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Propagation of Harmonic Waves in an Oblique Elastic Plate

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1. Introduction

The problem of a harmonic wave propagation in bounded elastic solids is of great interest because of the numerous possibilities for technological applications. At the same time it offers considerable difficulties connected mainly with satisfying the boundary conditions. A survey on many of the problems solved so far in this field as well as on the typical difficulties can be found, for instance, in [1]. As far as can be judged from this and other publications, the problem of wave propagation in oblique plates has not yet been investigated. An attempt to solve this problem was made in [2], but the substantial differences between the rectangular and non-rectangular plates concerning the reflection from the boundaries were not taken into account. That is why the solution given there cannot be accepted. It is the aim of this paper to investigate the same problem on the correct base.

2. First reflection from the boundary

We suppose that a harmonic wave is travelling along the plate in the direction parallel to one of the edges, say rightwards (Fig. 1). The wave may be both of the P-or SV-type. Unlike the case of rectangular plate, here reaching the right edge (inclined with respect to the direction of wave propagation) means an inclined (not normal) wave incidence on this edge. In the general case the incident P-wave reflects as P-and SV-waves (Fig. 2a), and the incident SV-wave gives rise to reflected SV-and P-waves according to Fig. 2b. The relation between the directions of the incident and reflected wave of different type is governed by the Snell's law

$$\frac{\sin \varphi_S}{\sin \varphi_P} = \frac{v_S}{v_P} = c, \quad c = \sqrt{\frac{1-2\nu}{2(1-\nu)}}, \tag{2.1 a,b}$$

where v_P and v_S are the propagation velocities of the P-and SV-waves in the given material, and ν is the Poisson's ratio.

Let us consider first the case of incident P-wave. The directions of the reflected P-and SV-waves under the condition $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$ are shown in Fig. 3. It follows from the Snell's law (for any α)

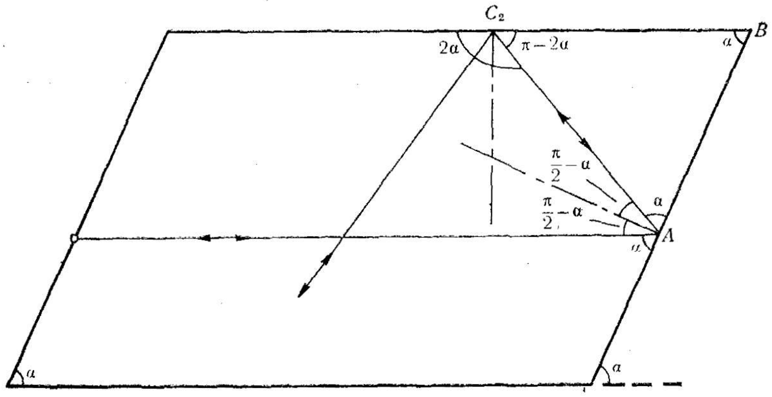

Fig. 1. The P-reflections of an incident P-wave at $\frac{\pi}{4}<\alpha<\frac{\pi}{2}$

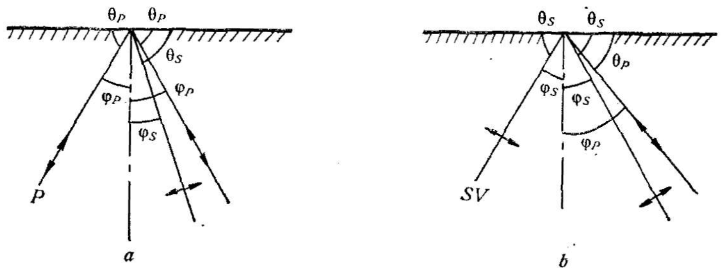

Fig. 2. Reflections of the incident waves from a plane boundary: a. P-wave, b. SV-wave

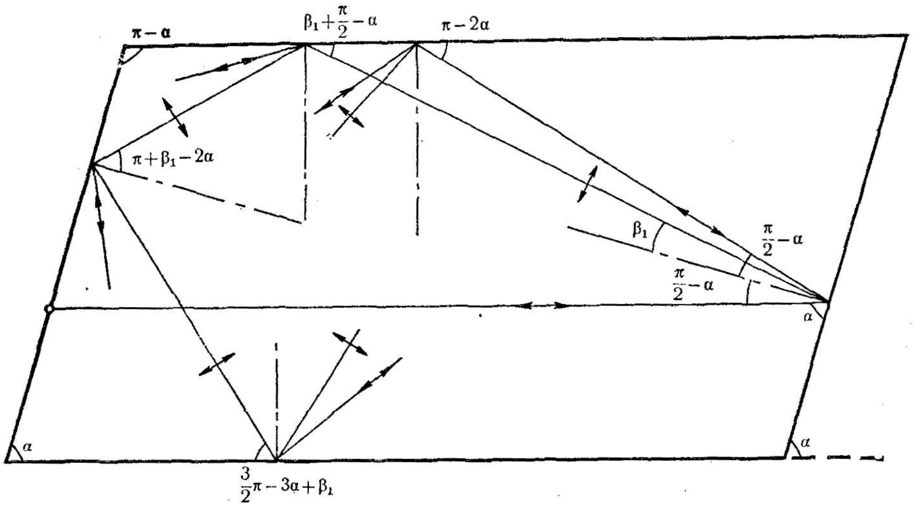

Fig. 3. First and following reflections of the incident P-wave

$$\frac{\sin \beta_1}{\cos \alpha} = c. \tag{2.2}$$

The question arises whether the right edge can reflect an SV-wave only. In terms of wave theory this means a full mode exchange. If the amplitude of the potential of the incident P-wave (Fig. 2a) is A_P then the amplitudes A_{PP} and A_{PS} of the reflected P-and SV-wave, respectively, will be given by the formulas*

$$\left.\begin{aligned} A_{PP} &= A_P \frac{4 s_1 r_1 - (s_1^2 - 1)^2}{4 s_1 r_1 + (s_1^2 - 1)^2} \\ A_{PS} &= A_P \frac{4 r_1 (1 - s_1^2)}{4 s_1 r_1 + (s_1^2 - 1)^2} \end{aligned}\right\}, \tag{2, 3a, b}$$

where $r_1 = \operatorname{tg} \theta_P$, $s_1 = \operatorname{tg} \theta_S$. The variation of A_{PP}/A_P for different values of ν is shown in Fig. 4. One can see that mode exchange takes place at

$$4 s_1 r_1 - (s_1^2 - 1)^2 = 0. \tag{2.4}$$

This equation has two real roots at $\nu < \nu^*$, where the critical value of the Poisson's ratio is $\nu^* = 0{,}2637$. For all materials which satisfy this condition there exist two angles of plate obliquity at which its right edge will reflect an *SV*-wave only. For instance, at $\nu = 0{,}25$ we obtain $\alpha \cong 30°$ and $\alpha \cong 12°$.

Further it is interesting to analyze the part of energy which is taken away by each of the reflected waves. In Fig. 5a the relative energy is shown which is taken away by the P-wave, and in Fig. 5b — the same for the *SV*-wave. For the case above considered ($\nu = 0.25$), if the obliquity of the plate α is between those two characteristic values 12° and 30°, the reflected *SV*-wave takes away the whole energy of the process.

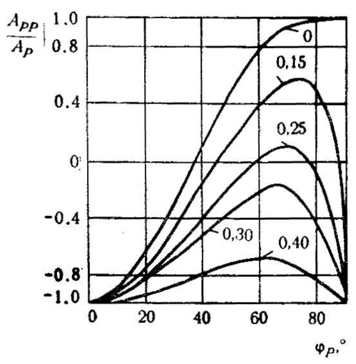

Fig. 4. The A_{PP}/A_P-ratio as a finction of the incidence angle

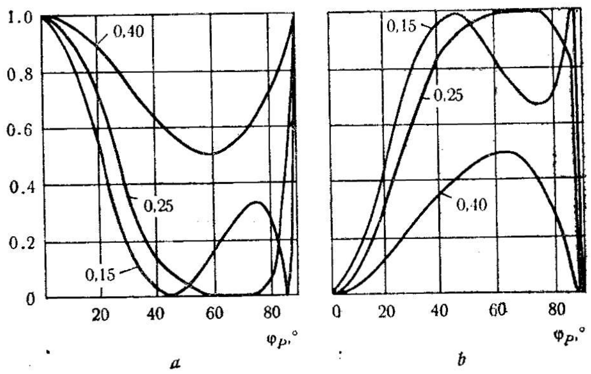

Fig. 5. The relative energies taken away by the reflecrted *P*-wave (a) and *SV*-wave (b) in the case of incident *P*-wave

* Our analysis is based upon some results of the wave theory (including also Fig. 4, 5, 7, 8) which are presented in [1] but supposedly are to be found in a number of other publications. However, they have not been used so far in the literature, known to the author, to analyze the wave propagation in oblique plates.

Naturally the counterquestion arises: is it possible in an oblique plate only a P-wave to be reflected? It follows from formula (2.3 b) $s_1=1$, i. e. $\theta_S=\pi/4$. But then Eq. (2.1) yields

$$\frac{\sqrt{2}}{2}=\sqrt{\frac{1-2\nu}{2(1-\nu)}}\cdot \sin\varphi_P \tag{2.5}$$

or

$$\sin\varphi_P=\sqrt{\frac{1-\nu}{1-2\nu}}>1,$$

which is obviously impossible.

Let us now consider the case of an SV-wave travelling rightwards along the plate, parallel to one of its edges (Fig. 6). For convenience now the angle of obliquity is labelled as β. We are looking for its values for which the mode exchange at the right boundary would take place. The relation between the amplitudes of the incident and reflected potentials is (see Fig. 2b)

$$A_{PS}=A_S\frac{4\,s_1(s_1^2-1)}{4\,s_1\,r_1+(s_1^2-1)^2}, \qquad \left.\begin{matrix} r_1=\operatorname{tg}\theta_P \\ s_1=\operatorname{tg}\theta_S \end{matrix}\right\} \tag{2.6}$$

$$A_{SS}=A_S\frac{4s_1\,r_1-(s_1^2-1)^2}{4s_1\,r_1+(s_1^2-1)^2}. \quad \Big\}$$

Unlike the case of incident P-wave, in this case there exists a critica incidence θ_S' such that at $\theta_S<\theta_S'$ the value r_1 becomes imaginary. This angle is defined by the equation $\cos\theta_S'=c$ as it follows from (2.1). The physical meaning of this phenomenon is that at $\theta_S<\theta_S'$ the process of reflecting the incident SV-waves cannot be understood any more within the frames of the ray ideas.

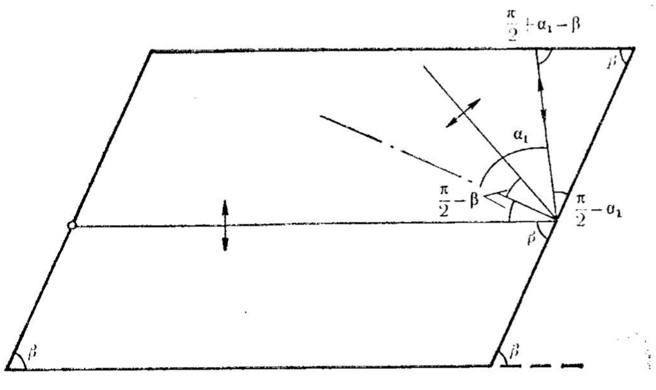

Fig. 6. Incidence of a *P*-wave on the upper edge after it had arised at the right inclined edge as a result of a reflection of an incident *SV*-wave

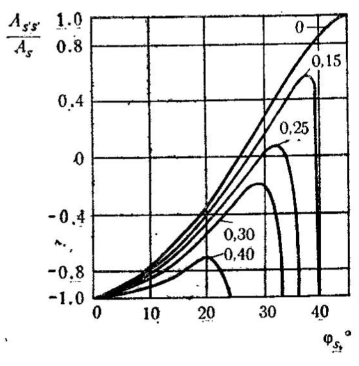

Fig. 7. The A_{SS}/A_S-ratio as a function of the incidence angle

For the oblique plate the angle β' is connected with the Poisson's ratio through the equation $\cos\beta'=\sqrt{(1-2\nu)/(2-2\nu)}$. For $\nu=0.25$ the critical obliquity is obtained to be $\beta'=54.8°$. At $\beta<\beta'$ the investigation of the wave propagation in the plate requires operating with imaginary numbers and a deeper physical insight into the problem.

As in the case of incident P-wave, here a full mode exchange (i. e. the reflected wave to be of the opposite kind with respect to the incident one) is possible provided the Poisson's ratio is smaller than the critical value $\nu^*=0{,}2637$. This can be seen from Fig. 7 where the ratio A_{SS}/A_S is displayed as a fun-

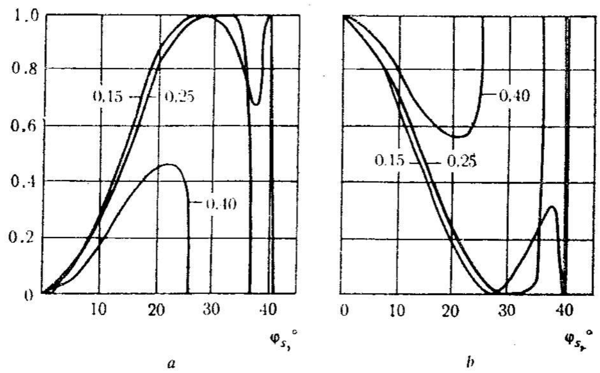

Fig. 8. The relative energies taken away by the reflected P-wave (a) and SV-wave (b) in the case of an incident SV-wave

ction of ν and φ_S for $\varphi_S<\varphi'_{*}$. At $\nu=0{,}25$ the two possible obliquities of the plate which give mode exchange are (see Fig. 6) $\beta=60°$ and $\beta=56°$.

Unlike the case of incident P-wave, at incident SV-wave it is possible to exist a reflected SV-wave only (i. e. a wave of the same type). The condition for this is $s_1=1$, i. e. $\varphi_S=45°$ which in this case does not lead to any absurd. So in a plate with $\beta=45°$ the right boundary will reflect the incident SV-wave also as SV.

The energy analysis in the case of incident SV-wave is of interest. In Fig. 8a and 8b are shown the relative energies of the reflected P-and SV-waves, respectively. One can see that for angles causing full mode exchange (at $\nu=0{,}25$ these are $\theta_S=60°$ and $\theta_S=56°$) the reflected P-wave takes away the whole energy which has come with the incidence. The same is true for plates for which β is between these two values (at least for this ν).

Let us summarize the results obtained. It turned out that the wave propagation in an oblique plate (unlike the rectangular one) is characterized by some interesting features. Already the first reflection from the right boundary generally leads to two reflected waves, and for a number of values of the obliquity angle of the plate α important effects arise. For example, at $\nu=0{,}25$ these are:

a. At $\alpha=60°$ the incident SV-wave is reflected into a P-wave only.

b. The same at $\alpha=56°$.

c. At $\alpha=54{,}8°$ the ray pattern of the reflection process from the right boundary ceases to be meaningful.

d. At $\alpha=45°$ the incident SV-wave is reflected into an SV-wave only.

e. At $\alpha=30°$ the incident P-wave is reflected as an SV-wave only.

f. The same at $\alpha=12°$.

3. Following reflections from the boundaries

The above performed analysis is true only for the **first reflection** from the right boundary. It is much more difficult to cover the picture of the **following reflections** which are displayed to some extent in Fig. 3 and 6. On the other hand, however, a great variety of effects arises at the following reflections. We are going to investigate some.

First let us follow the successive P-reflections of the incident P-wave (Fig. 1). It is assumed in this case that $\frac{\pi}{4}<\alpha<\frac{\pi}{2}$. Obviously after the first reflection the P-wave can get backwards if its incidence on the second boundary (upper horizontal) is normal. This leads to the condition $2\alpha=\frac{\pi}{2}$ or $\alpha=\frac{\pi}{4}$.

It is not difficult to see that this value for α can be obtained from another case when the plate is more sharply slanted (Fig. 9). Since at points A and B the angles are always equal to α the angles at C_2, C_3, $C_4, \ldots$ will increase as follows: 2α, 3α, $4\alpha, \ldots$. In order to start at some moment the backwards motion it is necessary one of the following conditions to be fulfilled: $2\alpha=\frac{\pi}{2}$, $3\alpha=\frac{\pi}{2}$, $4\alpha=\frac{\pi}{2}, \ldots$

or

$$\alpha=\pi/(2k), \quad k=1, 2, 3, \ldots. \tag{3.1}$$

Consequently, if the obliquity angle of the plate satisfies one of the numberless conditions (3.1), the P-wave will turn back, and this will take place after precisely k moves (the first move is the incidence on the right boundary). For example, if $\alpha=\frac{\pi}{60}$, the $30^{-\text{th}}$ move will be a normal incidence. The same feature is true for the successive SV-reflections of the incident SV-wave.

Another possible question is: is it possible that after the first reflection of the incident P-wave from the right boundary into a P-and an SV-wave, the SV-wave will be fully reflected back after the incidence on the upper edge?

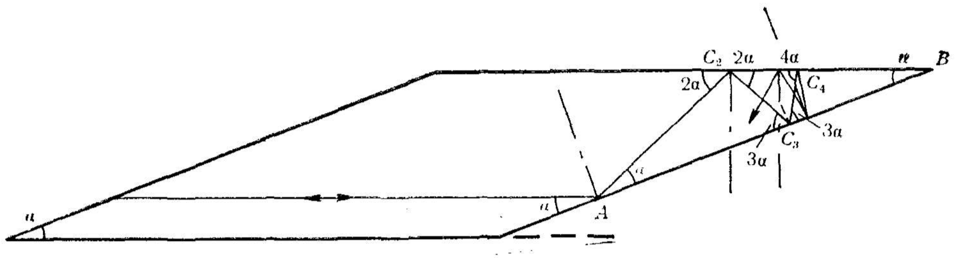

Fig. 9. Successive P-reflections of an incident P-wave at $\alpha<\frac{\pi}{4}$

It follows from Fig. 3 that this is possible if $\frac{\pi}{2}-\alpha+\beta_1=\frac{\pi}{2}$, i. e. $\alpha=\beta_1$. We obtain from Eq. (2.2) the condition

$$\operatorname{tg}\alpha=\sqrt{\frac{1-2\nu}{2(1-\nu)}}. \tag{3.2}$$

It is easy to be seen that in this case $\alpha<\frac{\pi}{4}$.

A similar question may be asked about the full reflection from the upper edge, of the P-wave incident on it after it has had arisen immediately before as a result of an incident SV-wave on the right edge of the plate (Fig. 6). Here also must be $\alpha_1=\beta$ but now

$$\frac{\sin\left(\frac{\pi}{2}-\beta\right)}{\sin \alpha_1}=c \quad \text{or} \quad \operatorname{tg}\beta=\sqrt{\frac{2(1-\nu)}{1-2\nu}}. \tag{3.3}$$

The condition (3.3) requires $\beta>\frac{\pi}{4}$. For example, at $\nu=0{,}30$ we obtain $\beta=61{,}7°$. It is curious that $\nu=0{,}25$ yields $\beta=60°$ what coincides with the case of a full mode exchange. In other words, for a plate with obliquity $\beta=60°$ and $\nu=0{,}25$ the incident SV-wave is reflected only into a P-wave but the latter is fully reflected from the upper edge, i. e. the whole energy of the process gets backwards. And since now the right boundary of the plate receives an incident P-wave under the angle $\frac{\pi}{2}-\alpha_1=30°$, i. e. $\alpha_1=60°$, it follows from Fig. 4 that a reverse mode exchange will take place. The same will happen at the left inclined and the lower horizontal edge, and so on — the process will be periodic.

There is another interesting question: is it possible after the initial P-wave, for the reflected SV-wave meet the upper edge under the critical angle beyond which the reflection will lose its ray character? It follows from Fig. 3 that this is possible when

$$\cos\left(\frac{\pi}{2}-\alpha+\beta_1\right)=c \quad \text{or} \quad \sin(\alpha-\beta_1)=c. \tag{3.4}$$

Eq. (3.4) and (2.2) yield

$$\sin\alpha.\sqrt{1-c^2\cos^2\alpha}-c.\cos^2\alpha=c \tag{3.5}$$

or, after some mathematical operations,

$$\cos\alpha=\sqrt{\frac{1-c^2}{1+3c^2}}. \tag{3.6}$$

For $\nu=0{,}3$ we obtain $\alpha=51{,}7°$.

The investigation of the later reflections is considerably more difficult, and, moreover, to some extent without practical value, if the interference at the numerous steps of the process is not taken into account. If necessary, this process can be followed through discretization both in the time and in the plate domains. Yet, some single conclusions can be drawn. For instance, at incident P-wave the three successive SV-reflections can end as a normal incidence on the lower edge (Fig. 3) provided

$$\frac{3}{2}\pi-3\alpha+\beta_1=\frac{\pi}{2} \quad \text{or} \quad \alpha=\frac{\pi}{3}+\frac{\beta_1}{3}. \tag{3.7}$$

In an analogous manner many features of similar nature can be established.

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