,31^2}\rangle \tag{14}$$

and the transformation

$$w_{4,\nu_3}=[a_4]\{\varrho_{TN12}\} \tag{15}$$

is introduced, where $[a_4]$ is the (1×10) row martix

$$[a_4]=[-2(1+\mu_3) \quad -4 \quad 4 \quad 2(1-\mu_3)$$
$$-\mu_3 \quad 1 \quad -1 \quad 1 \quad \mu_3 \quad -1]/4 \tag{16}$$

(equation (40) of ref. 1).

But for the same transformation matrix we derive from (13) the row matrix

$$[a_4]=[-2(1+\mu_3) \quad -4 \quad 4 \quad 2(1-\mu_3)$$
$$-0.5\mu_3 \quad 0.5 \quad -0.5 \quad 0.5 \quad 0.5\mu_3 \quad -0.5]/4. \tag{17}$$

As it will be shown further, it does not follow from expression (16) that the normal slope $w,_n$ will have a cubic variation along the edge l_3.

The row matrices $[a_5]$ and $[a_6]$ corresponding to the midpoint normal slopes w_{5,ν_1} and w_{6,ν_2} may be obtained from (17) by cyclic index changes. Then the (3×18) matrix $[C_0]$ in (7) is easily established by means of $[a_4]$, $[a_5]$ and $[a_6]$.

Having $[C_0]$ and following the above procedure, we derive instead of (6)

$$w=[\omega][A][C][\bar{X}_T]\{\bar{\varrho}_T\}. \tag{18}$$

The correctness of the row matrix (17) may be proved as follows. From (18) we find

$$w=[W]\{\bar{\varrho}_T\}=W_1w_1+W_2w_2+\cdots+W_{18}w_{3,yy}, \tag{19}$$

where $W_i(i=1,\ldots, 18)$ are the interpolation functions written in terms of area co-ordinates.

To simplify the calculations, the normal slope function $w,_{n3}$ along the edge l_3 of the finite element 1-2-3 (fig. 4) will be considered. It is evident that

$$\begin{aligned} w,_{n3}=w,_y=W_{1,y}w_1+\cdots+W_{10,y}w_{1,xx}+W_{11,y}w_{1,xy}+\\ +W_{12,y}w_{1,yy}+W_{13,y}w_{2,xx}+W_{14,y}w_{2,xy}+W_{15,y}w_{2,yy}+\cdots+W_{18,y}w_{3,yy}. \end{aligned} \tag{20}$$

The difference between the row matrices (16) and (17) affects only the functions W_i corresponding to the nodal degrees of freedom $w,_{xx}$, $w,_{xy}$ and $w,_{yy}$. Since $W_{10,y}\equiv W_{12,y}\equiv W_{13,y}\equiv W_{15,y}\equiv 0$ along the edge l_3, only the function $W_{11,y}$ or $W_{14,y}$ will be investigated. For example $W_{11,y}$ describes the edge normal slope, if $w_{1,xy}=1$ and the other nodal parameters are zero. The function $W_{11,y}$ must have a cubic variation. It may be shown that

$$W_{11,y}=\frac{\partial W_{11}}{\partial y}=\frac{1}{2\Omega}\left(a_1\frac{\partial W_{11}}{\partial\zeta_1}+a_2\frac{\partial W_{11}}{\partial\zeta_2}+a_3\frac{\partial W_{11}}{\partial\zeta_3}\right). \tag{21}$$

Using the area co-ordinate function W_{11}, equation (21) and the data of Fig. 4, we have the third order function

$$W_{11,y}=-\zeta_1^3+\zeta_1^2. \tag{22}$$

$W_{11,y}$ is plotted in Fig. 4, a. Obviously $W_{11,y}(e)+W_{11,y}(g)=1.5W_{11,y}(f)$ which demonstrates again the cubic variation of the normal slope function.

Using the row matrix (16) we obtain the wrong result

$$W_{11,y}=2\zeta_1^4-5\zeta_1^3+3\zeta_1^2 \tag{23}$$

and $W_{11,y}(e)+W_{11,y}(g)\neq 1.5W_{11,y}(f)$ (see Fig. 4, b).

All numerical results obtained with a computer program, and based on a correct finite element with eighteen degrees of freedom (with constant or variable thickness) show a monotonic convergence with mesh refinement and a good accuracy even for coarse mesh idealization. Our investigations include linear plate-bending and plate-buckling problems.

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