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Dispersion Equations of the Roundcylindrical Shell-Tank Hydroelasticity Problem (Part I)

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Preface

The "zero" variant of the present work was offered as a third part of one large investigation [2, 3] at the III International Conference on Differential Equations in Russe (Bulgaria), July 1985. In accordance with the state that a third paper could not be published in the conference volumes, the former two authors decided to modernize and to present it in the Journal TAM. In the process of this modernization V. Vassilev did take part, and some ideas of Ref. [5, 6] were used essentially. Secondly, the volume of the paper dilated in such degree that we decided to divide it into two parts. These two parts are not independent and that is why for the second of them we shall foresee the same preface.

I. Introduction

1.1. Preliminaries. Thin-walled structures containing ideal compressible liquid are very often subjected to technological dynamic loads. Here a circular cylindrical shell-tank is under consideration, assuming that it vibrates with small amplitude forms $w(x, t)$. In the vibration process following any technological prescription in the time the liquid level starts to lower, Fig. 1.

1.2. Co-ordinates. The problem will be investigated in cylindrical co-ordinates $O(x, r, \theta; t)$. Assuming a symmetry in circular direction, we shall take all sought functions independent on θ-co-ordinate. The co-ordinates accepted are connected with the tank structure as it is shown in Fig. 1. The figure is taken from [4].

1.3. Prerequisites: (*i*) the tank shell is represented by its mid surface $r=R$; (*ii*) the liquid wets namely this surface; (*iii*) the shell material is lineary elastic; (*iv*) the liquid is ideal, compressible and heavy; (*v*) in a part of the investigation the liquid depth is variable, i. e. $h=h(t)$; (*vi*) small shell vibrations $w(x, t)$ induce small acoustical waves in the liquid; (*vii*) the tank bottom is absolutely rigid.

1.4. Hydroelasticity and acoustical equation. Following the aforesaid, and particularly assumption (*vi*), we shall consider the classical acoustical equation

$$\Delta\Phi-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=0, \tag{1}$$

in case of constant liquid level, and its modification

$$\Delta\Phi-\frac{1}{c^2}\frac{D^2\Phi}{Dt^2}=0,\quad \frac{D}{Dt}=\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}, \tag{2}$$

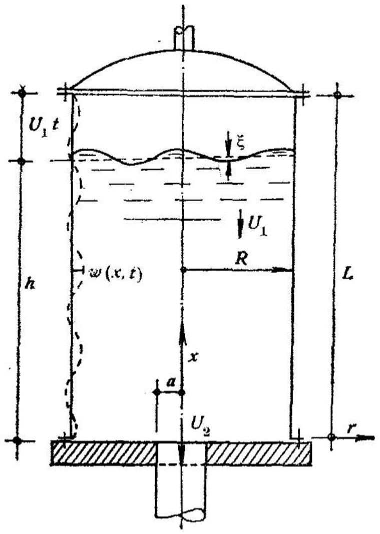

Fig. 1

in case of variable liquid level (and moving liquid with given velocity U along x-axis). The developed form of Eq. (2) is

$$(1-M^2)\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}-2\frac{M}{c}\frac{\partial^2\Phi}{\partial x\partial t}-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=0, \tag{3}$$

where Φ — velocity potential of all liquid motion; c — sound velocity in the liquid; $M=U/c$ — dimensionless Mach number; Δ — Laplassian. As the liquid level is variable, the surface of the dynamical shell-liquid interaction is also variable.

1.5. Wave number sphere. In classical acoustics of confined volumes, see f. e. Ref. [7, Vol. 1], it is an object of common knowledge that the wave numbers k_m, k_n and k_j along the three orthogonal Cartesian co-ordinates satisfy the sphere equation

$$k_m^2+k_n^2+k_j^2=(\omega_{mnj}/c)^2,\quad m, n, j=0, 1, 2,\ldots\infty, \tag{4}$$

where ω_{mnj} are natural frequences of the liquid volume. Relation (4) is a dispersion equation of the liquid motion problem. It plays role of a necessary and sufficient condition for variable separation in the integrals of Eq. (1), e. g.

$$\Phi=\sum_m\sum_n\sum_j A_{mnj}(i\omega_{mnj}).\cos k_m x.\cos k_n y.\cos k_j z.exp(i\omega_{mnj}t), \tag{5}$$

where A_{mnj} are integration constants, and $i=\sqrt{-1}$. The wave numbers are determinable from the boundary conditions

$$(6)\qquad \left.\frac{\partial\Phi}{\partial x}\right|_{x=L_1}=0,\quad \left.\frac{\partial\Phi}{\partial y}\right|_{y=L_2}=0,\quad \left.\frac{\partial\Phi}{\partial z}\right|_{z=L_3}=0,$$

which give the relations:

$$(7)\qquad k_m=(m\pi/L_1),\ k_n=(n\pi/L_2),\ k_j=(j\pi/L_3),$$

and L_1, L_2 and L_3 are the volume dimensions along the corresponding co-ordinates.

In case of symmetric cylindrical co-ordinates, given by relations

$$(8)\qquad r=\sqrt{y^2+z^2},\quad \theta=arctg\ (y/z)$$

the longitudinal (k_m) and radial (ψ_n) wave numbers satisfy the dispersion equation of cylindrical type

$$(9)\qquad k_m^2+\psi_n^2=(\omega_{mn}/c)^2,\ m,n=0,1,2,\ldots\infty,$$

if as an integral of the acoustical equation the function

$$(10)\qquad \Phi=\sum_m\sum_n A_{mn}\cdot(i\omega_{mn})\,.\,J_0(\psi_n r)\,.\cos k_m x\ \exp(i\omega_{mn}t)$$

is examined.

1.6. Independence on the time. The wave numbers in Eqs. (4) and (9) take discrete values, arranged by the corresponding indices, and they do not depend on the time coordinate (t).

II. The Problem

2.1. Liquid motion. We shall pay special attention to the problem of acoustical liquid motion expressed by Eqs. (1) and (3), considering different aspects of their construction, (and connected of course with different boundary conditions in various technological cases). The basical case will include, as it is shown in Fig. 1, a motion of the liquid with velocity $-U_1$, and supperposed to it acoustical motion with velocity potential φ_1.

2.2. Sound velocity and Mach number. The sound velocity for a broad variety of liquids is of order $c\sim10^3$ m/s. The real velocity of liquid lowering $|U_1|$ in large circle of technological problems is not more than 1 m/s. So, roughly said, the Mach number in our problem is of an order $M_1\sim10^{-3}$. At first sight it is seen that in Eq. (3) M^2 can be neglected in comparison with the unity. But the problem with the member $(2M/c)(\partial^2\Phi/\partial x\,\partial t)$ is not so simple, as the values of circular frequency ω_{mn} and wave number k_m influence to its general estimation.

2.3. The problem. The problem we shall solve here is to deduce a dispersion equation for the basical case of liquid motion formulated in P. 2. 1. With a purpose to give to our consideration some analytical completeness several additional cases are included. In all cases symmetric shell vibrations will be under consideration, accepting that in these cases the deformation energy is minimum, see P. 1. 2. and P. 1. 3.

Then the small shell vibration must be a solution of the equation [4]

$$(11)\qquad \frac{\partial^4 w}{\partial x^4}+\frac{1-\mu^2}{\gamma^2R^4}w+\frac{m_0}{D}\frac{\partial^2 w}{\partial t^2}=-\frac{p}{D}+\frac{q}{D},$$

where

$$(12)\qquad \gamma^2=\frac{\delta^2}{12R^2};\quad D=\frac{E\delta^3}{12(1-\mu^2)};\quad q=Q(x)e^{i\omega t};$$

E, μ — Young modulus and Poisson coefficient of the shell material respectively; m_0 — shell mass, kg/m^2; δ — shell thickness; p — hydrodynamical pressure; q — out dynamical loading; $Q(x)$ — amplitude value of q; σ — forced circular frequency.

Here the boundary conditions

$$[w(x, t)]_{(x=0\,;\,x=L)} = 0, \quad \left[\frac{\partial^2 w(x, t)}{\partial x^2}\right]_{(x=0,\, x=L)} = 0 \tag{13}$$

are accepted, although other combinations are also possible.

A second aim of our investigation is to show (for every case considered) how the dispersion equation can be used to analyse the corresponding technological parameters, having mechanical character.

For every case a full problem formulation will be given to have a local completeness and clearness.

III. Dispersion Equations (Incompressible Liquid)

3.1. Case $U_1 = 0$, $h = const \neq L$ (forced vibration).

3.1.1. We shall accept that the vibration function is representable in the form

$$w(x, t) = W(x) \exp(i\sigma t), \tag{14}$$

where $W(x)$ is the mode corresponding to σ.

$$\varphi = \varphi(r, x\,; t) \tag{15}$$

is a velocity potential function describing the acoustical vibration motion of the liquid caused by $w(x, t)$. It has to be a solution of the boundary problem

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial x^2} = 0\,; \tag{16}$$

$$\left.\frac{\partial \varphi}{\partial x}\right|_{x=0} = 0, \quad \left[\frac{\partial^2 \varphi}{\partial t^2} + g\frac{\partial \varphi}{\partial x}\right]_{x=h} = 0\,; \tag{17}(18)$$

$$\left.\frac{\partial \varphi}{\partial r}\right|_{r=R} = \frac{\partial w}{\partial t}, \tag{19}$$

and can be taken in the form

$$\varphi = \sum_m i\sigma A_m I_0(\psi_m r) \cos k_m x \exp(i\sigma t), \; m = 1, 2, \ldots \infty, \tag{20}$$

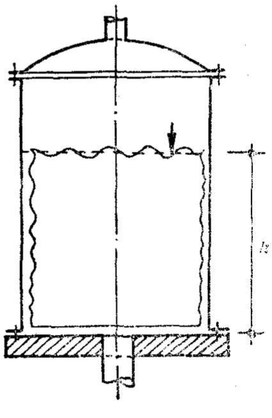

Fig. 2

where A_m is an integration constant; $I_0(\psi_m r)$ — modified Bessel function of zero order; k_m — longitudinal wave number; ψ_m — radial wave number; g — gravity acceleration.

3.1.2. The wave numbers k_m and ψ_m must satisfy the simplest possible dispersion equation

$$\psi_m^2 - k_m^2 = 0 \Rightarrow \psi_m = k_m \text{ and } \psi_m = -k_m. \tag{21}$$

In form (20) the velocity potential satisfies condition (17) only. Boudary condition (18) is satisfied, if for every σ the equation

$$(k_m h) \operatorname{tg}(k_m h) = -(\sigma^2 h / g) \tag{22}$$

is valid. The last equation determines a set of wave numbers

$$(k_m h)=k_m^* \Rightarrow k_m=(k_m^*/h),\ m=1,2,\ldots,\infty, \tag{23}$$

which take part in (21).

3.1.3. The last boundary condition has the form

$$\sum_m i\sigma A_m \psi_m I_0'(\psi_m R)\cos k_m x \exp(i\sigma t)=i\sigma W(x)\,.\exp(i\sigma t). \tag{24}$$

It is easy to prove that (22) plays role of necessary and sufficient condition to have a full set of eigenfunctions $\cos k_m x$ orthogonal in the interval $[0\leq x\leq h]$. Hence,

$$A_m=\frac{2}{\psi_m h I_0'(\psi_m R)}\int_0^h W(x)\cos k_m x\,dx\equiv A_m^*,\quad [A_m^*]=m^2. \tag{25}$$

So, the unknown integration constant was determined, but the function $W(x)$ was introduced into consideration*.

3.1.4. Let us rewrite (20) in the form

$$\varphi=\sum_m i\sigma A_m^* I_0(\psi_m r)\cos k_m x \exp(i\sigma t) \tag{26}$$

and find out the acoustical pressure having the form

$$p=-\rho(\partial\varphi/\partial t)=\sum_m \rho\sigma^2 A_m^* I_0(\psi_m r)\cos k_m x\ \exp(i\sigma t) \tag{27}$$

in an arbitrary point of the domain occupied by the liquid, and the form

$$p=\sum_m \rho\sigma^2 A_m^* I_0(\psi_m R)\cos k_m x \exp(i\sigma t) \tag{28}$$

when it is acting onto the shell wall. But it acts onto the shell wall only in the interval $[0\leq x\leq h]$. That is why the deflection function must satisfy equation (11), but in the form

$$\frac{\partial^4 w_1}{\partial x^4}+\frac{1-\mu^2}{\gamma^2 R^4}w_1+\frac{m_0}{D}\frac{\partial^2 w_1}{\partial t^2}=-\frac{p}{D}+\frac{q}{D} \tag{29}$$

in the interval $[0\leq x\leq h]$, and in the form

$$\frac{\partial^4 w_2}{\partial x^4}+\frac{1-\mu^2}{\gamma^2 R^4}w_2+\frac{m_0}{D}\frac{\partial^2 w_2}{\partial t^2}=\frac{q}{D},\quad (or=0) \tag{30}$$

in the interval $(h<x\leq L]$. After the solution of both equations (29) and (30) the forms w_1 and w_2 must be sewn together.

3.1.5. One can use another way, formulating the sinus integral transforms

$$\overline{W}_\nu=\int_0^L W(x)\sin\frac{\nu\pi x}{L}dx,\quad W(x)=\sum_{\nu=1}^{\infty}(2\overline{W}_\nu/L)\sin\frac{\nu\pi x}{L} \tag{31}$$

using boundary conditions (13). Further,

* If $W(x)$ is known, the problem is solved in explicit form. If it is unknown, the further transforms have formal character, and in the final result [see relations (32), (34) and (37)] one must keep in mind that A_m^* depends on $W(x)$.

$$
(32) \qquad \frac{d^4W}{dx^4}-\left(\frac{m_0\sigma^2}{D}-\frac{1-\mu^2}{\gamma^2R^4}\right)W
$$

$$
=\begin{cases} +\sum\limits_m \frac{\rho\sigma^2}{D}A_m^* I_0(\psi_m R)\cos k_m x+(Q/D), & 0\le x\le h,\\ 0, & h<x\le L,\end{cases}
$$

and applying here (31), introducing firstly the parameters:

$$
(33) \qquad \begin{gathered} \Omega^4=\{(m_0\sigma^2/D)-[(1-\mu^2)/\gamma^2R^4]\}, \quad [\Omega]=m^{-4};\\ S_m=(\rho\sigma^2 A_m^*/D), \quad [S_m]=m^{-3}, \end{gathered}
$$

we shall have

$$
(34) \qquad [(\nu^4\pi^4/L^4)-\Omega^4]\overline{W}_\nu=+\sum_m S_m I_0(\psi_m R)\mathcal{A}_{m\nu}+\overline{Q}_\nu,
$$

$$
(35) \qquad \mathcal{A}_{m\nu}=\int_0^h \cos k_m x\,.\sin(\nu\pi x/L)\,dx, \quad \overline{Q}_\nu=\int_0^h Q(x)\sin(\nu\pi x/L)\,dx.
$$

So, the determined transformant will be

$$
(36) \qquad \overline{W}_\nu^*=-\sum_m \frac{S_m I_0(\psi_m R)}{[\Omega^4-(\nu^4\pi^4/L^4)]}\mathcal{A}_{m\nu}-\frac{\overline{Q}_\nu}{[\Omega^4-(\nu^4\pi^4/L^4)]}
$$

and the original form of vibration —

$$
(37) \qquad W(x)=W_0+\sum_\nu \overline{W}_\nu^*(2/L)\,.\sin(\nu\pi x/L).
$$

3.1.6. It is important to mark the possibilities

$$
(38) \qquad \Omega^4=(\nu^4\pi^4/L^4), \quad k_m=(k_m^*/h)=(\nu\pi/L)
$$

which permit to identify the critical frequences σ_ν^* and critical liquid depths $h_{\nu m}^*$

$$
(39) \qquad \begin{gathered} \sigma_\nu^*=\sqrt{(D/m_0)\{(\nu^4\pi^4/L^4)+[(1-\mu^2)/(\gamma^2R^4)]\}},\\ h_{\nu m}^*=(k_m^* L/\nu\pi), \quad (\Rightarrow \mathcal{A}_{m\nu}\Rightarrow\infty), \end{gathered}
$$

causing resonance phenomena $\overline{W}_\nu^*\Rightarrow\infty$, if the given parameters σ and h become equal to them, i. e. if

$$
(40) \qquad \sigma=\sigma_\nu^* \text{ and } h=h_{\nu m}^*.
$$

A simple analysis shows that:

(*i*) as σ was taken as forced vibration frequency of the shell, its critical values depend on the shell geometric parameters only;

(*ii*) As the boundary conditions used [conditions (13)] are homogeneous, the case $Q\equiv 0$ provides a possibility to consider

$$
(41) \qquad W(x)=W_0-\sum_\nu\sum_m \{[S_m I_0(\psi_m R)\mathcal{A}_{m\nu}]/[\Omega^4-(\nu^4\pi^4/L^4)]\}(2/L)\sin(\nu\pi x/L)
$$

as a vibration mode of the free shell vibration, taking into account the influence of the liquid presence;

(*iii*) to every shell mode ν a set of critical depths is possible, e. g.:

(42)
$$\nu=1,\ h_{11}\sim L/2,\ h_{12}\sim 3L/2,\ldots;$$
$$\nu=2,\ h_{21}\sim L/4,\ h_{22}\sim 3L/4,\ h_{23}\sim 5L/4,\ldots;$$
$$\nu=3,\ h_{31}\sim L/6,\ h_{32}\sim 3L/6,\ h_{33}\sim 5L/6,\ldots,$$

the unreal values being truncated.

3.2. Case $U_1=0$, $h=\mathrm{const}\neq L$, **(free liquid vibration).**

3.2.1. If the shell under consideration stops to vibrate, the liquid will continue its vibrations, which will be already free ones. They can be described by the velocity potential

(43)
$$\varphi_0=\varphi_0(r,x;t)$$

being a solution of the boundary problem

(44)
$$\frac{\partial^2\varphi_0}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi_0}{\partial r}+\frac{\partial^2\varphi_0}{\partial x^2}=0;$$

(45) (46)
$$\left.\frac{\partial\varphi_0}{\partial x}\right|_{x=0}=0,\quad \left[\frac{\partial^2\varphi_0}{\partial t^2}+g\frac{\partial\varphi_0}{\partial x}\right]_{x=h}=0,$$

(47)
$$\left.\frac{\partial\varphi_0}{\partial r}\right|_{r=R}=0.$$

3.2.2. The velocity potential function

(48)
$$\varphi_0=\sum_n i\omega_n A_n\cdot J_0(\psi_n r)\,.\,ch\,k_n x\,\exp(i\omega_n t)$$

is a solution of the problem, if the system of dispersion equations

(49) (50)
$$k_n^2-\psi_n^2=0,\quad (k_n h)\,.\,th(k_n h)=(\omega_n^2 h/g)$$

is valid. The set of values of ψ_n is determined from condition (47) by the chain of transforms

(51)
$$\psi_n J_0'(\psi_n R)=0\Rightarrow(\psi_n R)=\psi_n^*;\quad \psi_n=(\psi_n^*/R);\quad n=0,1,\ldots,\infty.$$

So, from (49) the set of values of k_n is fully determined. Hence, the natural circular frequencies ω_n are determinable from equation (50), i. e.

(52)
$$\omega_n=+\sqrt{g\,.\,k_n th(k_n h)}.$$

3.3. The two cases (of forced and free liquid vibrations) $U_1\neq0$, $h=h(t)$ are considered in Ref. [4], and that is why here they will not be considered.

IV. Dispersion Equations (Compressible Liquid)

4.1. Case $U_1=0$, $h=\mathrm{const}\,t\neq L$, **(forced liquid vibrations).**

4.1.1. The equation of the liquid motion now is Eq. (1), developed in the form

(53)
$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\partial^2\varphi}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=0.$$

The velocity potential function has to satisfy additionally the boundary conditions (17), (18) & (19) which are independent of the liquid compressibility (or incompressibility).

As a solution one can examine again the function

$$\varphi=\sum_m i\sigma A_m J_0(\psi_m r)\cos k_m x \exp(i\sigma t), \tag{54}$$

which has wave numbers k_m & ψ_m satisfying the dispersion equation (9), but rewritten in the form

$$-\psi_m^2-k_m^2+(\sigma/c)^2=0 \Rightarrow \psi_m=\sqrt{(\sigma/c)^2-k_m^2}, \tag{55}$$

or the function

$$\varphi=\sum_m i\sigma A_m I_0(\psi_m r)\cos k_m x \exp(i\sigma t) \tag{56}$$

with dispersion equation

$$\psi_m^2-k_m^2+(\sigma/c)^2=0 \Rightarrow \psi_m=\sqrt{k_m^2-(\sigma/c)^2} \tag{57}$$

for the wave numbers k_m & ψ_m.

4.1.2. Which of both functions [(54), or (56)] must be taken is impossible to be said without additional analysis of equations (55) and (57). Shortly, this analysis contains the following points:

(*i*) the forced circular frequency σ must be qualified as low or high using the set of "dry" natural shell frequencies;

(*ii*) the wave number k_m for both cases must be determined from condition (18), i. e. from Eqs. (22) and (23); for real $(\sigma^2 h/g)$ we have $k_1^* \sim \pi/2$, $k_2^* \sim 3\pi/2$, etc. [5];

(*iii*) for low frequency σ, $(\sigma/c) < (k_m^*/h)$ showing that solution (56) must be accepted;

(*iv*) for high frequency σ, a number m^* exists, such that for $m<m^* \Rightarrow k_m < (\sigma/c)$; hence, solution (54) must be accepted; and for $m\geq m^* \Rightarrow k_m > (\sigma/c)$; hence, solution (56) has to be accepted.

After the function choice the solution can follow the scheme developed in P. 3. 1, (of course with some small variations).

4.1.3. We shall give here the second case when both functions (54) and (56) must be taken into account. Well, we have

$$\varphi=i\sigma\left[\sum_{m=1}^{<m^*} A_m J_0(\psi_m r)+\sum_{m^*}^{\infty} A_m I_0(\psi_m r)\right]\cos k_m x \exp(i\sigma t), \tag{58}$$

and

$$p_R=\rho\sigma^2\left[\sum_{m=1}^{<m^*} A_m^* J_0(\psi_m R)+\sum_{m^*}^{\infty} A_m^* I_0(\psi_m R)\right]\cos k_m x \exp(i\sigma t). \tag{59}$$

Condition (19), for A_m^* determination [if presentation (14) is valid] gives

$$A_m^*=\frac{2}{\psi_m h J_0'(\psi_m R)}\int_0^h W(x)\cos k_m x\, dx, \quad 1\leq m<m^*,$$

$$A_m^* = \frac{2}{\psi_m h I_0'(\psi_m R)} \int_0^h W(x) \cos k_m x \, dx, \quad m^* \leq m \leq \infty.$$

Let $Q=0$. Then two vibration forms $W(x)$ have place:
(i) case $m<m^*$ —

(61) $$W_1(x) = W_0 - \sum_\nu \sum_m \{[S_m J_0(\psi_m R) Я_{m\nu}]/[\Omega^4 - (\nu\pi/L)^4]\} (2/L) \sin(\nu\pi x/L),$$

where, following $(33)_2$

(62) $$S_m = (\rho\sigma^2 A_m^*/D), \qquad m<m^*,$$

$Я_{m\nu}$ being invariable, see $(33)_1$;
(ii) case $m \geq m^*$ —

(63) $$W_2(x) = W_0 - \sum_\nu \sum_m \{[S_m I_0(\psi_m R) Я_{m\nu}]/[\Omega^4 - (\nu\pi/L)^4]\} (2/L) \sin(\nu\pi x/L),$$

where, as above $Я_{m\nu}$ is invariable and S_m can be expressed by formula (62) but taking A_m^* from $(60)_2$, i. e. for $m \geq m^*$.

So, one can write generally

(64) $$W(x) = W_0 - \sum_\nu \overline{\overline{W}}_\nu (2/L) . \sin(\nu\pi x/L),$$

where

(65) $$\overline{\overline{W}}_\nu = \frac{\sum_{m=1}^{<m^*} S_m J_0(\psi_m R) Я_{m\nu} + \sum_{m^*}^{\infty} S_m I_0(\psi_m R) Я_{m\nu}}{[\Omega^4 - (\nu\pi/L)^4]}.$$

4.1.4. The analysis shows that:
(i) shell radius can not be arbitrary; the matter is that the set of values ψ_m is determined independently and for each of them a set of special radii R_τ, ($\tau = 1, 2, \ldots, \infty$) is posible such that

(66) $$J_0'(\psi_m R_\tau) = 0 \Rightarrow S_m(m<m^*) \Rightarrow \infty;$$

(ii) another set of shell radii $R_\varkappa$, ($\varkappa = 1, 2, \ldots, \infty$) is possible, such that

(67) $$J_0(\psi_m R_\varkappa) = 0, \quad (m<m^*),$$

but this possibility is not dangerous from a point of view of resonance occurrence;
(iii) if Eq. (67) has place, then

(68) $$[\partial\varphi_\varkappa/\partial x]_{r=R_\varkappa} = 0, \quad (m<m^*);$$

(iv) one can mark the resonance possibilities (38), (39) and (40), which do not change themselves here;
(v) again, if $W(x)$ is known form of vibration, the problem is solved in explicit form finding out the integration constants from expressions (60); otherwise, the transforms expressed by relations (61), (63), etc. have to be considered as formal ones.

4.1.5. Obviously, the case considered describes the free vibrations of the whole mechanical system (shell+liquid). Really, the radial boundary condition

(19), together with the dynamical boundary condition of type (32) unite the liquid and the shell in a general mechanical system. Its longitudinal boundary conditions, e. g. (17), (18) and (19), are homogeneous.

The problem is more clear if h is equal to L. Again $Q=0$. Then instead Eq. (32) one can write

$$(d^4W/dx^4)-\Omega^4 W=+\sum_m (\rho\sigma^2/D)\, A_m Z_m \cos k_m x, \tag{69}$$

where

$$\begin{aligned} &Z_m=J_0(\psi_m R), \quad Z'_m=J'_0(\psi_m R), \quad m<m^*,\\ &Z_m=I_0(\psi_m R), \quad Z'_m=I'_0(\psi_m R), \quad m\geq m^*, \end{aligned} \tag{70}$$

considering the constants as unknown. The solution of (69) is

$$\begin{aligned} W=&C_1\cos\Omega x+C_2\sin\Omega x+C_3\operatorname{ch}\Omega x+C^4\operatorname{sh}\Omega x\\ &+\sum_m [(\rho\sigma^2)/D(k_m^4-\Omega^4)]\, A_m Z_m \cos k_m x, \end{aligned} \tag{71}$$

where C_1, C_2, C_3 and C_4 are integration constants. In the form given above W does not satisfy the boundary conditions (13). That is why we shall accept $C_1=C_3=C_4=0$. Then

$$W=C_2\sin\Omega x+\sum_m [(\rho\sigma^2)/D(k_m^4-\Omega^4)]\, A_m\, Z_m \cos k_m x. \tag{72}$$

Here the constant C_2 and the multitude of constants A_m are unknown. But with this form of vibration condition (19) can be satisfied. This determines the constants

$$A_m=C_2\frac{D\,.\,K_m\,.\,(k_m^4-\Omega^4)}{[\psi_m Z'_m\cdot(k_m^4-\Omega^4)-\rho\sigma^2 Z_m]N_m}, \tag{73}$$

and the final vibration form

$$W=C_2\left\{\sin\Omega x+\sum_m\frac{\rho\sigma^2 Z_m K_m\cos k_m x}{[\psi_m Z'_m\cdot(k_m^4-\Omega^4)-\rho\sigma^2 Z_m]N_m}\right\}, \tag{74}$$

where

$$K_m=\int_0^L \sin\Omega x\,.\,\cos k_m x\,dx, \quad N_m=\int_0^L \cos^2 k_m x\,dx. \tag{75}$$

Again this form does not satisfy the boundary conditions (13), but now the sinus-transforms introduced, see formulae (31), permit to represent (74) in the form

$$W=\sum_\nu \overline{\overline{W}}_\nu\,.\,(2/L)\,.\sin(\nu\pi x/L), \tag{76}$$

taking

$$\overline{\overline{W}}_\nu=C_2\left\{P_\nu+\sum_m\frac{2\sigma^2 Z_m K_m \mathcal{A}_{m\nu}}{L[\psi_m Z'_m\cdot(k_m^4-\Omega^4)-\rho\sigma^2 Z_m]}\right\}, \tag{77}$$

where

(78) $$P_\nu = \int_0^L \sin\Omega x \,.\, \sin(\nu\pi x/L)\,dx.$$

The solution presented here can serve as an example showing the difficulties, if the formal scheme developed in P. 3. 1. 5 and P. 4. 1. 3 has to be accomplished. In both cases Eqs. (32) and (69) must be considered as integro-differential ones, and a special method of consequent approximations can be applied.

4.1.6. The resonance phenomena are identified by the multitudes of critical parameters

(79) $$\Omega = \nu\pi/L \Rightarrow \sigma_\nu^* = \sqrt{(D/m_0)\{(\nu\pi/L)^4 + [(1-\mu^2)/\gamma^2 R^4]\}}\,,$$

80) $$k_m = \nu\pi/L, \quad h_{\nu m}^* = (k_m^* L/\nu\pi),$$

(81) $$\psi_m Z'_m \,.\, (k_m^4 - \Omega^4) - \rho\sigma^2 Z_m = 0.$$

The anlysis of all these possibilities is very interesting, providing important data for the designers. The critical values of σ can be designated by ω_ν.

4.2. Case $U_1 = 0$, $h = \text{const} \neq L$, (free liquid vibrations).

4.2.1. Again the problem is to find the modes of the liquid vibrations if at a given moment the shell stops to vibrate. Then the velocity potential function φ_0 must be a solution of the familiar boundary-value problem

(82) $$\frac{\partial\varphi_0}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi_0}{\partial r} + \frac{\partial^2\varphi_0}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\varphi_0}{\partial t^2} = 0,$$

(83) (84) $$\left.\frac{\partial\varphi_0}{\partial x}\right|_{x=0} = 0, \quad \left[\frac{\partial^2\varphi_0}{\partial t^2} + g\frac{\partial\varphi_0}{\partial x}\right]_{x=h} = 0,$$

(85) $$[\partial\varphi_0/\partial r]_{r=R} = 0.$$

4.2.2. As a solution of the problem can be examined the function

(86) $$\varphi_0 = \sum_m \sum_n i\omega_{mn} B_{mn} J_0(\psi_n r) \cos k_m x \,.\, \exp(i\omega_{mn} t),$$

which was mentioned in P. 1. 5 {see Eq. (10), or [6]}. The radial wave numbers ψ_n are determinable again from the chain of transforms

(87) $$\psi_n J'_0(\psi_n R) = 0 \Rightarrow (\psi_n R) = \psi_n^* \Rightarrow \psi_n = \psi_n^*/R$$

the longitudinal wave numbers being determined from conditions (83) and (84), i. e. from equation

(88) $$(k_m h)\,\text{tg}(k_m h) = -(\omega_{mn}^2 h/g),$$

but here the frequency ω_{mn} is unknown. The wave numbers k_m and ψ_n must satisfy additionally the dispersion equation

(89) $$\omega_{mn}^2 = c^2(\psi_n^2 + k_m^2).$$

Substituting this value into the former equation, one can obtain the final relation

(90) $$(k_m h)\,\text{tg}(k_m h) = -\frac{c^2[(k_m h)^2 + (\psi_n h)^2]}{(gh)}$$

serving to determine a set of values k^*_m, $(m=1, 2, \ldots, \infty)$, and afterwards the wave numbers $k_m=(k^*_m/h)$. With this the natural frequencies ω_{mn} are fully determinable from relation (89).

4.2.3. It would be useful to mention that the eigenfunctions $J_0(\psi_n r)$ are mutually orthogonal in the interval $[0 \leq r \leq R)$. The functions $\cos k_m x$ are not mutually orthogonal in the interval $[0 \leq x \leq h]$. Really, they are solutions of the differential equation

$$X''+k_m^2 X=0, \quad (.)''=[d^2(.)/dx^2] \tag{91}$$

with boundary conditions

$$X'(0)=0, \quad [gX'-\omega_{mn}^2 X]_{x=h}=0, \tag{92}$$

but in accordance with (89) ω_{mn} depends on k_m. This introduces a peculiarity which shows that both functions X_1 and X_2 are not mutually orthogonal.

Let us consider two values k_1 and k_2 of the longitudinal wave numbers k_m and their corresponding functions $X_1(k_1 x)=\cos k_1 x$ and $X_2(k_2 x)=\cos k_2 x$. By using the procedure being an object of common knowledge one can prove the relation

$$\int_0^h (X_1''X_2-X_1X_2'')dx=-(k_1^2-k_2^2)\int_0^h X_1X_2 dx, \tag{93}$$

but

$$\int_0^h (X_1''X_2-X_1X_2'')dx=X_1'X_2-X_1X_2']_0^h, \tag{94}$$

the chain of transforms

$$[X_1'X_2-X_1X_2']_0^h=-k_1 \sin k_1 h \,.\cos k_2 h+k_2 \cos k_1 h \,.\sin k_2 h, \tag{95}$$

$$k_1 \operatorname{tg} k_1 h-k_2 \operatorname{tg} k_2 h=-(k_1^2-k_2^2)(c^2/g)$$

can be developed. On the last step Eq. (92) is used. So, using the results given above we can write

$$\int_0^h \cos k_1 x \,.\cos k_2 x \, dx=-\frac{c^2 \cos k_1 h \,.\cos k_2 h}{g}, \tag{96}$$

i. e. functions $\cos k_m x$ are not mutually orthogonal at conditions (92). They are orthogonal, if $\omega=\text{const}$, i. e. if the vibrations have one mode only, but of a forced type [8].

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