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Secondary Streaming Induced by the Coaxial Torsional Oscillations of Two Spheres in Viscous Liquid

N. Kovatcheva, Z. Zapryanov

1. Introduction

In this paper we consider the axisymmetrical viscous flow around two spheres which perform torsional oscillations. The spheres are external to each other and oscillate about their line of centres. Use is made of bispherical co-ordinates, which were employed by Jeffery [1] and Stimson and Jeffery [2]. The analytical solution is obtained for low amplitude and high frequency of oscillations. We employ the method of matched singular asymptotic expansions, described in details by Van Dyke [3].

Much attention has been given to the incompressible oscillatory flows, because of their importance in variety of applications, especially in chemical and biological technology. Most papers related to the rotational oscillations of bodies in viscous liquid study the oscillations of a single sphere in unbounded fluid. Stokes [4] first discussed the problem of torsional oscillations of a finite disk and gave the solution for transverse oscillations of a body in viscous liquid. Interest in problem of torsional oscillatory streamings was arisen with the purpose of measuring the viscousity of the fluid. Thus Helmholtz and Piotrowski [5] studied the torsional oscillations of spheres and cylinders in viscous liquid. Buchanan [6] treated the problem of transverse and torsional oscillations of a spheroid. The above-mentioned authors assumed that the fluid particles move in concentric circles whose centres lie on the axis of rotation. The existence of secondary flow in the plane containing axis of rotation was shown by Carrier and Di Prima [7]. Thus the fluid motion consists of a primary motion about the axis of rotation and a secondary motion in the meridiane plane.

Later on Stuart [8] revealed the generation of steady streaming in oscillating fluid flows and discussed its physical mechanism. Using the technique of matched asymptotic expansions Riley [9] developed a general theory for the flow induced by oscillating body in unbounded liquid and gave a detailed discussion for the nature of the flow.

In recent years there have been obtained interesting theoretical and experimental results for concentric torsional oscillating spheres and theoretical results for eccentric oscillating spheres. Munson and Douglass [10] considered viscous incompressible flow in concentric spherical annuli when the bounding spherical surfaces perform torsional oscillations. They obtained a perturbation solution for small frequency of oscillations and for case of high frequency of oscillations the experimental results were presented. Tabakova and Zapryanov [11] investigated the oscillatory viscous flow in concentric spherical container and gave theoretical solution for high frequency and low amplitude of oscillations. The numerical solution for this flow was obtained by Zapryanov and

Christov [12] and now the solution to the problem could be thought of as virtually completed.

The torsional oscillating viscous flow in eccentric spherical annuli was investigated theoretically by Kovatcheva, Christov and Zapryanov [13], where the analytical solution was obtained for high-frequency small-amplitude oscillations. In the present paper the technique devepoped in [13] is extended for the case of two external to each other spheres. The flow is created by the high-frequency small-amplitude coaxial torsional oscillations of the spheres.

2. Formulation of the problem

We consider two separate spheres I and II of radii a and b, respectively, torsionally oscillating with frequency ω and amplitude $\bar{\Omega}$ in unbounded fluid of kinematic viscousity ν. The spheres are external to each other and oscillate about their line of centres (see fig. 1). The fluid motion is described in bispherical co-ordinates (η, ξ, φ) (see Happel and Brenner [14], Appendix A.19). The relations between (η, ξ, φ) and appropriate polar cylindrical co-ordinates (ρ, z, φ) are

$$\rho=c\frac{\sin\eta}{\operatorname{ch}\xi-\cos\eta}, \quad z=c\frac{\operatorname{sh}\xi}{\operatorname{ch}\xi-\cos\eta}, \tag{2.1}$$

where $0\leq\eta\leq\pi$, $-\infty<\xi\leq\infty$ and c is a positive constant. The two spheres I and II are defined by $\xi=\alpha_1$, $\xi=a_2$ $(\alpha_1>0,\ \alpha_2<0)$.

Since the axisymmetrical flow is considered a stream function is introduced as follows

$$v'_\xi=\frac{(\operatorname{ch}\xi-\cos\eta)^2}{c^2\sin\eta}\frac{\partial\psi'}{\partial\eta}, \quad v'_\eta=-\frac{(\operatorname{ch}\xi-\cos\eta)^2}{c^2\sin\eta}\frac{\partial\psi'}{\partial\xi}. \tag{2.2}$$

As we consider only oscillations of small amplitude we expand the stream function in series with respect to the amplitude parameter $\varepsilon=\bar{\Omega}/\omega$ and retain only the terms that give order of approximation $o(\varepsilon)$. Thus we obtain from the Navier-Stokes equations the following set of equations (for details see, e. g. [13]):

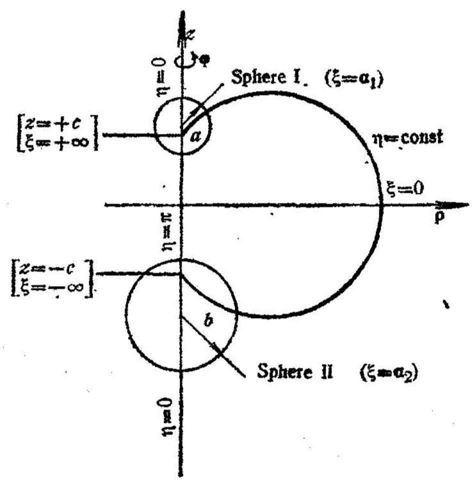

Fig. 1. Geometry of the problem and co-ordinate system.

(2.3) $$\frac{\partial\Omega}{\partial t}=\frac{1}{M^2}D^2\Omega,$$

(2.4) $$\frac{\partial(D^2\psi)}{\partial t}-\frac{\mathrm{Re}\,[\Omega]\,(\mathrm{ch}\,\xi-\mu)^4}{\sqrt{1-\mu^2}}\;\frac{\partial\left(\mathrm{Re}\,[\Omega],\ \frac{\sqrt{1-\mu^2}}{\mathrm{ch}\,\xi-\mu}\right)}{\partial(\xi,\ \mu)}=\frac{1}{M^2}D^4\psi$$

for the dimensionless angular velocity $\Omega=v'_\varphi/\varepsilon c^2\omega$ and stream function $\psi=\psi'/\varepsilon c^3\omega$. In the above equations is made use of the complex-valued function Ω for the sake of convenience. Respectively

$$D^2\equiv(\mathrm{ch}\,\xi-\mu)\left\{\frac{\partial}{\partial\xi}\left[(\mathrm{ch}\,\xi-\mu)\frac{\partial}{\partial\xi}\right]+(1-\mu^2)\frac{\partial}{\partial\mu}\left[(\mathrm{ch}\,\xi-\mu)\frac{\partial}{\partial\mu}\right]\right\},$$

$$\mu=\cos\eta,$$

and $M=\sqrt{\omega c^2/\nu}$ is called frequency parameter. In terms of the complex-valued functions the boundary conditions adopt the form

(2.5) $$\psi=\frac{\partial\psi}{\partial\xi}=0\quad\text{at}\quad\xi=\alpha_1,\ \xi=\alpha_2,$$

(2.6) $$\Omega=\frac{1-\mu^2}{(\mathrm{ch}\,\alpha_1-\mu)^2}\,e^{it}\quad\text{at}\quad\xi=\alpha_1,$$

(2.7) $$\Omega=\frac{1-\mu^2}{(\mathrm{ch}\,\alpha_2-\mu)^2}\,e^{it}\quad\text{at}\ \xi=\alpha_2,$$

(2.8) $$\psi=0,\quad\Omega=0\quad\text{at}\quad\mu=\pm1.$$

3. Method of matched asymptotic expansions

We consider large values of frequence parameter M, which means that there is a thin boundary layer of thickness $O(\nu/\omega)^{1/2}$ in the neighbourhood of the spheres. So the region occupied by the flow is divided into three regions — two boundary layers at spheres walls and a flow outside. In the boundary layers we introduce the inner variables as follows:

(3.1) $$\zeta=(\alpha_1-\xi)\frac{M}{\sqrt{2}}$$

for the sphere I ($\xi=\alpha_1$) and

(3.2) $$\bar{\zeta}=(\xi-\alpha_2)\frac{M}{\sqrt{2}}$$

for the sphere II ($\xi=\alpha_2$).

The asymptotic expansions for the two functions Ω and ψ are

(3.3) $$\Omega^{(i)}=\sum_{n=0}^{\infty}\Omega_n^{(i)}(\eta,\ \xi,\ t)\,\alpha_n^{(i)}(M^{-1})$$

and

(3.4) $$\psi^{(i)}=\sum_{n=0}^{\infty}\psi_n^{(i)}(\eta,\ \xi,\ t)\ \beta_n^{(i)}(M^{-1}),$$

where $i=0$ refers to the flow outside, $i=1$ — to the boundary layer at the sphere $\xi=\alpha_1$ and $i=2$ — to the sphere $\xi=\alpha_2$. The terms of the asymptotic series $\{\alpha_n^{(i)}(M^{-1})\}_{n=0}^{\infty}$ and $\{\beta_n^{(i)}(M^{-1})\}_{n=0}^{\infty}$ satisfy the following conditions:

$$\lim_{M^{-1}\to 0}\frac{\alpha_{n+1}^{(i)}(M^{-1})}{\alpha_n^{(i)}(M^{-1})}=0,\quad \lim_{M^{-1}\to 0}\frac{\beta_{n+1}^{(i)}(M^{-1})}{\beta_n^{(i)}(M^{-1})}=0.$$

Using the matching procedure (see [3]) we obtain for angular velocity the expressions

$$\Omega_0^{(1)}=\frac{1-\mu^2}{(\operatorname{ch}\alpha_1-\mu)^2}\exp\left(it-\frac{i+1}{\operatorname{ch}\alpha_1-\mu}\zeta\right),\quad \Omega_0^{(0)}=0,$$

$$\Omega_0^{(2)}=\frac{1-\mu^2}{(\operatorname{ch}\alpha_2-\mu)^2}\exp\left(it-\frac{1+\mu}{\operatorname{ch}\alpha_2-\mu}\bar{\zeta}\right),$$

$$\Omega_1^{(0)}=0,\quad \Omega_1^{(1)}=-\frac{(i+1)(1-\mu^2)\operatorname{sh}\alpha_2}{\sqrt{2}(\operatorname{ch}\alpha_1-\mu)^4}\zeta^2\exp\left(it-\frac{i+1}{\operatorname{ch}\alpha_1-\mu}\zeta\right),$$

$$\Omega_1^{(2)}=\frac{(i+1)(1-\mu^2)\operatorname{sh}\alpha_2}{\sqrt{2}(\operatorname{ch}\alpha_2-\mu)^4}\bar{\zeta}^2\exp\left(it-\frac{i+1}{\operatorname{ch}\alpha_2-\mu}\bar{\zeta}\right).$$

The equation (2.4) for ψ contains terms proportional to

$$\operatorname{Re}[e^{it}]\operatorname{Re}[e^{it}]=\frac{1}{2}(1+e^{2it}).$$

Thus the solution for ψ should be comprised of steady and unsteady parts, namely

$$\psi=\bar{\psi}(\xi,\mu)+e^{2it}\bar{\bar{\psi}}(\xi,\mu). \tag{3.5}$$

The steady streaming is of considerable interest and we focus our attention to obtain $\bar{\psi}_0^{(0)}$. For function $\bar{\psi}_0^{(0)}$ we obtain the equation

$$D^4\bar{\psi}_0^{(0)}=0. \tag{3.6}$$

The general solution of (3.6) (see [2]) is

$$\bar{\psi}_0^{(0)}=(\operatorname{ch}\xi-\mu)^{-3/2}\sum_{n=1}^{\infty}\Big[A_n\operatorname{ch}\left(n-\frac{1}{2}\right)\xi$$

$$+B_n\operatorname{sh}\left(n-\frac{1}{2}\right)\xi+C_n\operatorname{ch}\left(n+\frac{3}{2}\right)\xi+D_n\operatorname{sh}\left(n+\frac{3}{2}\right)\xi\Big]C_{n+1}^{-1/2}(\mu), \tag{3.7}$$

where $C_{n+1}^{-1/2}(\mu)$ are the Gegenbauer polynomials of order $n+1$ and degree $-1/2$. The constants A_n, B_n, C_n and D_n are determined by the boundary conditions (2.5) and by matching with the solutions in the boundary layers. Thus the system for unknowns A_n, B_n, C_n and D_n is

$$A_n\operatorname{ch}\left(n-\frac{1}{2}\right)\alpha_1+B_n\operatorname{sh}\left(n-\frac{1}{2}\right)\alpha_1+C_n\operatorname{ch}\left(n+\frac{3}{2}\right)\alpha_1+D_n\operatorname{sh}\left(n+\frac{3}{2}\right)\alpha_1=0,$$

$$(2n-1)\left[A_n\operatorname{sh}\left(n-\frac{1}{2}\right)\alpha_1+B_n\operatorname{ch}\left(n-\frac{1}{2}\right)\alpha_1\right]$$

$$+(2n+3)\left[C_n\operatorname{sh}\left(n+\frac{3}{2}\right)\alpha_1+D_n\operatorname{ch}\left(n+\frac{3}{2}\right)\alpha_1\right]=a_n(\alpha_1)$$

$$A_n\operatorname{ch}\left(n-\frac{1}{2}\right)\alpha_2+B_n\operatorname{sh}\left(n-\frac{1}{2}\right)\alpha_2+C_n\operatorname{ch}\left(n+\frac{3}{2}\right)\alpha_2+D_n\operatorname{sh}\left(n+\frac{3}{2}\right)\alpha_2=0, \tag{3.8}$$

$$(2n-1)\left[A_n \operatorname{sh}\left(n-\frac{1}{2}\right)\alpha_2+B_n \operatorname{ch}\left(n-\frac{1}{2}\right)\alpha_2\right]$$

$$+(2n+3)\left[C_n \operatorname{sh}\left(n+\frac{3}{2}\right)\alpha_2+D_n \operatorname{ch}\left(n+\frac{3}{2}\right)\alpha_2\right]=a_n(\alpha_2),$$

where

$$a_n(\alpha_1)=\frac{(2e^{-\alpha_1})^{3/2}}{2}\left[n(n+1)e^{-\alpha_1 n}+\frac{e^{-2\alpha_1}-1}{3}(n-1)n(n+1)e^{-\alpha_1(n-2)}\right] \tag{3.9}$$

and

$$a_n(\alpha_2)=\frac{(2e^{\alpha_2})^{3/2}}{2}\left[n(n+1)e^{\alpha_2 n}+\frac{e^{2\alpha_2}-1}{3}(n-1)n(n+1)e^{\alpha_2(n-2)}\right]. \tag{3.10}$$

4. Numerical results and discussion

Solving the system (3.8) offers certain difficulties because the value of its determinant is of several orders of magnitude smaller than the magnitudes of coefficients. For this reason it is solved analytically and the solution is manipulated in a manner to cancel the large terms in the analytical formulae priory to the numerical investigation. Having the solution of (3.8) one can calculate the value of stream function ψ for given (η, ξ) on the base of representation (3.7).

The most informative and convenient way to present the results concerning the secondary streaming is through the isolines of function ψ (the stream lines). To resolve the transcendental equation $\psi=\text{const}$ with ψ defined from Eq. (3.7) is a hard task to handle and requires a vast amount of computational time. For this reason we prefer an approximate approach in which the values of ψ are calculated in a number of grid points and then a standart procedure earlier devised by the authors is invoked to trace the streamlines by means of linear interpolation. It can be always done accurate enough since the analytical formula (3.7) is at our disposal and we can calculate the values of ψ at more dense mesh, if necessary.

The spacing in ξ-direction of the grid employed is

$$h_\xi=(\alpha_1-\alpha_2)/(M-1), \tag{4.1}$$

where M is the total number of grid points in ξ direction. In η-direction we use a non-uniform grid and hence an array of values of h_η which differ from point to point. The latter proved necessary, because of the fact that the bispherical co-ordinate system is more sparce in the regions at the top and bottom of fig. 1. We "correct" this by taking a mesh that is uniformly spaced along the surface of sphere for which $\xi=\alpha_2$ Then

$$\eta_j=2\operatorname{arctg}\left[\frac{\operatorname{sh}\alpha_2}{\operatorname{ch}\alpha_2+1}\operatorname{tg}\frac{s_j\cdot\operatorname{sh}\alpha_2}{2c}\right], \tag{4.2}$$

where

$$s_j=(j-1)\frac{c\pi}{(N-1)\operatorname{sh}\alpha_2}$$

and respectively $h_{\eta j}=\eta_{j+1}-\eta_j$.

Making use of the above described algorithm we have obtained results for various magnitude of focal distance c and ratio of the spheres radii $\lambda=b/a$. Unlike the case of eccentric spherical annuli (see [13], [15]) the pattern of secondary flow here is much complicated being governed by intricate hydrodynamic interaction.

On fig. 2 is presented the secondary streaming when the two spheres are of the same radius. The hydrodynamic interaction shows up here through the occurrence of

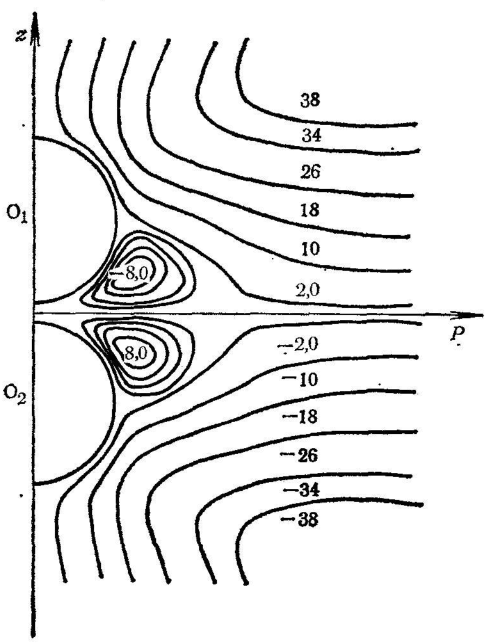

Fig. 2. Steady part of secondary flow $\psi \cdot 10^2$ for $c=1$ and $a=b=2{,}5$

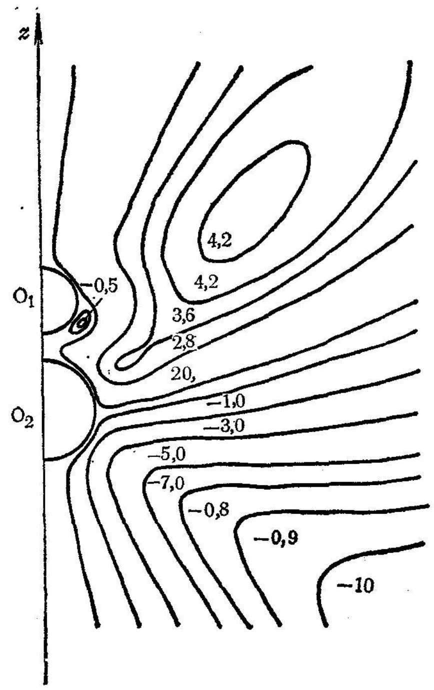

Fig. 3 Steady part of secondary flow $\psi \cdot 10^2$ for $c=1$ and $\lambda=1{,}5$

four-vortex structure (for comparison see the two-vortex structure of secondary flow for a single sphere [11, fig. 1]). Changing the value of focal distance c does not alter the qualitative outlook of the picture and hence the respective plots are not presented here for the sake of brevity.

The complexity of the flow patterns raises dramatically when considering unequal spheres. On fig. 3 is plotted the pattern of the flow for the same focal distance $c=1$ but for $\lambda=1{,}5$. It is seen that the positive vortices coalesce with each other and form a flow known as "cat-eye" (see for definition [16]). In stationary flows the presence of a cat-eye is a sign of developing instability while in the case under consideration the flow is completely stable. That is a demonstration of one of the differences between the secondary steady streaming in oscillatory flow and stationary main flow.

The above described phenomenon is observed also for some larger focal distances, e. g. $c=3$ (see fig. 4) but the cat-eye structure elongates and its singular point (where a streamline interacts itself) is moved further from the spheres. Increasing the focal distance c or the radii ratio λ results in still further removal of singular point and for $c=3$, $\lambda=2{,}5$ (fig. 5) it is not observed on the employed grid. The latter means that either it vanishes yielding to a simple vortex flow with positive rotation or it is so far away from the sphere and hence so close to infinity ($\eta=0$, $\xi=0$) that it can not be discerned.

In conclusion it is to be said that the flow under consideration offers very interesting and instructive example for hydrodynamic interaction in oscillatory fluid flows around systems of rigid bodies.

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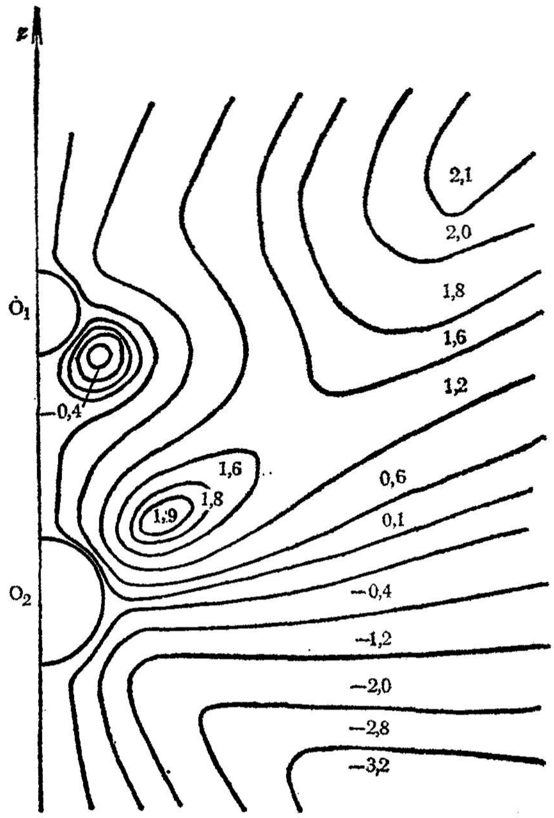

Fig. 4. Steady part of secondary flow $\psi \cdot 10^3$ for $c=3$ and $\lambda=1,5$

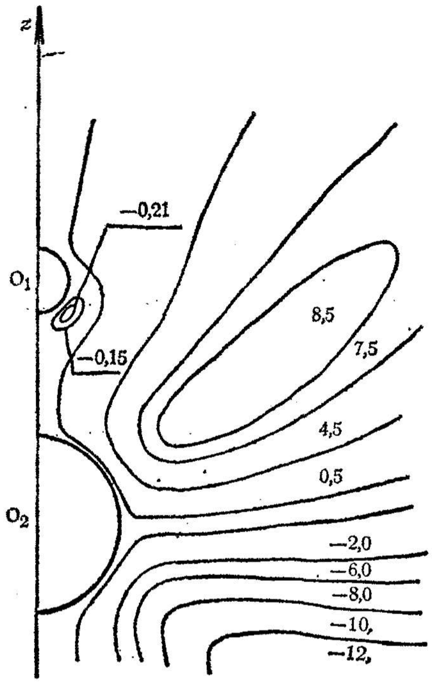

Fig. 5. Steady part of secondary flow $\psi \cdot 10^3$ for $c=3$ and $\lambda=2,5$

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