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Some properties of the potential energy functional in the theory of elastic stability

T. Ivanov

I. Introduction

The problem of stability of elastic bodies in equilibrium under the action of external conservative loads based on Lyapunov approach in stability is developed for mechanics of continua by means of suitably introduced norms in [1-5]. The ballistic energy functional which is the sum of the potential energy, the kinetic energy and a functional that depends only on the temperature is used in the proof of all results. In this way the stability problem is reduced to an examination of the potential energy functional that depends on mechanical properties of the elastic body and the external conservative loads.

In this paper some definitions are introduced and theorems are proved which we shall use in our further analysis. In some cases they are less restrictive for the potential energy than the definitions in [2-5]. Nevertheless the results in [2-5] and some new important results can be proved on the basis of these definitions and theorems.

II. Basic definitions and theorems

Let L be the real linear normed space of twice differentiable vector functions $\mathbf{u}(\mathbf{X})$, defined on the domain V of euclidean three space E_3, where $\mathbf{u}(\mathbf{X})$ and its first derivatives are continuous, they have limits when $\mathbf{X}$ tends to the boundary S of V and $\mathbf{u}(\mathbf{X})=\mathbf{0}$ on the portion $S_u \subset S$. With $\mathbf{X}$ we denote the radius vector of an arbitrary point of E_3 and X_i are its coordinates with respect to a rectangular Cartesian system. The function $\mathbf{u}(\mathbf{X})$ is called kinematically admissible displacement field and $\|\mathbf{u}\|$ is the norm of $\mathbf{u}$. Let $F[\mathbf{u}]$ be a real functional, defined on L such that $F[\mathbf{o}]=0$.

Definition 1. The functional $F[\mathbf{u}]$ is regular if there exist a constant $\alpha_0>0$ and the function $d(\alpha)=\inf\limits_{\|\mathbf{u}\|=\alpha} F[\mathbf{u}]$* for every sphere with radius $\alpha \in [0, \alpha_0]$.

Corollary 1. If the functional $F[\mathbf{u}]$ is lower bounded in the ball $\|\mathbf{u}\|\leq\alpha$ then it is regular.

* We suppose everywhere in the paper that the infimum or the minimum is finite.

Corollary 2. If the functional $F[\mathbf{u}]$ is regular and the function $d(\alpha)$ is lower bounded (or continuous) for $\alpha \in [0, \alpha_0]$, $\alpha_0 > 0$, then the functional is lower bounded in the ball $\|\mathbf{u}\| \leq \alpha_0$.

Definition 2. The functional $F[\mathbf{u}]$ is positive definite if there exist an $\alpha_0 > 0$ and the function $d(\alpha) = \inf\limits_{\|\mathbf{u}\| = \alpha} F[\mathbf{u}]$ such that $d(\alpha) > 0$ holds for every $\alpha \in (0, \alpha_0]$.

Corollary 3. If the functional $F[\mathbf{u}]$ is positive definite then it is regular.

Corollary 4. If for the functional $F[\mathbf{u}]$ there exist an $\alpha_0 > 0$ and the function $d(\alpha)$ such that $d(\alpha)$ is a strictly increasing function of $\alpha \in [0, \alpha_0]$ then the functional is positive definite.

Definition 3. The functional $F[\mathbf{u}]$ is indefinite if for every $\alpha > 0$ there exist an $\alpha_1 \in (0, \alpha)$ and a displacement field $\mathbf{u}_1$ such that $\|\mathbf{u}_1\| = \alpha_1$ and $F[\mathbf{u}_1] < 0$.

Corollary 5. If for the functional $F[\mathbf{u}]$ there exist an $\alpha_0 > 0$ and the function $d(\alpha) = \inf\limits_{\|\mathbf{u}\| = \alpha} F[\mathbf{u}]$ such that $d(\alpha)$ is a strictly decreasing function of $\alpha \in [0, \alpha_0]$ then the functional is indefinite.

Definition 4. The functional $F[\mathbf{u}]$ is semi-definite (positive) if there exist an $\alpha_0 > 0$ and the function $d(\alpha) = \inf\limits_{\|\mathbf{u}\| = \alpha} F[\mathbf{u}]$, such that $d(\alpha) \geq 0$ holds for every $\alpha \in (0, \alpha_0]$, and for every α there exists an $\alpha_* \in (0, \alpha]$ such that $d(\alpha_*) = 0$.

Corollary 6. If there exists an $\alpha_0 > 0$ such that $F[\mathbf{u}] > 0$ for every $\mathbf{u}$, where $0 < \|\mathbf{u}\| \leq \alpha_0$, then the functional $F[\mathbf{u}]$ is positive definite or semi-definite. The same conclusion is valid when $F[\mathbf{u}] \geq 0$ for $\|\mathbf{u}\| \leq \alpha_0$.

Definition 5. The functional $F_n[\mathbf{u}]$ is homogeneous of n-th degree $n > 0)$ if

$$F_n[a\mathbf{u}] = a^n F_n[\mathbf{u}]$$

holds for every $\mathbf{u} \in L$ and for any constant a.

Corollary 7. If the functional $F_n[\mathbf{u}]$ is homogeneous then $F_n[\mathbf{o}] = 0$.

Definition 6. If in the Definitions 1, 2 and 4 we substitute $d(\alpha)$ and $\inf\limits_{\|\mathbf{u}\| = \alpha} F[\mathbf{u}]$ with $m(\alpha)$ and $\min\limits_{\|\mathbf{u}\| = \alpha} F[\mathbf{u}]$ we shall obtain the definitions for a properly regular, a properly positive definite and a properly semi-definite functional correspondingly.

Corollary 8. If the functional is properly regular, properly positive definite or properly semi-definite it is also regular, positive definite or semi-definite correspondingly.

Corollary 9. If for the functional $F[\mathbf{u}]$ there exist an $\alpha_0 > 0$ and the function $m(\alpha) = \min\limits_{\|\mathbf{u}\| = \alpha} F[\mathbf{u}]$ for every $\alpha \in (0, \alpha_0]$ then the functional is properly positive definite if and only if $F[\mathbf{u}] > 0$ for every $\mathbf{u}$ where $0 < \|\mathbf{u}\| \leq \alpha_0$.

Corollary 10. If for the functional $F[\mathbf{u}]$ there exist an $\alpha_0 > 0$ and the function $m(\alpha) = \min\limits_{\|\mathbf{u}\| = \alpha} F[\mathbf{u}]$ for every $\alpha \in (0, \alpha_0)$ then the functional is properly semi-definite if and only if $F[\mathbf{u}] \geq 0$ for every $\mathbf{u}$, where $\|\mathbf{u}\| \leq \alpha_0$, and for every $\alpha \in (0, \alpha_0]$ there exist an $\alpha_* \in (0, \alpha]$ and a displacement field $\mathbf{u}^*$ such that $\|\mathbf{u}\| = \alpha_*$ and $F[\mathbf{u}^*] = 0$.

We would like to note that the classical definition of a positive definite functional used in calculus of variations ($F[\mathbf{u}] > 0$ for $0 < \|\mathbf{u}\| \leq \alpha_0$) is less restrictive than Definition 2 as it follows from Corollary 6. If the supplementary restriction from Corollary 9 is added we shall obtain a positive definite functional according to Definition 2. On the other hand the definition of a positive definite functional in [2-5] is more restrictive than Definition 2 as it follows from Corollary 4. Nevertheless we are able to prove all results in [2-5] and some other very important results based on our Definition 2. On the other hand further analysis shows that the classical definition of a positive definite functional is not useful for our purposes.

As a consequence from these definitions we shall prove some theorems which will be of use in our further considerations on stability.

Theorem 1. If for the homogeneous functional $F_n[\mathbf{u}]$ there exist an $\alpha_0>0$ and $\inf\limits_{\|\mathbf{u}\|=\alpha_0} F_n[\mathbf{u}]$ then

$$\inf_{\|\mathbf{u}\|=\alpha} F_n[\mathbf{u}]=\left(\frac{\alpha}{\alpha_0}\right)^n \inf_{\|\mathbf{u}\|=\alpha_0} F_n[\mathbf{u}]$$

holds for every $\alpha\geqq 0$.

Proof. Let $\mathbf{u}$ be an arbitrary displacement field with $\|\mathbf{u}\|=\alpha>0$. Then for the field $\mathbf{v}=\frac{\alpha_0}{\alpha}\mathbf{u}$ we obtain $\|\mathbf{v}\|=\frac{\alpha_0}{\alpha}\|\mathbf{u}\|=\alpha_0$ and therefore

$$F_n[\mathbf{u}]=\left(\frac{\alpha}{\alpha_0}\right)^n F_n[\mathbf{v}]\geqq\left(\frac{\alpha}{\alpha_0}\right)^n \inf_{\|\mathbf{v}\|=\alpha} F_n[\mathbf{v}]. \tag{1}$$

From this inequality follows that there exists $\inf\limits_{\|\mathbf{u}\|=\alpha} F_n[\mathbf{u}]$. It is equal to $\left(\frac{\alpha}{\alpha_0}\right)^n \inf\limits_{\|\mathbf{v}\|=\alpha_0} F_n[\mathbf{v}]$ since the displacement field $\mathbf{u}$ and its corresponding field $\mathbf{v}$ can be chosen so that $\|\mathbf{v}\|=\alpha_0$ and the values of $F_n[\mathbf{u}]$ to be arbitrary near to the value of $\left(\frac{\alpha}{\alpha_0}\right)^n \inf\limits_{\|\mathbf{v}\|=\alpha_0} F_n[\mathbf{v}]$.

Corollary 11. If for the homogeneous functional $F_n[\mathbf{u}]$ there exist an $\alpha_0>0$ and $\inf\limits_{\|\mathbf{u}\|=\alpha_0} F_n[\mathbf{u}]$ then the functional is regular and the continuous function $d_n(\alpha)=\inf\limits_{\|\mathbf{u}\|=\alpha} F_n[\mathbf{u}]$ exists for every $\alpha\geqq 0$.

Corollary 12. If for the homogeneous functional $F_n[\mathbf{u}]$ there exist an $\alpha_0>0$ and $\inf\limits_{\|\mathbf{u}\|=\alpha_0} F_n[\mathbf{u}]>0$ then the functional is positive definite, the function $d_n(\alpha)=\inf\limits_{\|\mathbf{u}\|=\alpha} F_n[\mathbf{u}]$ exists and it is a strictly increasing continuous function for $\alpha\geqq 0$.

Corollary 13. If for the homogeneous functional $F_n[\mathbf{u}]$ there exists a displacement field $\mathbf{u}_1$ for which $F_n[\mathbf{u}_1]<0$ then the functional is indefinite and for every $\alpha>0$ there exists a displacement field $\mathbf{u}$ such that $\|\mathbf{u}\|=\alpha$ and $F_n[\mathbf{u}]<0$.

Corollary 14. If for the homogeneous functional $F_n[\mathbf{u}]$ there exist an $\alpha_0>0$ and $\inf\limits_{\|\mathbf{u}\|=\alpha_0} F_n(\mathbf{u})=0$ then the functional is semi-definite, the function $d_n(\alpha)=\inf\limits_{\|\mathbf{u}\|=\alpha} F_n[\mathbf{u}]$ exists for every $\alpha\geqq 0$ and $d_n(\alpha)=0$.

Theorem 2. If for the homogeneous functional $F_n[\mathbf{u}]$ there exist an $\alpha_0>0$ and $\min\limits_{\|\mathbf{u}\|=\alpha_0} F_n[\mathbf{u}]$ then

$$\min_{\|\mathbf{u}\|=\alpha} F_n[\mathbf{u}]=\left(\frac{\alpha}{\alpha_0}\right)^n \min_{\|\mathbf{u}\|=\alpha_0} F_n[\mathbf{u}]$$

holds for every $\alpha\geqq 0$.

Proof. With the notations used in Theorem 1 we obtain that the inequality (1) is satisfied. Let $\mathbf{v}_1$ be the minimal field on the sphere $\|\mathbf{v}\|=\alpha_0$. Then

$$F_n[\mathbf{u}]\geqq\left(\frac{\alpha}{\alpha_0}\right)^n F_n[\mathbf{v}_1].$$

On the other hand the displacement field $\mathbf{u}_1=\frac{\alpha}{\alpha_0}\mathbf{v}_1$ satisfies the equality

$$F_n[\mathbf{u}_1]=\left(\frac{\alpha}{\alpha_0}\right)^n F_n[\mathbf{v}_1]$$

where $\|\mathbf{u}_1\|=\alpha$. Hence there exists $\min\limits_{\|\mathbf{u}\|=\alpha} F_n[\mathbf{u}]$ and it is equal to $\left(\frac{\alpha}{\alpha_0}\right)^n \min\limits_{\|\mathbf{v}\|=\alpha_0} F_n[\mathbf{v}]$.

Corollary 15. If for the homogeneous functional $F_n[\mathbf{u}]$ there exist an $\alpha_0>0$ and $\min\limits_{\|\mathbf{u}\|=\alpha_0} F_n[\mathbf{u}]$ then the functional is propertly regular and the continuous function $m_n(\alpha)=\min\limits_{\|\mathbf{u}\|=\alpha} F_n[\mathbf{u}]$ exists for every $\alpha\geq 0$.

Corollary 16. If for the homogeneous functional $F_n[\mathbf{u}]$ there exist an $\alpha_0>0$ and $\min\limits_{\|\mathbf{u}\|=\alpha_0} F_n[\mathbf{u}]>0$ then the functional is properly positive definite, the function $m_n(\alpha)=\min\limits_{\|\mathbf{u}\|=\alpha} F_n[\mathbf{u}]$ exists and it is a strictly increasing continuous function for every $\alpha\geq 0$.

Corollary 17. If for the homogeneous functional $F_n[\mathbf{u}]$ there exist an $\alpha_0>0$ and $\min\limits_{\|\mathbf{u}\|=\alpha_0} F_n[\mathbf{u}]=0$ then the functional is properly semi-definite, the function $m_n(\alpha)=\min\limits_{\|\mathbf{u}\|=\alpha} F_n[\mathbf{u}]$ exists for every $\alpha\geq 0$ and $m_n(\alpha)=0$.

III. Energy criteria for stability and instability

Let $\mathbf{u}(\mathbf{X}, t)\in L$ for every fixed t where t is the time. We could consider the functions $\mathbf{u}(\mathbf{X}, t)$ as an one-parametric family of displacement fields with parameter t. We denote the norm $\|\mathbf{u}(\mathbf{X}, t)\|$ by $\|\mathbf{u}\|_t$ and adopt the following

Assumption 1. The norm $\|\mathbf{u}\|_t$ is a continuous function of t.

Let L_2 be the real linear normed space of the velocity fields $\mathbf{v}(\mathbf{X}, t)$ where $\mathbf{v}(\mathbf{X}, t)=\frac{\partial \mathbf{u}(\mathbf{X}, t)}{\partial t}$ and $\mathbf{u}(\mathbf{X}, t)\in L$ are differentiable vector functions of t. The velocity field $\mathbf{v}(\mathbf{X}, t)$ is square integrable with the weight function $\rho(\mathbf{X})>0$ called mass density. We introduce the norm

$$\|\mathbf{v}\|_t=\left(\frac{1}{M}\int\limits_V \rho(\mathbf{X})\mathbf{v}(\mathbf{X}, t)\cdot\mathbf{v}(\mathbf{X}, t)dV\right)^{1/2}$$

where $M=\int\limits_V \rho(\mathbf{X})dV$ is the mass of the body. The dot denotes scalar product

Let Θ_2 be the real linear normed space of scalar functions $\theta(\mathbf{X}, t)$ where θ has a limit when $\mathbf{X}$ tends to S. We can consider the t as a parameter that determines an one-parametric family of temperature fields $\theta(\mathbf{X}, t)$ for every fixed function θ. The temperature field $\theta(\mathbf{X}, t)$ is square integrable with the weight function $\rho(\mathbf{X})$. We introduce the norm

$$\|\theta\|_t=\left(\frac{1}{M}\int\limits_V \rho(\mathbf{X})\theta^2(\mathbf{X}, t)dV\right)^{1/2}.$$

We consider an elastic body under the action of external conservative loads in an equilibrium state called fundamental state. Let V be the domain occupied by the body in the fundamental state. A material point of the body is identified by its radius vector $\mathbf{X}$ in this state. The absolute temperature of the body and its surrounding is necessarily uniform at the value θ_l. The motion of the body from the fundamental state is specified by the displacement field $\mathbf{u}(\mathbf{X}, t)$ and the absolute temperature field $\theta(\mathbf{X}, t)$. We assume that $\mathbf{u}(\mathbf{X},t)=\mathbf{0}$ on the portion $S_u \subset S$ where boundary condition for the displacement is given. When there is no boundary condition for the displacement we assume that L do not contain the displacemenl fields $\mathbf{u}$ $(\mathbf{X}, t)$ which contain or correspond to a rigid-body motion. Thus [2-6]

$$\frac{d}{dt}B[\mathbf{u}, \mathbf{v}, \theta] \leqq 0 \tag{2}$$

where the ballistic energy functional has the form

$$B[\mathbf{u}, \mathbf{v}, \theta]=P[\mathbf{u}(\mathbf{X}, t)]+K[\mathbf{v}(\mathbf{X}, t)]+\chi[\theta(\mathbf{X}, t)-\theta_l].$$

Here $P[\mathbf{u}(\mathbf{X}, t)]$, defined on L, is the functional of the potential energy change starting from the fundamental state, due to external conservative loads and the additionally stored isothermal elastic energy at temperature θ_l. This functional depends on $\mathbf{u}(\mathbf{X}, t)$ and its gradients with respect to X_i. The positive definite functional of the kinetic energy has the form

$$K[\mathbf{v}(\mathbf{X}, t)]=\frac{1}{2}\int_V \rho(\mathbf{X})\mathbf{v}(\mathbf{X}, t)\,.\,\mathbf{v}(\mathbf{X}, t)dV$$

and

$$\chi[\theta(\mathbf{X}, t)]=\frac{1}{2}\int_V \frac{C_E^*(\mathbf{X}, t)}{\theta^*(\mathbf{X}, t)}\theta^2(\mathbf{X}, t)dV,$$

where $\theta^*(\mathbf{X}, t)$ is an intermediate absolute temperature between $\theta(\mathbf{X}, t)$ and θ_l. $C_E^*(\mathbf{X}, t)$ is the specific heat at constant deformation, determined by the displacement field $\mathbf{u}(\mathbf{X}, t)$ and the intermediate temperature $\theta^*(\mathbf{X}, t)$. The proof of (2) is based on the inequality $(\theta-\theta_l)\mathbf{q}\,.\,\mathbf{n}\geqq 0$ where $\mathbf{q}(\mathbf{X}, t)$ is the heat flux vector and $\mathbf{n}$ is the unit outward normal vector on S. It follows from the boundary conditions on S for the temperature and/or for the normal component of the heat flux vector.

We adopt the following definition and assumptions [3-6]:

D e f i n i t i o n 7. The equlibrium in the fundamental state is stable if and only if for every triple of positive constants α_i $(i=1, 2, 3)$ a pair of positive constants β_0 and β_1 exists such that in every motion following after a disturbance at time $t=0$, determined by $B_0=B[\mathbf{u}(\mathbf{X}, 0), \mathbf{v}(\mathbf{X}, 0), \theta(\mathbf{X}, 0)]$ and $\|\mathbf{u}\|_0$, whose magnitude is restricted by the inequalities

$$B_0<\beta_0, \quad \|\mathbf{u}\|_0<\beta_1,$$

we have for the distance to the fundamental state

$$\|\mathbf{u}\|_t<\alpha_1, \quad \|\mathbf{v}\|_t<\alpha_2, \quad \|\theta-\theta_l\|_t<\alpha_3$$

for every $t\geqq 0$.

A s s u m p t i o n 2. There exists the constant $D^*=\inf\limits_{\substack{\mathbf{X}\in V\\ t\geqq 0}}\dfrac{C_E^*(\mathbf{X}, t)}{\rho(\mathbf{X})\theta^*(\mathbf{X}, t)}>0.$

C o r o l l a r y 18. The functional $\chi[\theta(\mathbf{X}, t)]$ is positive definite.

A s s u m p t i o n 3. There exists a constant $\alpha_0>0$ such that no equilibrium configuration exists at temperature θ_l with $P[\mathbf{u}(\mathbf{X})]<0$ when $0<\|\mathbf{u}\|\leqq\alpha_0$.

Assumption 4. Any motion of the body determined by the displacement field $\mathbf{u}(\mathbf{X}, t)$, where initial conditions with arbitrary $\mathbf{u}(\mathbf{X}, 0)$, $\mathbf{v}(\mathbf{X}, 0)=0$, $\theta(\mathbf{X}, 0)=\theta_I$ are assumed, comes at $t=t_*$ to an equilibrium state due to the energy dissipation, i. e. $\mathbf{v}(\mathbf{X}, t_*)=0$ and $\theta(\mathbf{X}, t_*)=\theta_I$.

If the potential energy functional $P[\mathbf{u}]$ is a continuous function of $\mathbf{u}\in L$ at $\mathbf{u}=\mathbf{O}$, when the norm $\|\mathbf{u}\|$ is used, then the inequality $B_0<\beta_0$ in Definition 7 could be replaced with $\|\mathbf{v}\|_0<\beta_2$, $\|\theta-\theta_I\|_0<\beta_3$ where β_2 and β_3 are appropriately chosen positive constants. Then the definition will coincide with the earlier definition for stability in [3].

We can prove the following sufficient conditions:

Theorem 3. If the potential energy functional is positive definite then the equilibrium in the fundamental state is stable.

Theorem 4. If the potential energy functional is indefinite then the equilibrium in the fundamental state is unstable.

The proofs of these theorems are similar to those in [6] nevertheless that now the requirements for positive definiteness and indefiniteness are less restrictive than those adopted in [6]. We note that these results are valid for any norm in the L space. Often it is chosen in the form [4]

$$\|\mathbf{u}\|_t=\left(\frac{1}{M}\int_V \rho(\mathbf{X})\mathbf{u}(\mathbf{X}, t)\,.\,\mathbf{u}(\mathbf{X}, t)dV\right)^{1/2}$$

where $\mathbf{u}(\mathbf{X}, t)$ is square integrable with the weight function $\rho(\mathbf{X})$.

The problem for a sufficient condition of stability on the basis of the second variation, announced by Koiter in [2], discussed ten years later in [5] and again in [7] could be solved with the help of here introduced definitions and proved theorems. The proof is similar to this one in [6] which is done at considerably stronger restrictions on the functionals under consideration.

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