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Dispersion Equations of the Roundcylindrical Shell-Tank Hydroelasticity Problem (Part II)

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Preface

To the Third International Conference on *Differential Equations, July 1985, Russe (Bulgaria)*, the former two authors had presented three papers. The third of them had the same headline as the present work. In accordance with the Statement only two papers could be published in the conference volume [2, 3]. The third paper was enlarged and revised essentially, and the present work is its second part (the first part being published in JTAM, Vol. XVII (1986), No 2, see Ref. [4]). The purpose of the present investigation is to show how the analysis of the dispersion equations gives a possibility to intervene into the technological parameter control of different acoustical processes of the circular cylindrical tanks.

The first part contains the following subdivisions: Preface; I. Introduction; II. The Problem; III. Dispersion Equations (Incompressible Liquid); IV. Dispersion Equations (Compressible Liquid). From the last subdivision two cases of a constant liquid level (forced and free liquid vibrations) are included. The next two cases concerning the forced and free vibrations of the liquid volume of variable level are an object of the present investigation. On Fig. 1 the scheme of the tank under consideration together with the cylindrical co-ordinates connected with the bottom are shown.

IV. Dispersion Equations (Compressible Liquid)

4.3. Case $U_1 \neq 0$, $h = const \neq L$ (forced liquid vibrations)

4.3.1. Obviously any fidding condition on the free liquid surface has to be introduced. As a such condition one can take the vapour condensation condition upon the surface $x=h$, Fig. 2. This case is very interesting from a technological (as well as an analytical) point of view, but as an example (here) a case when the liquid pours into the tank through a tube with radius b is considered. So, the liquid velocity U_0 in this tube is

$$U_0=(R^2/b^2)U_1, \tag{1}$$

the liquid velocity in the outlet tube being

$$U_2=(R^2/a^2)U_1, \tag{2}$$

where U_1 is a stationary transport liquid velocty in the tank. These two equations express the mass conservation principle in the specific technological conditions of the problem. And secondly, the three transport velocities U_0, U_1 and U_2 determine the

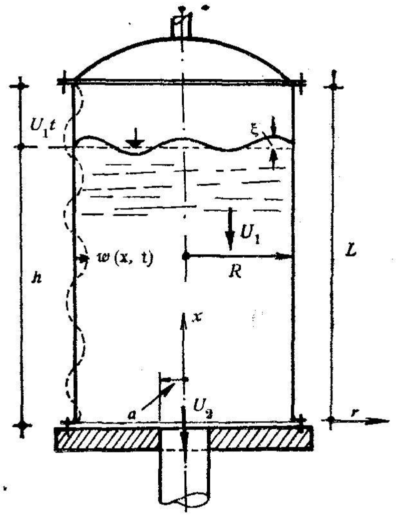

Fig. 1

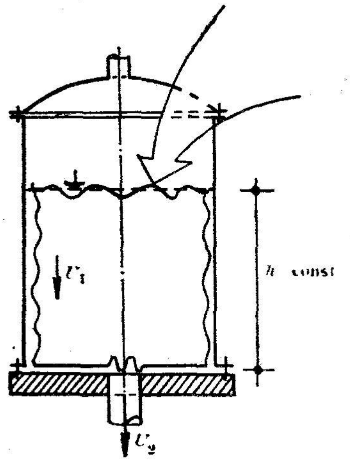

Fig. 2

part of the liquid motion (in the three structural components: inlet tube, tank, outlet tube) having a stationary character, as the liquid is taken ideal, compressible and potentially moving. The corresponding velocity potentials are premised in the forms:

$$-U_0 x\,; \quad -U_1 x\,; \quad -U_2 x. \tag{3}$$

The last assumption shows that the tank is supposed to be a long one. Otherwise, the boundary effects appearing around the inlet and outlet orifices will influence each other and velocity potential $(3)_2$ will be quite different, see Fig. 3.

Because of the small shell vibrations with forced circular frequency σ, described by the relation

$$w(x,\ t) = W(x)\, exp\,(i\sigma t) \tag{4}$$

in the liquid occupying the domain

$$[0 \leq x \leq h, \quad 0 \leq r \leq R, \quad 0 \leq \theta \leq 2\pi]$$

acoustical vibrations will be induced. They have a character of small perturbations superposed to $-U_1 x$-potential, and will be described by the function $\varphi_1(x,\ r;\ t)$ of the velocity potential, such that:

$$\Phi_1(x,\ r;\ t) = -U_1 x + \varphi_1(x,\ r,\ t)\,; \tag{5}$$

$$v_{1x} = -U_1 + (\partial\varphi_1/\partial x), \quad v_{1r} = (\partial\varphi_1/\partial r)\,; \tag{6}$$

$$p_1 = -\rho[(\partial\varphi_1/\partial t) - U_1(\partial\varphi_1/\partial x) + (U_1^2/2) + g(h+\zeta-x)], \tag{7}$$

where Φ_1 is the velocity potential of the general liquid motion; ρ — actual liquid density, g — gravity acceleration, $\zeta(r,\ t)$ — surface gravity wave function, $[\zeta] = m$; p — acoustical pressure.

Theoretically the acoustical perturbations will propagate along the liquid of the inlet tube and the velocity potential

$$\Phi_0(x,\ r;\ t) = -U_0 x + \varphi_0(x,\ r\,;\ t) \tag{8}$$

describes the general liquid motion in. The corresponding velocities and pressure are [8, vol. 2]:

(9) $$v_{0x} = -U_0 + (\partial\varphi_0/\partial x), \quad v_{0r} = (\partial\varphi_0/\partial r),$$

(10) $$p_0 = -\rho[(\partial\varphi_0/\partial t) - U_0(\partial\varphi_0/\partial x) + (U_0^2/2)].$$

Along the liquid in the outlet tube the velocity potential

(11) $$\Phi_2(x, r; t) = -U_2 x + \varphi_2(x, r; t)$$

is supposed. The corresponding velocities and pressure are:

(12) $$v_{2x} = -U_2 + (\partial\varphi_2/\partial x); \quad v_{2r} = (\partial\varphi_2/\partial r);$$

(13) $$p_2 = -\rho[(\partial\varphi_2/\partial t) - U_2(\partial\varphi_2/\partial x) + (U_2^2/2)].$$

In both last cases the static hydraulic pressure is neglected (is not taken into account).

4.3.2. The acoustical equation used in the previous part of the present investigation

(14) $$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0$$

was deduced in Eulerian co-ordinates $(x, r; t)$ but an arbitrary liquid particle had not any transport velocity. For the case under consideration such velocity exists, and it is different in the different structural components. In accordance with Ref. [8, Vol. 2 § 11, P.2] one can rewrite Eq. (14) in the form

(15) $$(\partial^2\varphi/\partial r^2) + (1/r)(\partial\varphi/\partial r) + (\partial^2\varphi/\partial x^2) - c^{-2}(D^2\varphi/Dt^2) = 0,$$

understanding $D\varphi/Dt$ as

(16) $$D\varphi/Dt = (\partial\varphi/\partial t) + U_{tr}(\partial\varphi/\partial x),$$

where $U_{tr} \equiv -U_0$ in the inlet tube, $U_{tr} \equiv -U_1$ in the tank, and $U_{tr} = -U_2$ in the outlet tube. So, if $j = 0, 1, 2$, then

(17) $$\frac{D^2\varphi_j}{Dt^2} = \frac{\partial^2\varphi_j}{\partial t^2} - 2U_j\frac{\partial^2\varphi_j}{\partial x\partial t} + U_j^2\frac{\partial^2\varphi_j}{\partial x^2}$$

which in combination with (15) results the governing acoustical equation considered, namely

(18) $$(1 - M_j^2)\frac{\partial^2\varphi_j}{\partial x^2} + \frac{\partial^2\varphi_j}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi_j}{\partial r} + 2\frac{M_j}{c}\frac{\partial^2\varphi_j}{\partial x\partial t} - \frac{1}{c^2}\frac{\partial^2\varphi_j}{\partial t^2} = 0,$$

where

(19) $$M_j = (U_j/c)$$

is Mach number, c being sound velocity in the liquid (do compare with Ref. [4, P. 1.4]).

4.3.3. Velocity potentials Φ_0, Φ_1 and Φ_2 have to satisfy boundary conditions described below.

(*i*). At $r = 0$ all three potentials Φ_0, Φ_1 and Φ_2 must be equal to limited constants, e. g.

(20) $$\Phi_0|_{r=0} = \text{const} < \infty; \quad \Phi_1|_{r=0} = \text{const} < \infty; \quad \Phi_2|_{r=0} = \text{const} < \infty.$$

(*ii*). At $r = R$, $r = b$ and $r = a$ the boundary conditions are

(21) $$\left.\frac{\partial\varphi_1}{\partial r}\right|_{r=R} = \frac{\partial w}{\partial t} - U_1\frac{\partial w}{\partial x}, \quad \left.\frac{\partial\Phi_0}{\partial r}\right|_{r=b} = 0, \quad \left.\frac{\partial\Phi_2}{\partial r}\right|_{r=a} = 0.$$

(*iii*). The intensity of the local liquid stream from volume V_1 (of the tank) into volume V_2 (of the outlet tube), see Fig. 3, at $x = 0$ must satisfy the condition

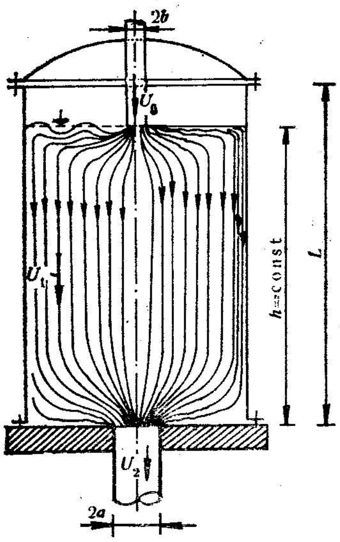

Fig. 3.

$$[\rho_1 v_{1x} r dr d\theta]_{x=0} = \begin{cases} [\rho_2 v_{2x} r dr d\theta]_{x=0}, & 0 \le r \le a. \\ 0, & a < r \le R. \end{cases} \tag{22}$$

We shall accept that $\rho_1 = \rho_2 = \rho_0$, where ρ_0 is an initial liquid density. So, Eq. (22) becomes

$$\int_0^{2\pi}\int_0^{R} [v_{1x}]_{x=0}\, r dr d\theta = \int_0^{2\pi}\int_0^{a} [v_{2x}]_{x=0}\, r dr\, d\theta. \tag{23}$$

Analogously, at $x=h$,

$$\int_0^{2\pi}\int_0^{R} [v_{1x}]_{x=h}\, r dr d\theta = \int_0^{2\pi}\int_0^{b} [v_{0x}]_{x=h}\, r\, dr\, d\theta. \tag{24}$$

(*iv*). Additionally, the intensity of the local power stream $(p_1^a v_{1x})$ at $x=0$ must satisfy the boundary conditon

$$[p_1^a v_{1x}]_{x=0}\, r\, dr\, d\theta = \begin{cases} [p_2^a v_{2x}]_{x=0}\, r\, dr\, d\theta, & 0 \le r \le a, \\ 0, & a < r \le R, \end{cases} \tag{25}$$

which in integral form turns into

$$\int_0^{2\pi}\int_0^{R} [p_1^a v_1]_{x=0}\, r d\, r\, d\theta = \int_0^{2\pi}\int_0^{a} [p_2^a v_{2x}]_{x=0}\, r\, dr\, d\theta. \tag{26}$$

Analogously at $x=h$—

$$\int_0^{2\pi}\int_0^{R} [p_1^a v_1]_{x=h}\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{b} [p_0^a v_{0x}]_{x=h}\, r\, dr\, d\theta. \tag{27}$$

Because of the circular symmetry in conditions (23), (24), (26) and (27) the integral along the circular co-ordinate can be rejected. However, the sought boundary conditions become:

$$(28) \qquad \int_0^R \left[\frac{\partial \Phi_1}{\partial x}\right]_{x=0} r\,dr = \int_0^a \left[\frac{\partial \Phi_2}{\partial x}\right]_{x=0} r\,dr,$$

$$(29) \qquad \int_0^R \left[\frac{\partial \Phi_1}{\partial x}\right]_{x=h} r\,dr = \int_0^b \left[\frac{\partial \Phi_0}{\partial x}\right]_{x=h} r\,dr;$$

$$(30) \qquad \int_0^R \left[\left(\frac{\partial \varphi_1}{\partial t} - U_1\frac{\partial \varphi_1}{\partial x}\right)\frac{\partial \varphi_1}{\partial x}\right]_{x=0} r\,dr = \int_0^a \left[\left(\frac{\partial \varphi_2}{\partial t} - U_2\frac{\partial \varphi_2}{\partial x}\right)\frac{\partial \varphi_2}{\partial x}\right]_{x=0} r\,dr,$$

$$(31) \qquad \int_0^R \left[\left(\frac{\partial \varphi_1}{\partial t} - U_1\frac{\partial \varphi_1}{\partial x}\right)\frac{\partial \varphi_1}{\partial x}\right]_{x=h} r\,dr = \int_0^b \left[\left(\frac{\partial \varphi_0}{\partial t} - U_0\frac{\partial \varphi_0}{\partial x}\right)\frac{\partial \varphi_0}{\partial x}\right]_{x=h} r\,dr.$$

The two additional conditions at the start of the inlet tube $x=L_0$ and at the end of outlet tube $x=-L_2$ must be taken into account. But this presupposes to know the corresponding technological conditions, e. g. how the pump is filling dynamically the inlet tube (at $x=L_0$) and how the liquid is pouring out from the outlet tube (at $x=-L_2$), etc.

We shall accept that both tubes are infinitely long, and then the acoustical perturbations described by $\varphi_0(x, r; t)$&$\varphi_2(x, r; t)$ must satisfy two conditions of the type of Sommerfeld attenuation principle, given by the relation

$$(32) \qquad \lim_{x\to\infty} [(\partial\varphi_j/\partial x) + ik_j\varphi_j] = 0,$$

if both potentials depend on the time as $\exp(i\sigma t)$. One can prove that the simple attenuation conditions

$$(33) \qquad \lim_{x\to\infty} [\partial\varphi_0/\partial x] = 0 \quad \& \quad \lim_{x\to-\infty} [\partial\varphi^2/\partial x] = 0$$

have a local character. In (32) k_j is a wave number, $j=0$, or 2.

4.3.4. Here we shall examine several functions fit to construct the general solution of the problem in the three different parts of the domain. All these functions can be obtained by the common separation of the variables

$$(34) \qquad \mathscr{C}_0(\psi r).X(kx).T(\sigma t),$$

where ψ is a radial wave number. To each of the examined functions any dispersion equation correspondes. Namely the analysis of these equations will show whether it is possible to use the function investigated in the construction of the general solution.

4.3.5. As a solution of the problem in the inlet tube one can examine the function

$$(35) \qquad \Phi_0 = -U_0 x + \sum_n i\sigma J_0(\psi_{0n} r)\,(A_{0n} e^{ik_{0n}x} + B_{0n}\, e^{-ik_{0n}x}) \exp(i\sigma t),$$

where the radial wave numbers ψ_{0n} are determinable form condition $(21)_2$ resulting

$$(36) \qquad \psi_{0n} J_1(\psi_{0n} b) = 0, \quad (\psi_{0n} b) = \psi_n^* \Rightarrow \psi_{0n} = \psi_{0n}^*/b,$$
$$n=0 \Rightarrow \psi_{00} = 0\,; \quad n = 1, 2, \ldots, \infty,$$

and keeping in mind that σ is known. Satisfying condition (32) one can prove that the constant A_{0n} must be equal to zero. So,

(37) $$\Phi_0=-U_0x+\sum_n(i\sigma)B_{0n}J_0(\psi_{0n}r)\,e^{-ik_{0n}x}e^{i\sigma t}.$$

The dispersion equation corresponding to solution (37) is

(38) $$(1-M_0^2)k_{0n}^2-2(M_0\sigma/c)k_{0n}-[(\sigma/c)^2-\psi_{0n}^2]=0,$$

and namely the satisfaction of this equation by the special combinations of the wave numbers and the geometrical parameters of the domain serves as a unique and sufficient condition to have variable separation expressed by (35) and (37). From (38) the unknown longitudinal wave numbers k_{0n} can be determined in the form

(39) $$(k_{0n})_{1,2}=\left[\frac{M_0\sigma}{c}\pm\sqrt{\left(\frac{\sigma}{c}\right)^2-\psi_{0n}^2(1-M_0^2)}\right](1-M_0^2)^{-1},$$

showing that the form of the solution (37) is possible only if

(40) $$\psi_{0n}^2(1-M_0^2)<(\sigma/c)^2,\quad M_0<1.$$

Obviously, if

(41) $$M_0>1,$$

(42) $$(k_{0n})_{1,2}=-\left[\frac{M_0\sigma}{c}\pm\sqrt{\left(\frac{\sigma}{c}\right)^2+\psi_{0n}^2(M_0^2-1)}\right](M_0^2-1)^{-1}$$

showing that solution (37) is valid for all ψ_{0n}. In accordance with Ref. [4], in cases of $M_0\ll 1$, its square can be neglected in comparison with the unity. Hence, then

(43) $$(k_{0n})_{1,2}=(M_0\sigma/c)\pm\sqrt{(\sigma/c)^2-\psi_{0n}^2}\,.$$

Here two cases are interesting to be considered.

(*i*). $n=0$, $\psi_{00}=0$, $J_0(\psi_{00}r)=1$. Then the velocity potential becomes

(44) $$\varphi_{00}=i\sigma B_{00}\,e^{-ik_{00}x}e^{i\sigma t},$$

describing both moving waves

(45) $$(B_{00})_{1,2}\exp\{-i[(k_{00})_{1,2}x-\sigma t]\},$$

where

(46) $$(k_{00})_{1,2}=M_0(\sigma/c)\pm(\sigma/c)=(\sigma/c)\,(M_0\pm 1).$$

(*ii*). $n=1, 2, \ldots, \infty$; $\psi_{0n}\neq 0$; $J_0(\psi_{0n}r)\neq 1$. A simple analysis shows [4], that in this case even the first radial wave number $\psi_{01}^*\approx 3{,}83$, and if f. e. $b=0{,}1\ m$, $\psi_{01}\sim 38{,}3\ m^{-1}$. So, if $(\sigma/c)^2\sim 0{,}49$ (which in technologies under consideration is almost possible), then $\psi_{01}>(\sigma/c)$, and

(47) $$(k_{0n})_{1,2}=(M_0\sigma/c)\pm i\sqrt{\psi_{0n}^2+(\sigma/c)^2(-1)^{-1}}.$$

These two wave numbers correspond to two systems of running waves

(48) $$B_{0n}\begin{cases}\exp\{i[-(M_0\sigma/c)x+\sigma t]\}\,.\exp[+\sqrt{\psi_{0n}^2-(\sigma/c)^2}\,x],\\ \exp\{i[-(M_0\sigma/c)x+\sigma t]\}\,.\exp\,]-\sqrt{\psi_{0n}^2-(\sigma/c)^2}\,x].\end{cases}$$

Obviously the first of them increases along x-co-ordinate and we shall reject it. Then the sought velocity potential in the inlet tube can be chosen in the form

(49) $$\Phi_0=-U_0x+(i\sigma)B_{00}\exp\{i[-(\sigma/c)(1+M_0)x+\sigma t]\}$$
$$+\sum_{n=1}^{\infty}(i\sigma)\,B_{0n}J_0(\psi_{0n}\,r)\exp\{i[-(M_0\sigma/c)x+\sigma t]+\sqrt{\psi_{0n}^2-(\sigma/c)^2}x\}.$$

4.3.6. The dispersion equation (but for both integrals in (35)) can be represented as it follows

(50) $$-(k_{0n}^2+\psi_{0n}^2)+[M_0k_{0n}\mp(\sigma/c)]^2=0.$$

Introducing the modified Mach numbers

(51) $$(M_{0n})_{1,2}=\{[U_0\mp(\sigma/k_{0n})]/c\}$$

and the new dimensionless number

(52) $$E_{0n}=\psi_{0n}/k_{0n}$$

one can deduce the curious relation

(53) $$(M_{0n})_{1,2}^2-(E_{0n})_{1,2}^2=1$$

showing that the wave number ratio can not be arbitrary. The signs (+ and —) in (51) show that the acoustical perturbations propagate with phase velocities $\pm(\sigma/k_{0n})$ along the inlet tube in both directions [1]. In form (53) the dispersion equation can be used for aims to control the wave propagation process in the tube.

4.3.7. For the outlet tube the velocity potential under examination is

(54) $$\Phi_2=-U_2x+\sum_n(i\sigma)J_0(\psi_{2n}r)\,(A_{2n}e^{ik_{2n}x}+B_{2n}e^{-ik_{2n}x})e^{i\sigma t},$$

and after a repetition of the whole procedure demonstrated in the previous point it can be selected in the form

(55) $$\Phi_2=-U_2x+(i\sigma)A_{20}\exp\{i[-(\sigma/c)\ (1+M_2)x+\sigma t]\}$$

$$+\sum_{n=1}^{\infty}(i\sigma)A_{2n}J_0(\psi_{2n}r)\exp\{i[-(M_2\sigma/c)x+\sigma t]+\sqrt{\psi_{2n}^2-(\sigma/c)^2}x\},$$

where the negative sign of the co-oridinate x is to be taken into account.

4.3.8. As far as Φ_1-determination is concerned we shall accept

(56) $$\Phi_1=-U_1x+\varphi_{11}+\varphi_{12},$$

(57) $$\varphi_{11}=\sum_{n=0}^{\infty}(i\sigma)J_0(\psi_{1n}r)\,(A_{1n}e^{ik_{1n}x}+B_{1n}e^{-ik_{1n}x})\exp(i\sigma t),$$

(58) $$\varphi_{12}=\sum_{m=0}^{\infty}(i\sigma)I_0(\psi_{1m}r)\,.\,A_{1m}\cos k_{1m}x\exp(i\sigma t),$$

satisfying the boundary conditions following the prescription given bellow.

Boundary condition $(21)_1$ is decomposed taking

(59) $$\left[\frac{\partial\varphi_{11}}{\partial r}\right]_{r=R}=0\ \&\ \left[\frac{\partial\varphi_{12}}{\partial r}\right]_{r=R}=\frac{\partial w}{\partial t}-U_1\frac{\partial w}{\partial x}.$$

The first of these two conditions determines the set of wavenumbers ψ_{1n} by the chain of transforms, see (36)

(60) $$\psi_{1n}J_1(\psi_{1n}R)=0,\quad(\psi_{1n}R)=\psi_{1n}^*\Rightarrow\psi_{1n}=(\psi_{1n}^*/R),$$

$$n=0\Rightarrow\psi_{10}=0;\quad n=1,\ 2,\ldots,\ \infty.$$

From the second condition the constants A_{1m} are determinable. The set of wavenumbers k_{1m} will be determined from the pair of conditions

$$\frac{\partial\varphi_{12}}{\partial x}\Big|_{x=0}=0 \ \& \ \frac{\partial\varphi_{12}}{\partial x}\Big|_{x=h}=0. \tag{61}$$

Obviously these two conditions exclude velocity potential φ_{12} from the satisfaction of the boundary conditions (28)—(31). However, boundary conditions (28)—(31) must be satisfied independently only by φ_{11}. Conditions (61) are satisfied, if

$$k_{1m}\sin(k_{1m}h)=0, \quad (k_{1m}h)=k^{*}_{1m} \Rightarrow k_{1m}=(k^{*}_{1m}/h) \tag{62}$$

$$m=0 \Rightarrow k_{10}=0; \quad m=1,\ 2,\dots,\ \infty.$$

4.3.9. In P. 4.1.5 of Ref. [4] the determination of $W(x)$, see (4), was demonstrated for a case of $h=L$. Here we shall consider the form $W(x)$ as fully determined. So $w(x,\ t)$ is full determined, as we did assume that σ is known, see (4). Let us satisfy condition $(59)_2$. After some simple transforms one can obtain

$$A^{*}_{1m}=\frac{2[K_{1m}+i(U_1/\sigma)K_{2m}]}{\psi_{1m}hI'_0(\psi_{1m}R)}, \tag{63}$$

where

$$K_{1m}=\int_0^h W(x).\cos k_{1m}x dx, \quad K_{2m}=\int_0^h W'(x)\cos k_{1m}x dx,$$

$$W'(x)=[dW(x)/dx], \tag{64}$$

$$I'_0(\psi_{1m}R)=(1/\psi_{1m})\,[dI_0(\psi_{1m}r)/dr]_{r=R},$$

and the asterisk shows that A_{1m} is already determined.

4.3.10. Both velocity potentials were independently determined (until now). We shall make an attempt to sew them together, determining the constants A_{1n} and B_{1n} by A^{*}_{1m}. Let us put (57) and (58) into governing equation (18), obtaining

$$\sum_n(\mathfrak{A}_{1n}A_{1n}e^{ik_{1n}x}+\mathfrak{A}_{2n}B_{1n}e^{-ik_{1n}x})J_0(\psi_{1n}r) \tag{65}$$

$$=-\sum_m A^{*}_{1m}(\mathfrak{B}_{1m}\cos k_{1m}x+i\mathfrak{B}_{2m}\sin k_{1m}x)I_0(\psi_{1m}r),$$

where

$$\begin{aligned}\mathfrak{A}_{1n}&=-(k^2_{1n}+\psi^2_{1n})+[(\sigma/c)-M_1k_{1n}]^2,\\ \mathfrak{A}_{2n}&=-(k^2_{1n}+\psi^2_{1n})+[(\sigma/c)+M_1k_{1n}]^2,\\ \mathfrak{B}_{1m}&=-(k^2_{1m}-\psi^2_{1m})+(M_1k_{1m})^2+(\sigma/c)^2,\\ \mathfrak{B}_{2m}&=-2(M_1/c)\sigma k_{1m}.\end{aligned} \tag{66}$$

Averaging along the radial co-oridinate, and introducing the designations:

$$\mathfrak{C}_{1n}=\int_0^R J^2_0(\psi_{1n}r)rdr=(R^2/2)J^2_0(\psi_{1n}R), \tag{67}$$

$$i\mathfrak{D}_{mn}=\int_0^R J_0(\psi_{1n}r)I_0(\psi_{1m}r)rdr=\frac{i(\psi_{1m}R)J_0(\psi_{1n}R)I_1(\psi_{1m}R)}{\psi^2_{1n}+\psi^2_{1m}}$$

one can write

(68)
$$(A_{1n}\mathfrak{A}_{1n}e^{ik_{1n}x}+B_{1n}\mathfrak{A}_{2n}e^{-ik_{1n}x})\mathfrak{C}_{1n}$$
$$=-i\sum_m A^*_{1m}(\mathfrak{B}_{1m}\cos k_{1m}x+i\mathfrak{B}_{2m}\sin k_{1m}x)\mathfrak{D}_{mn}.$$

Averaging along the longitudinal co-ordinate and introducing the designations:

(69)
$$d_{1mn}=\int_0^h \cos k_{1n}x \sin k_{1m}x\,dx,\quad d_{2mn}=\int_0^h \sin k_{1n}x \sin k_{1m}x\,dx,$$
$$d_{3mn}=\int_0^h \sin k_{1n}x \cos k_{1m}x dx,\quad d_{4mn}=\int_0^h \cos k_{1n}x \cos k_{1m}x dx$$

one can determine the constants A_{1n} and B_{1n} in the form:

(70)
$$A^*_{1n}=\frac{hA^*_{1m}\mathfrak{D}_{mn}[\mathfrak{B}_{2m}(d_{3mn}+id_{4mn})-\mathfrak{B}_{1m}(d_{1mn}-id_{2mn})]}{4\mathfrak{C}_{1n}\mathfrak{A}_{1n}(d_{1mn}d_{3mn}-d_{2mn}d_{4mn})},$$
$$B^*_{1n}=\frac{hA^*_{1m}\mathfrak{D}_{mn}[\mathfrak{B}_{2m}(d_{3mn}-id_{4mn})+\mathfrak{B}_{1m}(d_{1mn}+id_{2mn})]}{4\mathfrak{C}_{1n}\mathfrak{A}_{2n}(d_{1mn}d_{3mn}-d_{2mn}d_{4mn})},$$

and again the asterisk shows that the constants are determined already. As a matter of fact the constants determined must have a subscript m, but we shall miss it. One can conclude that:

(*i*) if constants A_{1n} and B_{1n} are determinable by relations (70), both potentials (57) and (58) are general solution of the governing equation (18), *Q. E. D*;

(*ii*) the independent dispersion equations

(71)
$$\mathfrak{A}_{1n}=0 \quad\text{and}\quad \mathfrak{A}_{2n}=0$$

identify resonance phenomena;

(*iii*) the dispersion relations (for preliminary given m)

(72)
$$d_{1mn}d_{3mn}-d_{2mn}d_{4mn}=0$$

determine resonances also (for every pair m, n).

Generally said, the three equations (71) and (72) identify a multitude of k_{1n} by three different ways (for every given m), but we shall determine k_{1n} from other relations playing role of real dispersion equations.

4.3.11. Obviously, conditions (25)—(27) are very natural ones, and they take into account yet once more the quality of the jump-like cross section change between two subdomains. But as it does they postulate the equality of the corresponding acoustical pressures

(73)
$$p^a_0(x=h,\ r,\ t)=p^a_1(x=h,\ r,\ t),$$

(74)
$$v^a_1(x=0,\ r,\ t)=p^a_2(x=0,\ r,\ t),$$

which is seen from the comparison between conditions (25) and (22). Finally both boundary conditions additional to (28) and (29) are taken as it follows

(75)
$$\left[\frac{\partial\varphi_0}{\partial t}-U_0\frac{\partial\varphi_0}{\partial x}\right]_{x=h}=\left[\frac{\partial\varphi_1}{\partial t}-U_1\frac{\partial\varphi_1}{\partial x}\right]_{x=h}$$

(76) $$\left[\frac{\partial\varphi_1}{\partial t}-U_1\frac{\partial\varphi_1}{\partial x}\right]_{x=0}=\left[\frac{\partial\varphi_2}{\partial t}-U_2\frac{\partial\varphi_2}{\partial x}\right]_{x=0},$$

but not in the form of relations (30) and (31) which are conditions of second order [9, Vol. I, § 14.6].

*4.3.12. **However**,* our purpose is to demonstrate how the dispersion equations of complicated cases can be constructed. That is why we shall underline here that we have four boundary conditions with four unknown constants. Let us write down this system in the table form given below

(77)

B_{0n}	A_{1n}	B_{1n}	A_{2n}	R.H.S.
f_{11mn}	f_{12mn}	f_{13mn}	0	0
f_{21mn}	f_{22mn}	f_{23mn}	0	$\mathfrak{P}_{1mn}$
0	f_{32mn}	f_{33mn}	f_{34mn}	0
0	f_{42mn}	f_{43mn}	f_{44mn}	$\mathfrak{P}_{2mn}$

where f_{ijmn} are coefficients obtained after any averaging of the boundary condition equations. For the first two equations

(78) $$f_{ijmn}=f_{ij}(k_{0n},\ \psi_{0n},\ b;\ k_{1n},\ \psi_{1n},\ k_{1m},\ \psi_{1m},\ R,\ h;\ \sigma,\ c),$$

and for the second pair of equations

(79) $$f_{ijmn}=f_{ij}(k_{1n},\ \psi_{1n},\ k_{1m},\ \psi_{1m},\ R,\ h;\ k_{2n},\ \psi_{2n},\ a;\ \sigma,\ c).$$

Both loading members $\mathfrak{P}_{1mn}$ and $\mathfrak{P}_{2mn}$ are connected with the action of constants A^*_{1m}, etc.

Using the Cramer's rule one can obtain

(80) $$A^*_{1mn}=A^*_{1n}=\frac{\Delta_{2mn}}{\Delta_{mn}},\quad B^*_{1mn}=B^*_{1n}=\frac{\Delta_{3mn}}{\Delta_{mn}};$$
$$B^*_{0n}=\frac{\Delta_{1mn}}{\Delta_{mn}},\quad A^*_{2n}=\frac{\Delta_{4mn}}{\Delta_{mn}}.$$

Equating A^*_{1n} and B^*_{1n} from (70) and (80) we shall have the sought dispersion equations for k_{1n} and ψ_{1m} determination.

*4.4. **Case** $U_1\neq0$, **but** $h=h(t)$.* This is the case initiating the present cycle of investigations. Fig. 1 is which can give a graphical representation of the problem. The solution must be integral of Eq. (18). Again we shall presume a validity of relation (56), but taken in the form

(81) $$\Phi_1=-U_1x+\varphi_1,\quad \varphi_1=\varphi_{11}+\varphi_{12}.$$

Then the boundary conditions which must be added are [4]:

(82) $$-U_1+\left.\frac{\partial\varphi_1}{\partial x}\right|_{x=0}=\begin{cases}-U_2+\left.\dfrac{\partial\varphi_2}{\partial x}\right|_{x=0}, & 0\le r\le a,\\ 0, & a<r\le R,\end{cases}$$

(83) $$\left[\frac{\partial^2\varphi_1}{\partial t^2}+g\frac{\partial\varphi_1}{\partial x}\right]_{x=h(t)}=0,$$

(84) $$\left.\frac{\partial\varphi_1}{\partial r}\right|_{r=R}=\frac{\partial w}{\partial t}-U_1\frac{\partial w}{\partial x},\quad \left.\frac{\partial\varphi_2}{\partial r}\right|_{r=a}=0,\quad etc.$$

From (82) condition (22) is obtained. In Ref. [7, 4] was proved that the condition of type (83) (but with constant liquid level h) is adequate to condition

(85) $$p_1^q(x=h,\ r,\ t)=\rho\left.\frac{\partial\varphi_1}{\partial t}\right|_{x=h}=0$$

with very good degree of approximation. That is why we shall use here namely condition (85), (but with variable level h).

The general solution φ_1 must be constructed again with the help of velocity potentials (57) and (58), but taken independently. But we are not sure that a full separation of the variables is possible. The matter is that wave numbers k_{1n}, because of the condition on the free surface (85), depend on the time, changing the sense of the whole problem. That is why, firstly, we shall isolate the variable r (and its functions) taking

(86) $$\varphi_{11}=\sum_n X_{1n}(x,\ t).J_0(\psi_{1n}r),\qquad \varphi_{12}=\sum_m X_{2m}(x,\ t).I_0(\psi_{1m}r),$$

and considering afterwards the independent equations corresponding to both functions $X_{1n}(x,\ t)$ & $X_{2n}(x,\ t)$:

(87) $$\begin{aligned}&(1-M_1^2)X''_{1n}-\psi_{1n}^2X_{1n}+2(M_1/c)\dot{X}'_{1n}-(1/c^2)\ddot{X}_{1n}=0,\\&(1-M_1^2)X''_{2m}+\psi_{1m}^2X_{2m}+2(M_1/c)\dot{X}'_{2m}-(1/c^2)\ddot{X}_{2m}=0.\end{aligned}$$

The attempt to seek the solutions of (87) in terms of moving waves shows that the integration constants must depend on the time also. Then no one of the previous, dispersion equations is valid here. That is why we shall try to deduce new dispersion equations. Again they must play role of necessary and sufficient conditions for a variable separation. However, let us substitute the functions (nonlinear on the time co-ordinate, but expressing standing waves):

(88) $$X_{1n}(x,\ t)=A_{1n}(t)\cos[k_{1n}(t)x]+B_{1n}(t)\sin[k_{1n}(t)x]$$

in Eq. $(87)_1$. After some transforms the sought equations are obtained in the form (the subscript "1" is rejected):

(89) $$\begin{aligned}&\ddot{A}_n+[k_n^2c^2(1-M^2)+\psi_n^2c^2-2Mck_n(\dot{k}_nx)-(\dot{k}_nx)^2]A_n\\&-[2Mck_n-(2\dot{k}_nx)]\,\dot{B}_n-[2\,Mc\dot{k}_n-(\ddot{k}_nx)]B_n=0,\\&\ddot{B}_n+[k_n^2c^2(1-M^2)+\psi_n^2c^2+2Mck_n(\dot{k}_nx)-(\dot{k}_nx)^2]B_n\\&+[2Mck_n-2(\dot{k}_nx)]\dot{A}_n+[2Mc\,\dot{k}_n-(\ddot{k}_nx)]A_n=0.\end{aligned}$$

The nature of these equations is not quite clear at first sight, because the boundary conditions at $x=0$ and $x=h(t)$ can be different, and so, the time dependence of A_n & B_n can be different also. The peculiar case

(90) $$A_n(t)=A_{n^*}\cos\sigma t,\quad B_n(t)=B_{n^*}\sin\sigma t,\quad \dot{k}_n=0,\quad \ddot{k}=0$$

results the dispersion equations of known type, and this plays role of verification control of Eqs (89).

The next very interesting peculiar case is when wave numbers k_n along x-axis are not preliminary determined functions on t, i. e.

(91) $$A_n(t)=A_n \cos \sigma t, \quad B_n(t)=B_n \sin \sigma t, \quad \dot{k}_n \neq 0, \quad \ddot{k}_n \neq 0.$$

Substituting (91) in 89) one can obtain the system of equations

(92) $$\begin{aligned} &-k_n^2c^2(M_{1,2}^2-1)\pm\psi_n^2c^2-2Mck_n(\dot{k}_n\bar{x})-(\dot{k}_n\bar{x})^2 \mp 2\sigma(\dot{k}_n\bar{x})=0, \\ &(\ddot{k}_n\bar{x})-2Mc\dot{k}_n=0, \end{aligned}$$

M_2 being excluded further, and $\bar{x}$ determines constant phase characteristics. As this systems must be solved for arbitrary $\bar{x}$, it must be eliminated, and so, the nonlinear differential equation

(93) $$[(M_1^2-1)k_n^2\pm\psi_n^2]\,(\ddot{k}_n)^2-4MM_1k_n(\dot{k}_n)^2\,(\ddot{k}_n)+4M^2(\dot{k}_n)^4=0$$

is obtained. Its solution (relatively said) for velocity potentials of type (58), but rewritten down in the form

(94) $$\varphi_{11}=\sum_n \sigma(A_n \cos k_nx \cos \sigma t+B_n \sin k_nx \sin \sigma t)J_0(\psi_nr)$$

will be (the signs are emphasized)

(95) $$(\ddot{k}_n)_{1,2}=2M(\dot{k}_n)^2[(M_1^2-1)k_n^2+\psi_n^1]^{-1}[M_1k_n\pm\sqrt{k_n^2-\psi_n^2}].$$

Analogously, for velocity potentials of type (58), but rewritten down in fhe form

(96) $$\varphi_{12}=\sum_m \sigma(A_m \cos k_mx \cos\sigma t+B_m \sin k_m x\sin \sigma t)I_0(\psi_mr)$$

we shall have the solution

(97) $$(\ddot{k}_m)_{1,2}=2M(\dot{k}_m')^2[(M_1^2-1)k_m^2-\psi_m^2]^{-1}[M_1k_m\pm\sqrt{k_m^2+\psi_m^2}].$$

It is important to underline that because of the nonlinearity of Eqs (92) and (93) the values for k_n and k_m are not discrete. However, (92) and (93) role of dispersion equations of the last problem under investigation.

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