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The Flow Field Induced by the Torsional Oscillations of Small Particle in a Spherical Container

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1. Introduction

It has been experimentally established that when a fluid is set into oscillations, as in the presence of a sound source or an oscillating boundary, steady streaming motions are generated under certain conditions. Streaming of this kind had already been reported by Faraday [1] and Dvorak [2] in the last century. A theoretical explanation of the phenomena associated with the above experiments was given for the first time by Rayleigh [3] and Carriére [4].

In 1932, in a theoretical work, Schlichting [5] found out a solution periodic in time by the approximation method of the two dimensional nonsteady boundary layer equations. It was found that the boundary conditions at the oscillating cylinder can be satisfied, and that at the large distance the tangential component of velocity is finite but not zero. Unable to satisfy that condition he relaxes it to a stipulation that the tangential velocity remains bounded at the edge of the boundary layer. This result indicates that the flow in the boundary layer which is periodic with respect to time induces a steady motion at a large distance from the wall of the body. Following the same method of analysis Lane [6] has studied the problem of an oscillating sphere in an unbounded viscous fluid.

If a body of typical dimension a oscillates with velocity $U_\infty \cos \omega_0 t$ in a fluid of kinematic viscosity ν, which is otherwise at rest, we can display the four dimensionless parameters which have appeared

$$\varepsilon = \frac{U_\infty}{\omega_0 a}, \quad \overline{M}^2 = \frac{a^2 \omega_0}{\nu}, \quad Re = \frac{U_\infty a}{\nu}, \quad Res = \frac{U_\infty^2}{\omega_0 \nu}.$$

We are concerned furthermore with the situation $\varepsilon \ll 1$. Physically this condition implies that the amplitude of the oscillation is small compared with a.

In 1965 Wang [7] examined the case of an oscillating sphere when $R_{es} = 0(1)$, but he did not determine the structure of the outer steady streaming.

Recently Riley [8—11] has studied many problems related with the vibrating bodies in an unbounded fluid. However, many points remain to be answered. One of them is what is the behaviour of the steady streaming in the case of the bounded flow?

In 1976 Di Prima and Liron [12] have solved the problem of torsionally oscillating sphere in an unbounded viscous fluid.

In this paper an examination of the transient motion of the fluid contained between two concentric spheres is presented. The outer sphere is assumed to remain at rest, while the inner one is forced to execute a torsional oscillation. The unsteady motion of the viscous fluid due to the torsional oscillation of the particle (inner sphere) in a container (outer sphere) is of interest as a guide to the importance of wall effects in the motion of a single particle. Of particular interest is the behaviour of the steady streaming in the case of the bounded flow.

2. Formulation of the problem

Let a viscous incompressible fluid be contained between two concentric spheres of radii $r=a$ and $r=b$ $(a<b)$. The inner sphere is forced to execute a torsional oscillation with frequency ω_0 and angular amplitude ε; thus, the angular velocity of the particle is $\Omega_0=\varepsilon\omega_0 e^{i\omega_0 t}$, where t is the time. Here and what follows the real part of any complex quantity is to be understood.

It is convenient to use spherical polar coordinates r, θ, φ fixed in the container, which is assumed to remain at rest. Here r is the radial coordinate, θ is the latitudinal angle, and φ is the longitudinal angle. We shall suppose that the amplitude of the oscillation is small and seek a solution of the equations that is independent of φ.

The velocity components (v'_r, v'_θ) are related to the stream function ψ by:

$$(1) \qquad v'_r=\frac{1}{r'^2\sin\theta}\cdot\frac{\partial\Psi'}{\partial\theta}, \qquad v'_\theta=-\frac{1}{r'\sin\theta}\frac{\partial\Psi'}{\partial r}.$$

Non-dimensional variables and parameters are now introduced according to the scheme

$$(2) \qquad r'=ar, \quad t'=\omega_0\tau, \quad \Psi'=\varepsilon a^3\omega_0\Psi, \quad v'=\varepsilon\omega a v, \quad M^2=\frac{\omega_0 a^2}{\nu}.$$

The governing equations of unsteady-state motion for an incompressible Newtonian fluid in terms of non-dimensional variables are

$$(3) \qquad \frac{\partial}{\partial\tau}(D^2\Psi)+\varepsilon\left[\frac{\partial\Omega}{r^2\sin^2\theta}\left(\frac{\partial\Omega}{\partial\tau}\cos\theta-\frac{1}{r}\frac{\partial\Omega}{\partial\theta}\sin\theta\right)-\frac{1}{r^2\sin\theta}\frac{\partial(\Psi,D^2\Psi)}{\partial(r,\theta)}\right.$$

$$\left.+\frac{2D^2\Psi}{r^2\sin^2\theta}\left(\frac{\partial\Psi}{\partial\tau}\cos\theta-\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\sin\theta\right)\right]=\frac{1}{|M|^2}D^4\Psi$$

$$(4) \qquad \frac{\partial\Omega}{\partial\tau}-\frac{\varepsilon}{r^2\sin\theta}\left[\frac{\partial\Psi}{\partial r}\frac{\partial\Omega}{\partial\theta}-\frac{\partial\Psi}{\partial\theta}\frac{\partial\Omega}{\partial r}\right]=\frac{1}{|M|^2}D^2\Omega.$$

Here

$$D^2\Psi=\frac{\partial^2\Psi}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right), \qquad \Omega=r\sin\theta\, w,$$

and w is the rotational speed.

The boundary conditions are

$$(5) \qquad \psi=\frac{\partial\psi}{\partial r}=0 \quad \text{at } r=1 \text{ and } r=\lambda=\frac{b}{a}$$

$$(6) \qquad \Omega=r^2\sin^2\theta e^{i\tau}(w=r\sin\theta e^{i\tau}) \quad \text{at} \quad r=1$$

(7) $\Omega=0 \qquad (w=0) \qquad \text{at} \quad r=\lambda.$

Reference to equations (3) and (4) shows that except the primary flow in torsional direction there will be a secondary flow in the planes containing the axis of oscillation. Mathematical complication of the problem (3)—(7) do not permit an analysis for arbitrary frequency. The parameter $\frac{1}{M}$ measures the thickness of the Stokes layer $O\left(\sqrt{\frac{\nu}{\omega_0}}\right)$ relative to the radius of the inner sphere. When $M \gg 1$ the space between the spheres is divided into three separate but overlapping regions: i) two Stokes layers of thickness $O\left[\left(\frac{\omega_0}{2\nu}\right)^{1/2}\right]$ adjacent to the particle and container, and ii) an intermediate region between the two boundary layers.

We introduce the following variables

(7) $$\eta=(r-1)\frac{M}{\sqrt{2}}, \quad \psi=\Psi\frac{M}{\sqrt{2}}, \quad \omega$$

in the boundary layer around the particle and

(8) $$\zeta=(\lambda-r)\frac{M}{\sqrt{2}}, \quad \bar{\psi}=\Psi\frac{M}{\sqrt{2}}, \quad \bar{\omega}$$

in the boundary layer at the container.

For small ε and large M, the boundary value problem given by equations (3)—(7) can be solved by expanding in ε and $\frac{1}{M}$. We suppose that

(9) $$F=F_{00}+\frac{1}{M}F_{01}+\ldots+\varepsilon^2\left(F_{10}+\frac{1}{M}F_{11}+\ldots\right)+\ldots,$$

where

$$F=[\omega, \Omega, \omega],$$

(10) $$G=\varepsilon\left[G_{00}+\frac{1}{M}G_{01}+\ldots\right]+\varepsilon^3\left[G_{10}+\frac{1}{M}G_{11}+\ldots\right]+\ldots,$$

where

$$G=[\psi, \Psi, \bar{\psi}].$$

3. Construction of the solution and results

Substituting (7) and (9) in equation (4) and equating the terms of equal powers of the angular displacement ε and frequency parameter $\frac{1}{M}$ we obtain the equations:

(11) $$2\frac{\partial\omega_{00}}{\partial\tau}-\frac{\partial^2\omega_{00}}{\partial\eta^2}=0$$

(12) $$\frac{\partial\omega_{01}}{\partial\tau}-\frac{\partial^2\omega_{01}}{\partial\eta^2}=0$$

(13) $$\frac{\partial\omega_{02}}{\partial\tau}-\frac{\partial^2\omega_{02}}{\partial\eta^2}=2(1-\mu^2)\frac{\partial^2\omega_{00}}{\partial\mu^2}$$

(14) $$\frac{\partial \Omega_{0i}}{\partial \tau}=\frac{1}{M^2}\left[\frac{\partial^2 \Omega_{0i}}{\partial r^2}+\frac{1-\mu^2}{r^2}\frac{\partial^2 \Omega_{0i}}{\partial \mu^2}\right] \quad (i=0,\ 1,\ 2).$$

First, the zero approximation of the circumferential velocity is computed. Using the method of matched asymptotic expansions we have found

(15) $$\omega_{00}=E(1-\mu^2)e^{i\tau},\quad \omega_{01}=0,\quad \omega_{02}=-(1-i)\eta E(1-\mu^2)e^{i\tau}$$

$$\Omega_{00}=\Omega_{01}=\Omega_{02}=0,\quad \bar{\omega}_{00}=\bar{\omega}_{01}=\bar{\omega}_{02}=0,$$

where $\mu=\cos\theta$ and $E=e^{-(1+i)\eta}$.

Then, these results are used in computing the first approximation to the pumping motion.

The stream function of the secondary flow satisfies the equations

(16) $$\frac{1}{2}\frac{\partial^4}{\partial \eta^4}\chi_{00}-\frac{\partial^3 \chi_{00}}{\partial \tau \partial \eta^2}=\frac{2\cos\theta}{\sin^2\theta}\omega_{00}\frac{\partial \omega_{00}}{\partial \eta}$$

(17) $$\frac{1}{2}\frac{\partial^4}{\partial \eta^4}\chi_{01}-\frac{\partial^3 \chi_{01}}{\partial \tau \partial \eta^2}=-\frac{4\sqrt{2}\cos\theta}{\sin^2\theta}\omega_{00}\eta\frac{\partial \omega_{00}}{\partial \eta}$$

$$+\frac{2\cos\theta}{\sin^2\theta}\omega_{00}\frac{\partial \omega_{01}}{\partial \eta}+\frac{2\cos\theta}{\sin^2\theta}\omega_{01}\frac{\partial \omega_{00}}{\partial \eta}-\frac{2\sqrt{2}}{\sin\theta}\omega_{00}\frac{\partial \omega_{00}}{\partial \theta}$$

(18) $$\frac{1}{2}\frac{\partial^4 \chi_{02}}{\partial \eta^4}-\frac{\partial^3 \chi_{02}}{\partial \tau \partial \eta^2}=-2(1-\mu^2)\frac{\partial^4 \chi_{00}}{\partial \eta^2 \partial \mu^2}+2(1-\mu^2)\frac{\partial^3 \chi_{00}}{\partial \tau \theta \mu^2}$$

$$+12\eta^2\frac{\partial \omega_{00}}{\partial \eta}\omega_{00}\frac{\cos\theta}{\sin^2\theta}+\frac{2\cos\theta}{\sin^2\theta}\omega_{00}\frac{\partial \omega_{02}}{\partial \eta}$$

$$+\frac{2\cos\theta}{\sin^2\theta}\omega_{02}\frac{\partial \omega_{00}}{\partial \eta}+\frac{4\eta}{\sin\theta}\omega_{00}\frac{\partial \omega_{00}}{\partial \theta}+\frac{8\eta}{\sin^2\theta}\omega_{00}\frac{\partial \omega_{00}}{\partial \theta}.$$

The equations for $\bar{\chi}_{00}$, $\bar{\chi}_{01}$ and $\bar{\chi}_{02}$ are like equations (16)—(18).

In the region between two boundary layers we will have

(19) $$\frac{\partial}{\partial \tau}(D^2\Psi_0)=\frac{1}{|M|^2}D^4\Psi_0,$$

where

$$\Psi_0=\Psi_{00}+\frac{1}{|M|}\Psi_{01}+\frac{1}{|M|^2}\Psi_{02}+\ \dots$$

We expect Ψ_0 to contain a term independent of τ in addition to the oscillatory one:

(20) $$\Psi_0=\Psi_0^{(s)}+e^{i\tau}\Psi_0^{(u)},$$

where the superscript (s) denotes steady and (u) unsteady. The equations for the unsteady and steady parts of Ψ_0 are

(21) $$D^2\Psi_{00}^{(u)}=0,\quad D^2\Psi_{01}^{(u)}=0,\quad D^2\Psi_{02}^{(u)}=-\frac{i}{2}D^4\Psi_{00}^{(u)}$$

$$D^4\Psi_{0k}^{(s)}=0,\quad k=0,\ 1,\ 2.$$

The corrections ω_1, Ω_1 and $\bar{\omega}_1$ of the circumferential velocity of the primary flow hold the equations

(22) $$2\frac{\partial\omega_{10}}{\partial\tau}-\frac{\partial^2\omega_{10}}{\partial\eta^2}=\frac{2}{\sin\theta}\,\frac{\partial(\chi_{00},\,\omega_{00})}{\partial(\eta,\,\theta)}$$

(23) $$2\frac{\partial\omega_{11}}{\partial\tau}-\frac{\partial^2\omega_{11}}{\partial\eta^2}=-\frac{\sqrt{2}\,\eta}{\sin\theta}\,\frac{\partial(\chi_{00},\,\omega_{00})}{\partial(\eta,\,\theta)}+\frac{2}{\sin\theta}\,\frac{\partial(\chi_{00},\,\omega_{00})}{\partial(\eta,\,\theta)}$$

(24) $$2\frac{\partial\omega_{12}}{\partial\tau}-\frac{\partial^2\omega_{12}}{\partial\eta^2}=2(1-\mu^2)\frac{\partial^2\omega_{10}}{\partial\mu^2}+\frac{12\eta^2}{\sin\theta}\,\frac{\partial(\chi_{00},\,\omega_{00})}{\partial(\eta,\,\theta)}$$

$$-\frac{4\sqrt{2}\,\eta}{\sin\theta}\,\frac{\partial(\chi_{00},\,\omega_{00})}{\partial(\eta,\,\theta)}+\frac{2}{\sin\theta}\,\frac{\partial(\chi_{02},\,\omega_{00})}{\partial(\eta,\,\theta)}+\frac{2}{\sin\theta}\,\frac{\partial(\chi_{00},\,\omega_{00})}{\partial(\eta,\,\theta)}$$

(25) $$\frac{\partial\Omega_{10}}{\partial\tau}=0,\qquad\frac{\partial\Omega_{11}}{\partial\tau}=0,\qquad\frac{\partial\Omega_{12}}{\partial\tau}=D^2\Omega_{10}$$

(26) $$2\frac{\partial\bar{\omega}_{10}}{\partial\tau}-\frac{\partial^2\bar{\omega}_{10}}{\partial\zeta^2}=0$$

(27) $$2\frac{\partial\bar{\omega}_{11}}{\partial\tau}-\frac{\partial^2\bar{\omega}_{11}}{\partial\zeta^2}=0$$

(28) $$2\frac{\partial\bar{\omega}_{12}}{\partial\tau}-\frac{\partial^2\omega_{12}}{\partial\zeta^2}=\frac{2(1-\mu^2)}{\lambda^2}\,\frac{\partial^2\bar{\omega}_{10}}{\partial\mu^2}.$$

Some terms of our solution are given by the next equations

(29) $$\chi_{01}=\Big\{-\frac{1}{2\sqrt{2}}(1-e^{-2\eta})+\frac{1}{2\sqrt{2}}\,\eta(1+e^{-2\eta})$$

$$+\frac{(3\lambda^{10}-21\lambda^5+25\lambda^3-7)\eta^2}{\sqrt{2}\,(4\lambda^{10}-25\lambda^7+42\lambda^5-25\lambda^3+4)}$$

$$+e^{2i\tau}\Big[\frac{1}{2\sqrt{2}}(1-E^2)-\frac{3(1+i)}{4\sqrt{2}}\,\eta-\frac{1+i}{4\sqrt{2}}\,\eta E^2$$

$$-(1-e^{-(1+i)\sqrt{2}\,\eta}-(1+i)\sqrt{2}\eta\Big]\frac{\sqrt{2}\,(3+2\lambda^5)-8\lambda^5}{16(1-\lambda^5)}\Big\}\,\mu(1-\mu^2)$$

(30) $$\chi_{01}=\Big\{\frac{-15\lambda^8+36\lambda^6-35\lambda^3+15\lambda}{\sqrt{2}\,(4\lambda^{10}-25\lambda^7+42\lambda^5-25\lambda^2+4)}$$

$$+e^{2i\tau}\Big[1-e^{(1+i)\sqrt{2}\,\zeta}-(1+i)\sqrt{2}\,\zeta\,\frac{5\lambda^2(\sqrt{2}-1)}{16(1-\lambda^5)}\Big]\Big\}\mu(1-\mu^2)$$

(31) $$\Psi_{01}=\frac{1}{2\sqrt{2}(4\lambda^{10}-25\lambda^7+42\lambda^5-25\lambda^3+4)}[2(-6\lambda^5+5\lambda^3+1)r^5$$

$$+2(10\lambda^7-7\lambda^5-3)r^3-\lambda^3(8\lambda^7+7\lambda^2-15)+(4\lambda^{10}+5\lambda^7-9\lambda^5)r^{-2}]$$

$$+e^{2i\tau}\frac{(1+i)(\sqrt{2}-1)}{1-\lambda^5}[-r^3+\lambda^5r^{-2}]\Big\}Q_2(\mu),$$

where

$$Q_2(\mu)=-\frac{1}{2}\,\mu(1-\mu^2)$$

(32) $$\Omega_{10}=\Omega_{11}=\Omega_{12}=0$$

(33) $$\bar{\omega}_{10}=\bar{\omega}_{11}=\bar{\omega}_{12}=0.$$

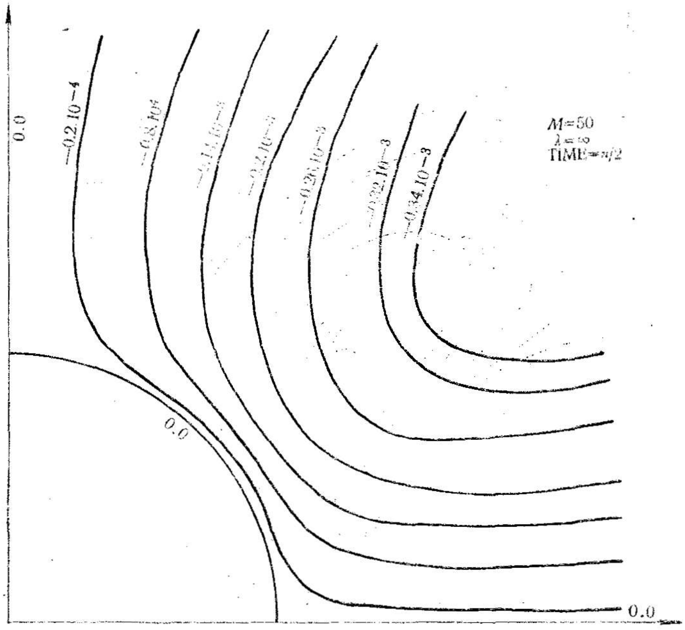

Fig. 1. M=50, $\lambda=\infty$ TIME$=\pi/2$
The secondary flow streamline pattern

For conciseness the other terms of the solution are not recorded here.

It is interesting to note that according to the equations (15) and (29)—(33) the rotational speed is not equal to zero only in the Stokes shear layer at the surface of the oscillating particle. At the same time the secondary flow in the planes containing the axis of oscillation has the velocity components which are not equal to zero in the two Stokes boundary layers and between them. The reason for this is the existence of the steady streaming in the periodic boundary layer. At a large distance from the wall of particle the velocity of this streaming does not vanish.

Fig. 1 shows the secondary flow streamline pattern at $M=50$ in the case when there is no container. Since the centrifugal force is greatest in the neighborhood of the equator of the oscillating particle, the velocity of the fluid is directed inward along the pole $(\theta=0)$ and outward at the equator $\left(\theta=\frac{\pi}{2}\right)$. In the case when there is a container (fig. 2) the fluid is sucked in along the poles, expelled at the equator to the container's wall and after that again approaches the poles. In this way, combined with the motion about the axis, there will be a circulatory motion in planes containing the axis of oscillation.

At leading order the induced secondary flow consists a steady streaming and a flow proportional to the first harmonic of oscillation, $\exp i2\tau$. Figs. 3 and 4 show the stationary streamline pattern at $M=20$, $\lambda=1{,}5$ and $M=20$, $\lambda=2{,}5$, respectively. The effect of finite amplitude on the torque exerted on a torsionally oscillating particle in a viscous fluid in a container is given by the equation.

$$T=T_0+A,$$

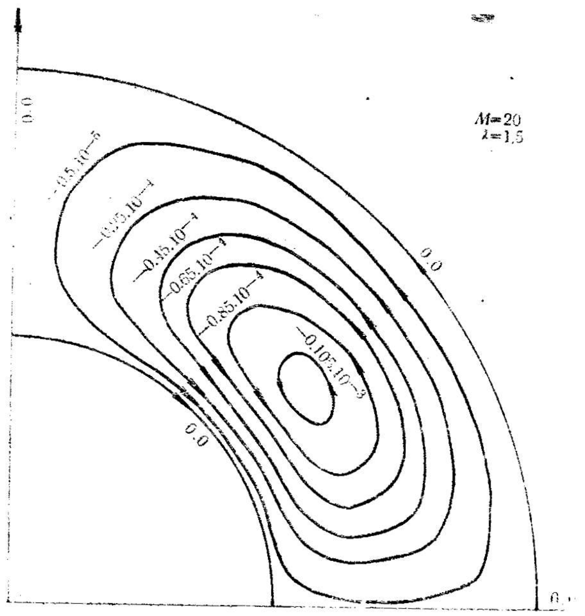

Fig. 2. M=50, $\lambda=\infty$, TIME=$\pi/2$
The secondary flow stre mline pattern

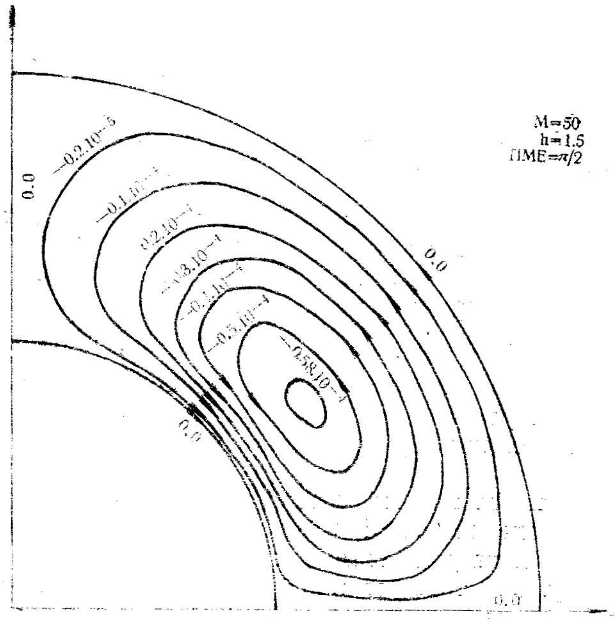

Fig. 3. M=20, $\lambda=1{,}5$
The stationary streamline pattern

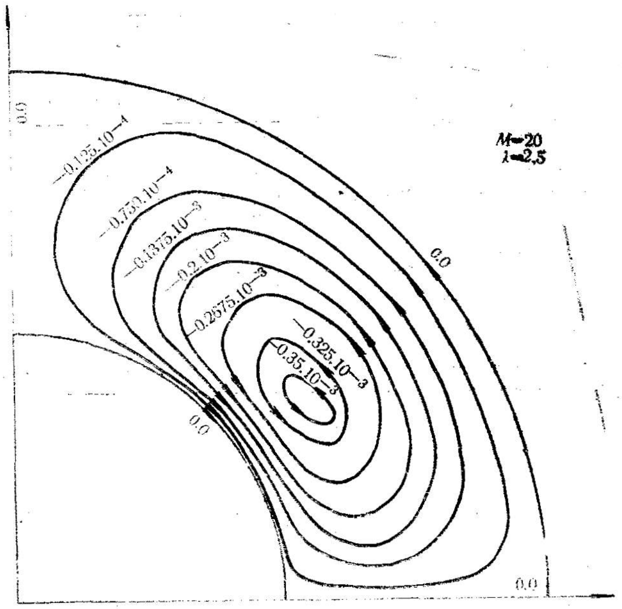

Fig. M=20, $\lambda=2,5$
The stationary flow streamline pattern

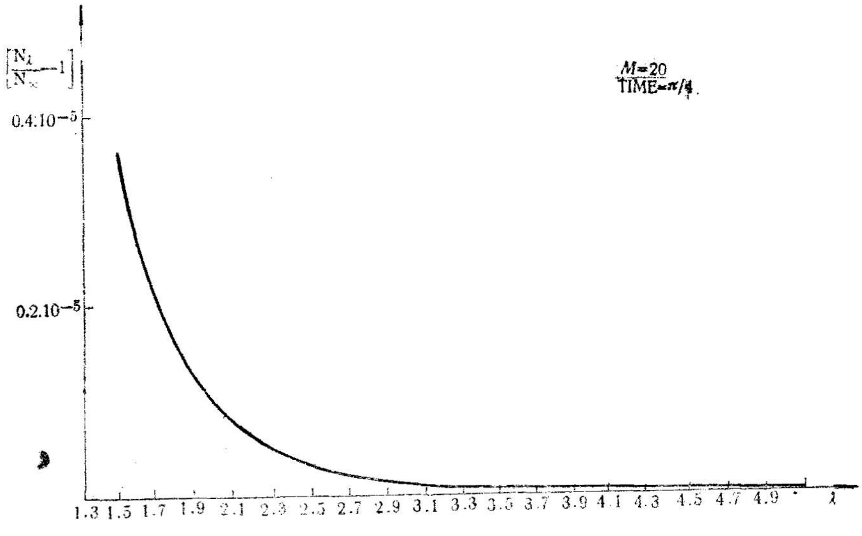

Fig. 5 M=20 TIME$=\pi/4$ $\left[\frac{N\lambda}{N_\infty}-1\right]$
The forgue N_λ as a function of λ, where N_∞ is the torque for $\lambda \to \infty$

where $T_0 = -\frac{8\pi\mu}{3\sqrt{2}}|M|$ is the torque of the torsionally oscillating sphere in unbounded viscous fluid and

$$A = |T_0| \frac{\varepsilon^2}{|M|\lambda^3}\Big\{0,33146\Big(1 - \frac{4,24259}{|M|} - \frac{2,74474}{\lambda^2} + \frac{8,68299}{|M|\lambda^2}\Big)\cos(\tau + \alpha_1) + \frac{0,00579}{\lambda^2}\Big(1 - \frac{5,85319}{|M|}\Big) \cdot \cos(3\tau + \alpha_2) + 0\Big(\frac{1}{|M|^2}, \frac{1}{\lambda_3}\Big)\Big\}.$$

Here

$$\alpha_1 = -\frac{0,02286}{\lambda^2} - \frac{1,41197}{|M|} - \frac{0,84705}{|M|\lambda^2} + 0\Big(\frac{1}{|M|^2}, \frac{1}{\lambda^3}\Big)$$

and

$$\alpha_2 = \frac{5,85319}{|M|} + 0\Big(\frac{1}{|M|^2}\Big).$$

This formula gives us the wall effect on the torque of the particle which is too weak. Fig. (5) shows as we can expect that when λ increases the effect of the container's wall on the torque decreases and asymptotically tends to the torque of the torsionally oscillating particle in an unbounded flow. So, our analysis of the flow field induced by the torsional oscillations of a small particle in a spherical container includes the Di Prima and Liron's analysis as a particular case — when $\lambda \to \infty$.

References

1. Faraday, M. Trans. Roy. Soc., London 121, 1831.
2. Dvorak, V., Ann. Physics, 157, 1875.
3. Lord Reyleigh, Phil. Trans., A 175, 1884.
4. Carrière, Z. J. Phys. Radium, 10, 1929.
5. Schlichting H., Phys. Z., 33, 1933.
6. Lane, C., J. Acous. Coc. Amer., 27, 1955.
7. Wang C. Y. J. Sound Vib., 2, 1965.
8. Riley N., J. Inst. Math. Appl., 3, 1967.
9. Riley N. J. Fluid Mech., 53, 1972.
10. Riley N., S. I. A. M. Review, 17, 1975.
11. Riley N., J. Fluid Mech., 68, 1975.
12. R. C. Di Prima and N. Liron, The Physics of Fluids 19, 1976.

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